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CANONICAL ALMOST GEODESIC MAPPINGS $\pi_2(e),\ e = \pm 1$, OF SPACES WITH AFFINE CONNECTION ONTO m-SYMMETRIC SPACES

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Abstract. In the paper we consider canonical almost geodesic mappings $\pi_2(e)$, $e = \pm 1$, of spaces with affine connection onto 2-symmetric, 3-symmetric and m-symmetric spaces. The main equations for the mappings have been obtained as closed systems of PDEs of Cauchy type in covariant derivatives. We have found the maximum numbers of essential parameters which the general solutions of the systems depend on.

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1. INTRODUCTION

The paper is devoted to further study of the theory of almost geodesic mappings of affinely connected spaces. The theory goes back to the paper [14], by T. Levi-Civita, in which the problem on the search for Riemannian spaces with common geodesics was stated and solved in a special coordinate system. We note a remarkable fact that this problem is related to the study of equations of dynamics of mechanical systems.

The theory of geodesic mappings has been developed by T. Thomas, J. Thomas, H. Weyl, L. P. Eisenhart, P.A. Shirokov, A.S. Solodovnikov, N.S. Sinyukov, A.V. Aminova, J. Mikeš, and others, see [1, 16, 20].

Issues arisen by the exploration were studied by V.F. Kagan, D.V. Vedenyapin, G. Vrançeanu, Ya.L. Shapiro and others. The authors discover special classes of $(n-2)$-projective spaces.

In [17], A.Z. Petrov introduced the notion of quasi-geodesic mappings. In particular, holomorphically projective mappings of Kählerian spaces are special quasi-geodesic mappings; they were examined by T. Otsuki and Y. Tashiro, M. Prvanović, J. Mikeš, and others, see [16, 20].

A natural generalization of these classes of mappings is the class of almost geodesic mappings introduced by Sinyukov (see [19, 20]). He also specified three types of almost geodesic mappings π_1, π_2, π_3.

The theory of almost geodesic mappings was developed by V.S. Sobchuk [22], N.Y. Yablonskaya [25], V.E. Berezovski, J. Mikeš [2–11, 15, 16], Lj.S. Velimirović, N. Vesić, M.S. Stankovič, [24] et al.

In this paper we consider almost geodesic mappings of the second type $\pi_2(e)$, $e = \pm 1$, of spaces with affine connection onto 2-symmetric, 3-symmetric and m-symmetric spaces. The main equations for the mappings have been obtained as closed systems of PDEs of Cauchy type in covariant derivatives. Also we have found the maximum numbers of essential parameters which the solutions of the systems depend on.

2. Basic definitions of almost geodesic mappings of spaces with affine connections.

Let us recall the basic definition, formulas and theorems of the theory presented in [4, 15, 16, 19, 21].

Consider a space A_n with an affine torsion-free connection $\Gamma^h_{ij}(x)$. The space is referred to a local coordinate system $x^1, x^2, \ldots, x^n$.

A curve $l\colon\ x^h = x^h(t)$ in the space A_n is a *geodesic* if its tangent vector $\lambda^h(t) = dx^h(t)/dt$ satisfies the equations

$$\lambda^h_1 = \rho(t)\cdot\lambda^h,$$

where

$$\lambda^h_1 \equiv \lambda^h_{,\alpha}\lambda^\alpha = d\lambda^h(t)/dt + \Gamma^h_{\alpha\beta}(x(t))\lambda^\alpha(t)\lambda^\beta(t),$$

and $\rho(t)$ is a function of t. We denote by comma "," the covariant derivative with respect to the connection of the space A_n.

A curve $l\colon\ x^h = x^h(t)$ in the space A_n $(n > 2)$ is an *almost geodesic* if its tangent vector $\lambda^h(t)$ satisfies the equations

$$\lambda^h_2 = a(t)\cdot\lambda^h + b(t)\cdot\lambda^h_1,$$

where $\lambda^h_2 \equiv \lambda^h_{1,\alpha}\lambda^\alpha$, $a(t)$ and $b(t)$ are functions of t.

We say that a mapping $f : A_n \to \overline{A}_n$ is an *almost geodesic mapping* if any geodesic curve of A_n is mapped under f onto an almost geodesic curve in $\overline{A}_n$.

Suppose, that a space A_n with affine connection $\Gamma^h_{ij}(x)$ admits a mapping f onto a space $\overline{A}_n$ with affine connection $\overline{\Gamma}^h_{ij}(x)$, and the spaces are referred to a common coordinate system $x^1, x^2, \ldots, x^n$.

The tensor

$$P^h_{ij}(x) = \overline{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x) \tag{2.1}$$

is called a *deformation tensor* of the connections $\Gamma^h_{ij}(x)$ and $\overline{\Gamma}^h_{ij}(x)$ with respect to the mapping f. The symbols $\Gamma^h_{ij}(x)$ and $\overline{\Gamma}^h_{ij}(x)$ are components of affine connections of the spaces A_n and $\overline{A}_n$ respectively. The components are expressed in the common local coordinate system.

It is known [16, 20, 21] that a necessary and sufficient condition for the mapping of a space A_n onto a space $\overline{A}_n$ to be almost geodesic is that the deformation tensor $P^h_{ij}(x)$ of the mapping f in the common coordinate system $x^1, x^2, \dots, x^n$ has to satisfy the condition

$$A^h_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma = a\cdot P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + b\cdot\lambda^h,$$

where λ^h is an arbitrary vector, a and b are certain functions of variables $x^1, x^2, \dots, x^n$ and $\lambda^1, \lambda^2, \dots, \lambda^n$. The tensor A^h_{ijk} is defined as

$$A^h_{ijk} \stackrel{\text{def}}{=} P^h_{ij,k} + P^\alpha_{ij}P^h_{\alpha k}.$$

According to the character of a and b, i. e. depending on how the functions involve the coordinates $\lambda^1, \lambda^2, \dots, \lambda^n$ of the vector λ, N.S. Sinyukov [16, 20] distinguished three kinds of almost geodesic mappings, namely π_1, π_2 and π_3.

A mapping $f : A_n \to \overline{A}_n$ is called *almost geodesic of type* π_1, if the conditions

$$A^h_{(ijk)} = \delta^h_{(i}a_{jk)} + b_{(i}P^h_{jk)}$$

are satisfied, where a_{ij} is a symmetric tensor, b_i is a covariant vector, and δ^h_i is the Kronecker delta. We denote by the round parentheses an operation called *symmetrization* without division with respect to the indices i, j and k.

A mapping $f\colon A_n \to \overline{A}_n$ is called *almost geodesic of type* π_2, if the conditions

$$P^h_{ij} = \delta^h_{(i}\psi_{j)} + F^h_{(i}\varphi_{j)}, \tag{2.2}$$

$$F^h_{(i,j)} + F^h_\alpha F^\alpha_{(i}\varphi_{j)} = \delta^h_{(i}\mu_{j)} + F^h_{(i}\rho_{j)} \tag{2.3}$$

holds. Here ψ_i, φ_i, μ_i, ρ_i are some covectors, F^h_i is a tensor of type $(1,1)$.

We consider mappings $\pi_2 : A_n \to \overline{A}_n$ characterized locally in a common coordinate system, by the equations (2.2) and (2.3) as corresponding to $F^h_i(x)$.

A mapping π_2 satisfies the *mutuality condition* if the inverse mapping is also an almost geodesic of type π_2 and corresponding to the same affinor $F^h_i(x)$.

The mappings π_2 satisfying the mutuality condition will be denoted as $\pi_2(e)$, where $e = -1, 0, 1$.

As it was proved in [23], in the case when $e = \pm 1$ the basic equations of the mappings $\pi_2(e)$ can be written as (2.2), the differential equations

$$F^h_{i,j} = F^h_{ij}, \qquad F^h_{ij,k} = \overset{6}{\Theta}{}^h_{ijk}, \qquad \mu_{i,j} = \mu_{ij}, \qquad \mu_{ij,k} = \overset{7}{\Theta}_{ijk}, \tag{2.4}$$

and algebraic equations

$$F^h_{(ij)} = F^h_{(i}\mu_{j)} - \delta^h_{(i}F^\alpha_{j)}\mu_\alpha, \qquad F^h_\alpha F^\alpha_i = e\delta^h_i, \qquad \mu_{(ij)} = \overset{5}{\Theta}_{ij}, \tag{2.5}$$

where

$$\overset{1}{\Theta}{}^h_{ijk} \equiv \overset{2}{\Theta}{}^h_{ijk} + \overset{2}{\Theta}{}^h_{kji} - \overset{2}{\Theta}{}^h_{jki} + 2F^h_\alpha R^\alpha \alpha_{kji} - F^\alpha_i R^h_{\alpha jk} + F^\alpha_j R^h_{\alpha ik} + F^\alpha_k R^h_{\alpha ij},$$

$$\overset{2}{\Theta}{}^h_{ijk} \equiv \mu_{(i} F^h_{j)k} - \delta^h_{(i} F^\alpha_{j)k} \mu_\alpha,$$

$$\overset{3}{\Theta}{}^h_{ijk} \equiv \overset{2}{\Theta}{}^h_{ijk} - \overset{2}{\Theta}{}^h_{kji} + F^\alpha_j R^h_{\alpha ik} - F^h_\alpha R^\alpha_{jik},$$

$$\overset{4}{\Theta}_{jk} \equiv F^\alpha_\beta \overset{1}{\Theta}{}^\beta_{\alpha jk} + 2F^\alpha_{\beta j} F^\beta_{\alpha k},$$

$$\overset{5}{\Theta}_{jk} \equiv \frac{1}{(n-1-F^\alpha_\alpha)^2 - 1}\big((n-1-F^\alpha_\alpha)\overset{4}{\Theta}_{ij} + \overset{4}{\Theta}_{\alpha\beta} F^\alpha_i F^\beta_j\big),$$

$$\overset{6}{\Theta}{}^h_{ijk} \equiv \frac{1}{2}\big(F^h_i \mu_{(jk)} + F^h_j \mu_{[ik]} + F^h_k \mu_{[ij]} - \delta^h_i m_{(jk)} - \delta^h_j m_{[ik]} - \delta^h_k m_{[ij]} + \overset{1}{\Theta}{}^h_{ikj}\big),$$

$$\overset{7}{\Theta}_{ijk} \equiv \mu_\alpha R^\alpha_{kji} + \frac{1}{2}\big(\overset{5}{\Theta}_{ij,k} + \overset{5}{\Theta}_{ik,j} - \overset{5}{\Theta}_{jk,i}\big), \qquad m_{ij} \equiv F^\alpha_i \mu_{\alpha j},$$

$F^h_i(x)$, $F^h_{ij}(x)$, $\mu_i(x)$, $\mu_{ij}(x)$ are unknown functions, $R^h_{ijk}(x)$ is the Riemann tensor of the space A_n. We denote by the brackets $[ik]$ an operation called *antisymmetrization (or, alternation)* without division with respect to the indices i and k.

Obviously, right hand sides of the equations (2.4) depend on unknown functions $F^h_i(x)$, $F^h_{ij}(x)$, $\mu_i(x)$, $\mu_{ij}(x)$, and on the components $\Gamma^h_{ij}(x)$ of the space A_n.

The equations (2.4) and (2.5) form a closed mixed system of differential equations of Cauchy type in covariant derivatives with respect to the functions $F^h_i(x)$, $F^h_{ij}(x)$, $\mu_i(x)$, $\mu_{ij}(x)$. Also the mapping $\pi_2(e)$ depends on unknown functions $\psi_i(x)$, $\varphi_j(x)$ (see the equations (2.2)).

An almost geodesic mapping π_2 for which $\psi_i \equiv 0$ is called *canonical*. It is known [21] that any almost geodesic mapping π_2 can be written as the composition of a canonical almost geodesic mapping of type π_2 and a geodesic mapping. The latter may be referred to as a trivial almost geodesic mapping.

Hence canonical almost geodesic mappings $\pi_2(e)$, $e = \pm 1$, are determined by the equations

$$P^h_{(ij)} = F^h_{(i}\varphi_{j)}, \tag{2.6}$$

and also by the equations (2.4) and (2.5).

A mapping $f : A_n \to \overline{A}_n$ is *almost geodesic of type* π_3, if the conditions

$$P^h_{ij} = \delta^h_{(i}\psi_{j)} + \theta^h a_{ij}, \qquad \theta^h_{,i} = \rho \cdot \delta^h_i + \theta^h a_i$$

holds. Here θ^h is a certain vector, ψ_i, a_i are certain covectors, a_{ij} is a certain symmetric tensor and ρ is a certain function.

The types of almost geodesic mappings π_1, π_2, π_3 can intersect. The problem of completeness of classification had long remained unresolved. V. Berezovsky and

J. Mikeš [7] proved that for $n > 5$ other types of almost geodesic mappings except π_1, π_2, and π_3 do not exist.

3. Canonical almost geodesic mappings $\pi_2(e)$, $e = \pm 1$, of spaces A_n with affine connections onto 2-symmetric spaces

A space $\overline{A}_n$ with an affine connection is called *2-symmetric* if its Riemann tensor $\overline{R}^h_{ijk}$ satisfies the condition

$$\overline{R}^h_{ijk|m\rho} = 0. \tag{3.1}$$

By the symbol "$|$" we denote covariant derivative respect to the connection of the space $\overline{A}_n$.

We recall that *symmetric* spaces $\overline{A}_n$ are characterized by $\overline{R}^h_{ijk|m} = 0$. Symmetric spaces were introduced by P.A. Shirokov [18] and É. Cartan [12], see also S. Helgason [13].

Let us consider the canonical almost geodesic mappings of type $\pi_2(e)$, $e = \pm 1$, of spaces A_n with affine connection onto 2-symmetric spaces $\overline{A}_n$, which are determined by the equations (2.4), (2.6) and (2.5). Suppose, that the spaces A_n and $\overline{A}_n$ are referred to a common coordinate system $x^1, x^2, \ldots, x^n$.

Since

$$\overline{R}^h_{ijk|m} = \frac{\partial \overline{R}^h_{ijk}}{\partial x^m} + \overline{\Gamma}^h_{m\alpha}\overline{R}^\alpha_{ijk} - \overline{\Gamma}^\alpha_{mi}\overline{R}^h_{\alpha jk} - \overline{\Gamma}^\alpha_{mj}\overline{R}^h_{i\alpha k} - \overline{\Gamma}^\alpha_{mk}\overline{R}^h_{ij\alpha},$$

then taking account of (2.1) we can obtain

$$\overline{R}^h_{ijk|m} = \overline{R}^h_{ijk,m} + P^h_{m\alpha}\overline{R}^\alpha_{ijk} - P^\alpha_{mi}\overline{R}^h_{\alpha jk} - P^\alpha_{mj}\overline{R}^h_{i\alpha k} - P^\alpha_{mk}\overline{R}^h_{ij\alpha}. \tag{3.2}$$

Since according to the definition of covariant derivative

$$\left(\overline{R}^h_{ijk|m}\right)_{,\rho} = \frac{\partial \overline{R}^h_{ijk|m}}{\partial x^\rho} + \Gamma^h_{\alpha\rho}\overline{R}^\alpha_{ijk|m} - \Gamma^\alpha_{i\rho}\overline{R}^h_{\alpha jk|m} - \Gamma^\alpha_{j\rho}\overline{R}^h_{i\alpha k|m} - \Gamma^\alpha_{k\rho}\overline{R}^h_{ij\alpha|m} - \Gamma^\alpha_{m\rho}\overline{R}^h_{ijk|\alpha},$$

then taking account of (2.1), we have

$$\begin{aligned}\left(\overline{R}^h_{ijk|m}\right)_{,\rho} = \overline{R}^h_{ijk|m\rho} - P^h_{\alpha\rho}\overline{R}^\alpha_{ijk|m} + P^\alpha_{i\rho}\overline{R}^h_{\alpha jk|m} + P^\alpha_{j\rho}\overline{R}^h_{i\alpha k|m} \\ + P^\alpha_{k\rho}\overline{R}^h_{ij\alpha|m} + P^\alpha_{m\rho}\overline{R}^h_{ijk|\alpha}.\end{aligned} \tag{3.3}$$

Differentiating (3.2) with respect to x^ρ in the space A_n, we get

$$\begin{aligned}\left(\overline{R}^h_{ijk|m}\right)_{,\rho} = \overline{R}^h_{ijk,m\rho} + P^h_{m\alpha,\rho}\overline{R}^\alpha_{ijk} + P^h_{m\alpha}\overline{R}^\alpha_{ijk,\rho} - P^\alpha_{mi,\rho}\overline{R}^h_{\alpha jk} - P^\alpha_{mi}\overline{R}^h_{\alpha jk,\rho} \\ - P^\alpha_{mj,\rho}\overline{R}^h_{i\alpha k} - P^\alpha_{mj}\overline{R}^h_{i\alpha k,\rho} - P^\alpha_{mk,\rho}\overline{R}^h_{ij\alpha} - P^\alpha_{mk}\overline{R}^h_{ij\alpha,\rho}.\end{aligned} \tag{3.4}$$

Substituting in (3.3) from (3.4), we have

$$\begin{aligned}\overline{R}^h_{ijk,m\rho} = \overline{R}^h_{ijk|m\rho} - P^h_{\alpha\rho}\overline{R}^\alpha_{ijk|m} + P^\alpha_{i\rho}\overline{R}^h_{\alpha jk|m} + P^\alpha_{j\rho}\overline{R}^h_{i\alpha k|m} + P^\alpha_{k\rho}\overline{R}^h_{ij\alpha|m} \\ + P^\alpha_{m\rho}\overline{R}^h_{ijk|\alpha} - P^h_{m\alpha,\rho}\overline{R}^\alpha_{ijk} - P^h_{m\alpha}\overline{R}^\alpha_{ijk,\rho} + P^\alpha_{mi,\rho}\overline{R}^h_{\alpha jk} + P^\alpha_{mi}\overline{R}^h_{\alpha jk,\rho}\end{aligned}$$

$$+P^{\alpha}_{mj,\rho}\overline{R}^{h}_{i\alpha k}+P^{\alpha}_{mj}\overline{R}^{h}_{i\alpha k,\rho}+P^{\alpha}_{mk,\rho}\overline{R}^{h}_{ij\alpha}+P^{\alpha}_{mk}\overline{R}^{h}_{ij\alpha,\rho} \tag{3.5}$$

Suppose that the space $\overline{A}_n$ is a 2-symmetric space. Then the identity (3.1) holds. Hence from (3.5) we obtain

$$\begin{aligned}\overline{R}^{h}_{ijk,m\rho}=&-P^{h}_{\alpha\rho}\overline{R}^{\alpha}_{ijk|m}+P^{\alpha}_{i\rho}\overline{R}^{h}_{\alpha jk|m}+P^{\alpha}_{j\rho}\overline{R}^{h}_{i\alpha k|m}+P^{\alpha}_{k\rho}\overline{R}^{h}_{ij\alpha|m}\\&+P^{\alpha}_{m\rho}\overline{R}^{h}_{ijk|\alpha}-P^{h}_{m\alpha,\rho}\overline{R}^{\alpha}_{ijk}-P^{h}_{m\alpha}\overline{R}^{\alpha}_{ijk,\rho}+P^{\alpha}_{mi,\rho}\overline{R}^{h}_{\alpha jk}+P^{\alpha}_{mi}\overline{R}^{h}_{\alpha jk,\rho}\\&+P^{\alpha}_{mj,\rho}\overline{R}^{h}_{i\alpha k}+P^{\alpha}_{mj}\overline{R}^{h}_{i\alpha k,\rho}+P^{\alpha}_{mk,\rho}\overline{R}^{h}_{ij\alpha}+P^{\alpha}_{mk}\overline{R}^{h}_{ij\alpha,\rho}.\end{aligned} \tag{3.6}$$

Let us introduce the tensor $\overline{R}^{h}_{ijkm}$ defined by

$$\overline{R}^{h}_{ijk,m}=\overline{R}^{h}_{ijkm}. \tag{3.7}$$

It is known [16, 20] that the Riemann tensors of the spaces A_n and $\overline{A}_n$ are related to each other by the equations

$$\overline{R}^{h}_{ijk}=R^{h}_{ijk}+P^{h}_{ik,j}-P^{h}_{ij,k}+P^{\alpha}_{ik}P^{h}_{\alpha j}-P^{\alpha}_{ij}P^{h}_{\alpha k}. \tag{3.8}$$

Since the deformation tensor of the mapping P^{h}_{ij} is represented by the equations (2.6), it follows from (3.8) that

$$\varphi_{i,j}F^{h}_{k}+\varphi_{k,j}F^{h}_{i}-\varphi_{i,k}F^{h}_{j}-\varphi_{j,k}F^{h}_{i}=C^{h}_{ijk}, \tag{3.9}$$

where

$$\begin{aligned}C^{h}_{ijk}=\overline{R}^{h}_{ijk}-R^{h}_{ijk}-\varphi_i\big(F^{h}_{kj}+\varphi_\alpha F^{\alpha}_{k}F^{h}_{j}+e\delta^{h}_{k}\varphi_j-F^{h}_{jk}-\varphi_\alpha F^{\alpha}_{j}F^{h}_{k}-e\delta^{h}_{j}\varphi_k\big)\\+\varphi_k\big(F^{h}_{ij}+\varphi_\alpha F^{\alpha}_{i}F^{h}_{j}\big)-\varphi_j\big(F^{h}_{ik}+\varphi_\alpha F^{\alpha}_{i}F^{h}_{k}\big).\end{aligned} \tag{3.10}$$

Let us multiply (3.9) by F^{m}_{ρ} and contract for ρ and h. Hence we have

$$\delta^{m}_{k}\varphi_{i,j}+\delta^{m}_{i}\varphi_{k,j}-\delta^{m}_{j}\varphi_{i,k}-\delta^{m}_{i}\varphi_{j,k}=eC^{\alpha}_{ijk}F^{m}_{\alpha}. \tag{3.11}$$

Contracting the equations (3.11) for m and i we get

$$\varphi_{k,j}-\varphi_{j,k}=\frac{e}{n+1}C^{\alpha}_{\beta jk}F^{\beta}_{\alpha}. \tag{3.12}$$

Again, contracting the equations (3.11) for m and k we get

$$n\varphi_{i,j}-\varphi_{j,i}=eC^{\alpha}_{ij\beta}F^{\beta}_{\alpha}. \tag{3.13}$$

Taking account of (3.12) the equations (3.13) can be written as

$$\varphi_{i,j}=\frac{e}{n-1}\big(C^{\alpha}_{ij\beta}-\frac{1}{n+1}C^{\alpha}_{\beta ji}\big)F^{\beta}_{\alpha}. \tag{3.14}$$

And finally, taking account of (2.4), (2.6) and (3.7) the equations (3.6) can be written as

$$\overline{R}^{h}_{ijkm,\rho}=\Theta^{h}_{ijkm\rho}, \tag{3.15}$$

where

$$\Theta^h_{ijkm\rho} = -F^h_{(\alpha}\varphi_{\rho)}\overline{R}^\alpha_{ijk|m} + F^\alpha_{(i}\varphi_{\rho)}\overline{R}^h_{\alpha jk|m} + F^\alpha_{(j}\varphi_{\rho)}\overline{R}^h_{i\alpha k|m} + F^\alpha_{(k}\varphi_{\rho)}\overline{R}^h_{ij\alpha|m}$$
$$+ F^\alpha_{(m}\varphi_{\rho)}\overline{R}^h_{ijk|\alpha} - \big(F^h_{(m|\rho|}\varphi_{\alpha)} + F^h_{(m}\varphi_{\alpha),\rho}\big)\overline{R}^\alpha_{ijk} - F^h_{(m}\varphi_{\alpha)}\overline{R}^\alpha_{ijk\rho} + \big(F^\alpha_{(m|\rho|}\varphi_{i)}$$
$$+ F^\alpha_{(m}\varphi_{i),\rho}\big)\overline{R}^h_{\alpha jk} + F^\alpha_{(m}\varphi_{i)}\overline{R}^h_{\alpha jk\rho} + \big(F^\alpha_{(m|\rho|}\varphi_{j)} + F^\alpha_{(m}\varphi_{j),\rho}\big)\overline{R}^h_{i\alpha k}$$
$$+ F^\alpha_{(m}\varphi_{j)}\overline{R}^h_{i\alpha k\rho} + \big(F^\alpha_{(m|\rho|}\varphi_{k)} + F^\alpha_{(m}\varphi_{k),\rho}\big)\overline{R}^h_{ij\alpha} + F^\alpha_{(m}\varphi_{k)}\overline{R}^h_{ij\alpha\rho}.$$

Suppose, that in the above formula the tensors $\overline{R}^h_{ijk|m}$ and $\varphi_{i,j}$ are expressed according to (3.2) and (3.14). Also we suppose that $\overline{R}^h_{ijk,m} = \overline{R}^h_{ijkm}$. Obviously, in the space A_n the equations (2.4), (3.7), (3.14) and (3.15) form a system of PDE's of Cauchy type in covariant derivatives with respect to functions $F_i^h(x)$, $F^h_{ij}(x)$, $\mu_i(x)$, $\mu_{ij}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$, $\varphi_i(x)$, and the functions $F_i^h(x)$, $F^h_{ij}(x)$, $\mu_i(x)$, $\mu_{ij}(x)$ must satisfy the algebraic conditions (2.5). The algebraic conditions for the functions $\overline{R}^h_{ijk}(x)$ are

$$\overline{R}^h_{i(jk)} = 0, \quad \overline{R}^h_{(ijk)} = 0. \tag{3.16}$$

Since $\overline{R}^h_{i(jk|m)} = 0$ and taking account of (2.6) and (3.2), we obtain another algebraic condition

$$\overline{R}^h_{i(jkm)} = -F^h_{(m}\varphi_{\alpha)}\overline{R}^\alpha_{ijk} - F^h_{(k}\varphi_{\alpha)}\overline{R}^\alpha_{imj} - F^h_{(j}\varphi_{\alpha)}\overline{R}^\alpha_{ikm}$$
$$+ F^\alpha_{(m}\varphi_{i)}\overline{R}^h_{\alpha jk} + F^\alpha_{(k}\varphi_{i)}\overline{R}^h_{\alpha mj} + F^\alpha_{(j}\varphi_{i)}\overline{R}^h_{\alpha km}. \tag{3.17}$$

Hence we have the following theorem.

Theorem 1. *In order that a space A_n with affine connection admit almost geodesic mapping of type $\pi_2(e)$, $e = \pm 1$, onto a 2-symmetric space $\overline{A}_n$, it is necessary and sufficient that the closed mixed system of differential equations of Cauchy type in covariant derivatives* (2.4)*,* (3.7)*,* (3.14)*,* (3.15)*,* (2.5)*,* (3.16)*,* (3.17) *have a solution with respect to functions $F_i^h(x)$, $F^h_{ij}(x)$, $\mu_i(x)$, $\mu_{ij}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$, $\varphi_i(x)$.*

Consequence 1. *The general solution of the closed mixed system of Cauchy type* (2.4)*,* (3.7)*,* (3.14)*,* (3.15)*,* (2.5)*,* (3.16) *and* (3.17) *depends on no more than*

$$n(n^2+2n+2) + \frac{1}{3}n^2(n+1)(n^2-1)$$

essential parameters.

4. Canonical almost geodesic mappings $\pi_2(e)$, $e = \pm 1$, of spaces with affine connections onto 3-symmetric spaces

A space $\overline{A}_n$ with an affine connection is called 3-*symmetric* if its Riemann tensor $\overline{R}^h_{ijk}$ satisfies the condition

$$\overline{R}^h_{ijk|m\rho l} = 0. \tag{4.1}$$

We differentiate the equations (3.5) covariantly with respect to x^l and the connection of A_n. Then in the left-hand side we express the covariant derivative with respect to the connection of A_n in terms of the covariant derivative with respect to the connection of $\overline{A}_n$, using the formula

$$(\overline{R}^h_{ijk|m\rho})_{,l} = \overline{R}^h_{ijk|m\rho l} - P^h_{\alpha l}\overline{R}^\alpha_{ijk|m\rho} + P^\alpha_{il}\overline{R}^h_{\alpha jk|m\rho} + P^\alpha_{jl}\overline{R}^h_{i\alpha k|m\rho}$$
$$+ P^\alpha_{kl}\overline{R}^h_{ij\alpha|m\rho} + P^\alpha_{ml}\overline{R}^h_{ijk|\alpha\rho} + P^\alpha_{\rho l}\overline{R}^h_{ijk|m\alpha}.$$

Let us introduce the tensor $\overline{R}^h_{ijkm\rho}$ defined by

$$\overline{R}^h_{ijkm,\rho} = \overline{R}^h_{ijkm\rho}. \tag{4.2}$$

Suppose that the space $\overline{A}_n$ is a 3-symmetric space. Then from the obtained equation, if we take account of the equations (4.1) and (4.2), we have

$$\overline{R}^h_{ijkm\rho,l} = \Theta^h_{ijkm\rho l}, \tag{4.3}$$

where

$$\Theta^h_{ijkm\rho l} = -P^h_{\alpha l}\overline{R}^\alpha_{ijk|m\rho} + P^\alpha_{il}\overline{R}^h_{\alpha jk|m\rho} + P^\alpha_{jl}\overline{R}^h_{i\alpha k|m\rho} + P^\alpha_{kl}\overline{R}^h_{ij\alpha|m\rho} + P^\alpha_{ml}\overline{R}^h_{ijk|\alpha\rho}$$
$$+ P^\alpha_{\rho l}\overline{R}^h_{ijk|m\alpha} + \big(-P^h_{\alpha\rho}\overline{R}^\alpha_{ijk|m} + P^\alpha_{i\rho}\overline{R}^h_{\alpha jk|m} + P^\alpha_{j\rho}\overline{R}^h_{i\alpha k|m}$$
$$+ P^\alpha_{k\rho}\overline{R}^h_{ij\alpha|m} + P^\alpha_{m\rho}\overline{R}^h_{ijk|\alpha} - P^h_{m\alpha,\rho}\overline{R}^\alpha_{ijk} - P^h_{m\alpha}\overline{R}^\alpha_{ijk,\rho} + P^\alpha_{mi,\rho}\overline{R}^h_{\alpha jk} + P^\alpha_{mi}\overline{R}^h_{\alpha jk,\rho}$$
$$+ P^\alpha_{mj,\rho}\overline{R}^h_{i\alpha k} + P^\alpha_{mj}\overline{R}^h_{i\alpha k,\rho} + P^\alpha_{mk,\rho}\overline{R}^h_{ij\alpha} + P^\alpha_{mk}\overline{R}^h_{ij\alpha,\rho}\big)_{,l}.$$

We have assumed that in the last formula the covariant derivatives of the tensors are expressed according to the formulas (3.5), (3.4), (3.3), (3.2), (2.6) and (3.14).

Obviously, in the space A_n the equations (2.4), (3.7), (3.14), (4.2) and (4.3) form a system of PDEs of Cauchy type with respect to the functions $F^h_i(x)$, $F^h_{ij}(x)$, $\mu_i(x)$, $\mu_{ij}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$, $\overline{R}^h_{ijkm\rho}(x)$, $\varphi_i(x)$. The functions $F^h_i(x)$, $F^h_{ij}(x)$, $\mu_i(x)$, $\mu_{ij}(x)$ must satisfy the algebraic conditions (2.5). In turn the functions $\overline{R}^h_{ijk}(x)$ and $\overline{R}^h_{ijkm}(x)$ must satisfy the algebraic conditions (3.16) and (3.17).

Hence we proved the theorem:

Theorem 2. *In order that a space A_n with affine connection admit almost geodesic mapping of type $\pi_2(e)$, $e = \pm 1$, onto a 3-symmetric space $\overline{A}_n$, it is necessary and sufficient that the closed mixed system of differential equations of Cauchy type in covariant derivatives (2.4), (3.7), (3.14), (4.2), (4.3), (2.5), (3.16), (3.17) have a solution with respect to functions $F^h_i(x)$, $F^h_{ij}(x)$, $\mu_i(x)$, $\mu_{ij}(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$, $\overline{R}^h_{ijkm\rho}(x)$, $\varphi_i(x)$.*

Consequence 2. *The general solution of the closed mixed system of Cauchy type* (2.4)*,* (3.7)*,* (3.14)*,* (4.2)*,* (4.3)*,* (2.5)*,* (3.16)*,* (3.17) *depends on no more than*

$$n(n^2+2n+2)+\frac{1}{3}n^2(n+1)(n^3-1)$$

essential parameters.

5. Canonical almost geodesic mappings $\pi_2(e)$, $e=\pm 1$, of spaces with affine connections onto m-symmetric spaces

A space $\overline{A}_n$ with an affine connection is called *m-symmetric* if its Riemann tensor $\overline{R}^h_{ijk}$ satisfies the condition

$$\overline{R}^h_{ijk|\rho_1\rho_2\ldots\rho_m}=0. \tag{5.1}$$

Of course 2-symmetric spaces and 3-symmetric spaces are special cases of m-symmetric spaces.

Let us introduce the tensors $\overline{R}^h_{ijk\rho_1\rho_2\rho_3},\ldots,\overline{R}^h_{ijk\rho_1\rho_2\rho_3\ldots\rho_{m-2}\rho_{m-1}}$ defined by

$$\begin{aligned}\overline{R}^h_{ijk\rho_1\rho_2,\rho_3} &= \overline{R}^h_{ijk\rho_1\rho_2\rho_3},\\ &\ldots\\ \overline{R}^h_{ijk\rho_1\rho_2\rho_3\ldots\rho_{m-2},\rho_{m-1}} &= \overline{R}^h_{ijk\rho_1\rho_2\rho_3\ldots\rho_{m-2}\rho_{m-1}}.\end{aligned} \tag{5.2}$$

We differentiate (4.3) covariantly $(m-2)$ times with respect to the connection of the space A_n and in the left-hand side express the covariant derivatives with respect to the connection of A_n in terms of the covariant derivatives with respect to the connection of $\overline{A}_n$, using the formula

$$\begin{aligned}(\overline{R}^h_{ijk|\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}})_{,\rho_\tau} &= \overline{R}^h_{ijk|\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}\rho_\tau} - P^h_{\alpha\rho_\tau}\overline{R}^\alpha_{ijk|\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}} + P^\alpha_{i\rho_\tau}\overline{R}^h_{\alpha jk|\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}}\\ &\quad + P^\alpha_{j\rho_\tau}\overline{R}^h_{i\alpha k|\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}} + P^\alpha_{k\rho_\tau}\overline{R}^h_{ij\alpha|\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}}\\ &\quad + P^\alpha_{\rho_1\rho_\tau}\overline{R}^h_{ijk|\alpha\ldots\rho_{\tau-2}\rho_{\tau-1}} + \cdots + P^\alpha_{\rho_{\tau-1}\rho_\tau}\overline{R}^h_{ijk|\rho_1\ldots\rho_{\tau-2}\alpha}.\end{aligned}$$

This equation was obtained making use of (2.1).

Let us assume that the space $\overline{A}_n$ is m-symmetric $(m>2)$. Hence, from the obtained equation because of (5.3), using substitutions and transformations, taking account of (5.1), we get

$$\overline{R}^h_{ijk\rho_1\ldots\rho_{m-2}\rho_{m-1},\rho_m} = \Theta^h_{ijk\rho_1\ldots\rho_{m-1}\rho_m}, \tag{5.3}$$

where $\Theta^h_{ijk\rho_1\ldots\rho_{m-1}\rho_m}$ is a tensor which involves unknown tensors F^h_i, F^h_{ij}, μ_i, μ_{ij}, φ_i, $\overline{R}^h_{ijk}$, $\overline{R}^h_{ijk\rho_1}$, $\ldots$, $\overline{R}^h_{ijk\rho_1\ldots\rho_{m-1}}$. The tensor $\Theta^h_{ijk\rho_1\ldots\rho_{m-1}\rho_m}$ also involves some given tensors.

Obviously, in the space A_n the equations (2.4), (3.7), (3.14), (4.2), (5.2), (5.3) form a closed system of PDEs of Cauchy type with respect to the functions $F^h_i(x)$, $F^h_{ij}(x)$,

$\mu_i(x)$, $\mu_{ij}(x)$, $\varphi_i(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijk\rho_1}(x)$, $\ldots$, $\overline{R}^h_{ijk\rho_1\ldots\rho_{m-1}}(x)$. The functions must satisfy the algebraic conditions (2.5), (3.16) and (3.17).

Finally, we obtain the following theorem.

Theorem 3. *In order that a space A_n with an affine connection admit almost geodesic mapping of type $\pi_2(e)$, $e = \pm 1$, onto an m-symmetric space $\overline{A}_n$, it is necessary and sufficient that the closed mixed system of differential equations of Cauchy type in covariant derivatives* (2.4), (3.7), (3.14), (4.2), (5.2), (5.3), (2.5), (3.16) *and* (3.17) *have a solution with respect to functions $F^h_i(x)$, $F^h_{ij}(x)$, $\mu_i(x)$, $\mu_{ij}(x)$, $\varphi_i(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijk\rho_1}(x)$, $\ldots$, $\overline{R}^h_{ijk\rho_1\ldots\rho_{m-1}}(x)$.*

Consequence 3. *The general solution of the closed mixed system of Cauchy type* (2.4), (3.7), (3.14), (4.2), (5.2), (5.3), (2.5), (3.16) *and* (3.17) *depends on no more than*

$$n(n^2+2n+2)+\frac{1}{3}n^2(n+1)(n^m-1)$$

essential parameters.

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References

[1] A. V. Aminova, "Projective transformations of pseudo-Riemannian manifolds," *J. Math. Sci., New York*, vol. 113, no. 3, pp. 367–470, 2003, doi: 10.1023/A:1021041802041.

[2] V. Berezovski, Y. Cherevko, J. Mikeš, and L. Rýparová, "Canonical almost geodesic mappings of the first type of spaces with affine connections onto generalized m-Ricci-symmetric spaces," *Mathematics*, vol. 9, no. 4, 437, 2021, doi: 10.3390/math9040437.

[3] V. Berezovski, N. Guseva, and J. Mikeš, "Geodesic mappings of equiaffine and Ricci symmetric spaces," *Math. Notes*, vol. 110, no. 2, pp. 293–296, 2021, doi: 10.1134/S0001434621070312.

[4] V. E. Berezovski and J. Mikeš, "Almost geodesic mappings of type π_1 onto generalized Ricci-symmetric spaces," *Uch. Zap. Kazan. Gos. Univ., Ser. Fiz.-Mat. Nauki*, vol. 151, no. 4, pp. 9–14, 2009.

[5] V. E. Berezovski, J. Mikeš, and A. Vanžurová, "Fundamental PDE's of the canonical almost geodesic mappings of type $\tilde{\pi}_1$," *Bull. Malays. Math. Sci. Soc. (2)*, vol. 37, no. 3, pp. 647–659, 2014.

[6] V. Berezovski, S. Bácsó, and J. Mikeš, "Almost geodesic mappings of affinely connected spaces that preserve the Riemannian curvature," *Ann. Math. Inform.*, vol. 45, pp. 3–10, 2015.

[7] V. Berezovski and J. Mikeš, "On a classification of almost geodesic mappings of affine connection spaces," *Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math.*, vol. 35, pp. 21–24, 1996.

[8] V. E. Berezovskii and J. Mikeš, "On canonical almost geodesic mappings of the first type of affinely connected spaces," *Russ. Math.*, vol. 58, no. 2, pp. 1–5, 2014, doi: 10.3103/S1066369X14020017.

[9] V. E. Berezovskii and J. Mikeš, “Almost geodesic mappings of spaces with affine connection,” *J. Math. Sci., New York*, vol. 207, no. 3, pp. 389–409, 2015, doi: 10.1007/s10958-015-2378-5.

[10] V. E. Berezovskii, J. Mikeš, H. Chudá, and O. Y. Chepurna, “On canonical almost geodesic mappings which preserve the Weyl projective tensor,” *Russ. Math.*, vol. 61, no. 6, pp. 1–5, 2017, doi: 10.3103/S1066369X17060019.

[11] V. Berezovsky, J. Mikes, and A. Vanzurová, “Almost geodesic mappings onto generalized Ricci-symmetric manifolds,” *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, vol. 26, no. 2, pp. 221–230, 2010.

[12] E. Cartan, “Les espaces riemanniens symétriques,” Verh. Internat. Math.-Kongr. 1, 152-161, 1932.

[13] S. Helgason, “Differential geometry and symmetric spaces,” Pure and Applied Mathematics, 12. New York-London: Academic Press. XIV, 1962.

[14] T. Levi-Civita, “Sulle trasformazioni dello equazioni dinamiche.” *Annali di Mat. (2)*, vol. 24, pp. 255–300, 1896, doi: 10.1007/BF02419530.

[15] J. Mikeš, V. E. Berezovski, E. Stepanova, and H. Chudá, “Geodesic mappings and their generalizations,” *J. Math. Sci., New York*, vol. 217, no. 5, pp. 607–623, 2016, doi: 10.1007/s10958-016-2993-9.

[16] J. Mikeš, E. Stepanova, and A. Vanžurová, *Differential geometry of special mappings*. Olomouc: Palacký University, Faculty of Science, 2015.

[17] A. Z. Petrov, “Modeling of physical fields,” *Gravitation and General Relativity*, vol. 4, no. 5, pp. 7–21, 1968.

[18] A. P. Shirokov, “P. A. Shirokov’s work on the geometry of symmetric spaces,” *J. Math. Sci., New York*, vol. 89, no. 3, pp. 1253–1260, 1998, doi: 10.1007/BF02414870.

[19] N. S. Sinyukov, “Almost geodesic mappings of affinely connected and Riemannian spaces,” *Sov. Math., Dokl.*, vol. 4, pp. 1086–1088, 1963.

[20] N. S. Sinyukov, “Geodesic mappings of Riemannian spaces,” Moskva: Nauka, 1979.

[21] N. S. Sinyukov, “Almost-geodesic mappings of affinely connected and Riemann spaces,” *J. Sov. Math.*, vol. 25, pp. 1235–1249, 1984, doi: 10.1007/BF01084672.

[22] V. S. Sobchuk, “Almost geodesic mappings of Riemannian spaces onto symmetric Riemannian spaces,” *Mat. Zametki*, vol. 17, pp. 757–763, 1975.

[23] H. Vavříková, J. Mikeš, O. Pokorná, and G. Starko, “On fundamental equations of almost geodesic mappings of type $\pi_2(e)$,” *Russ. Math.*, vol. 51, no. 1, pp. 8–12, 2007, doi: 10.3103/S1066369X07010021.

[24] N. O. Vesić, L. S. Velimirović, and M. S. Stanković, “Some invariants of equitorsion third type almost geodesic mappings,” *Mediterr. J. Math.*, vol. 13, no. 6, pp. 4581–4590, 2016, doi: 10.1007/s00009-016-0763-z.

[25] N. V. Yablonskaya, “Special groups of almost geodesic transformations of spaces with affine connection,” *Sov. Math.*, vol. 30, no. 1, pp. 105–108, 1986.

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