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NONLINEAR INVERSE PROBLEM FOR IDENTIFYING A COEFFICIENT OF THE LOWEST TERM IN HYPERBOLIC EQUATION WITH NONLOCAL CONDITIONS

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Abstract. In this paper, a nonlinear inverse boundary value problem for the second-order hyperbolic equation with nonlocal conditions is studied. To investigate the solvability of the original problem, we first consider an auxiliary inverse boundary value problem and prove its equivalence (in a certain sense) to the original problem. Then using the Fourier method and contraction mappings principle, the existence and uniqueness theorem for auxiliary problem is proved. Further, on the basis of the equivalency of these problems the existence and uniqueness theorem for the classical solution of the considered inverse coefficient problem is proved for the smaller value of time.

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1. INTRODUCTION AND PROBLEM STATEMENT

Let $0 < T < +\infty$ be some fixed number and D_T be a rectangular region in the xt-plane defined by $0 \le x \le 1,\ 0 \le t \le T$. Consider the problem of determining the unknown functions $u = u(x,t)$ and $a = a(t)$ such that the pair $\{u,a\}$ satisfies the following hyperbolic equation of second order

$$u_{tt}(x,t) - u_{xx}(x,t) = a(t)u(x,t) + f(x,t) \quad (x,t) \in D_T, \tag{1.1}$$

with the nonlocal initial conditions

$$u(x,0) + \delta_1 u(x,T) = \varphi(x), \quad u_t(x,0) + \delta_2 u_t(x,T) = \psi(x), \ 0 \le x \le 1, \tag{1.2}$$

the boundary conditions

$$u(0,t) = \beta u(1,t), \quad 0 \le t \le T, \tag{1.3}$$

$$\int_0^1 u(x,t)dx = 0, \qquad 0 \le t \le T, \tag{1.4}$$

and overdetermination condition

$$u\left(\frac{1}{2},t\right)=h(t),\ 0\le t\le T, \tag{1.5}$$

in which $\delta_1,\delta_2\ge 0$ and $\beta\ne\pm 1$ are given numbers, $f(x,t),\varphi(x)$, and $\psi(x)$ are known functions of $x\in[0,1]$ and $t\in[0,T]$.

In the present work we investigate a nonlinear inverse problem for identifying a coefficient of the lowest term in hyperbolic equation from the overdetermination data. Such problems are called inverse problems in mathematical physics. The applied importance of inverse problems is so great (seismology, mineral exploration, biology, medicine, desalination of seawater, movement of liquid in a porous medium, acoustics, electromagnetics, fluid dynamics, calculating the density of the Earth from measurements of its gravity field, for example) which puts them a series of the most actual problems of modern mathematics. Strictly speaking, the inverse problems for hyperbolic/wave equations are of prime interest in seismology. Besides, vibrations of structures (as buildings and beams) are modeled by hyperbolic differential equations. However, it should be noted that an inverse problem is called linear if the recovery function enters the given equation linearly, and nonlinear otherwise.

Nowadays, inverse problems for hyperbolic equations have been well studied by many authors using different methods (see, e.g., [2, 5–7, 9, 10, 19–22], and the references given therein). Moreover, in [15–18] the authors present a regularity result for solutions of partial differential equations in the framework of mixed Morrey spaces.

A distinctive feature of presented article is the investigation of an inverse hyperbolic problem with both spatial and time nonlocal conditions.

This article is based on ideas close to those used in [1, 3, 8, 13].

Definition 1. A pair of functions $\{u(x,t),a(t)\}$ is said to be a classical solution of problem (1.1)–(1.5) if all three of the following conditions are satisfied:

a. The function $u(x,t)$ with the derivatives $u_{xx}(x,t)$ and $u_{tt}(x,t)$ are continuous in the domain D_T.
b. The function $a(t)$ is continuous on the interval $[0,T]$.
c. The Eq. (1.1) and conditions (1.2)–(1.5) are satisfied in the classical (usual) sense.

Now, to study problem (1.1)–(1.5), we consider the following auxiliary inverse boundary value problem: it is required to find a pair of functions $u(x,t)\in C^2(D_T)$, $a(t)\in C[0,T]$ from (1.1)–(1.3) and

$$u_x(0,t)=u_x(1,t),\ 0\le t\le T, \tag{1.6}$$

$$h''(t)-u_{xx}\left(\frac{1}{2},t\right)=a(t)h(t)+f\left(\frac{1}{2},t\right),\ 0\le t\le T. \tag{1.7}$$

Similarly (see [12], Theorem 2.2, p.4) it can be proved that

Theorem 1. *Suppose that* $\varphi(x),\psi(x)\in C[0,1]$, $h(t)\in C^2[0,T]$, $h(t)\neq 0$, $f(x,t)\in C(D_T)$, $\int_0^1 f(x,t)dx=0$, $0\le t\le T$ *and the compatibility conditions*

$$\int_0^1 \varphi(x)dx=0,\ \int_0^1 \psi(x)dx=0,$$

$$h(0)+\delta_1 h(T)=\varphi\left(\frac{1}{2}\right),\ h'(0)+\delta_2 h'(T)=\psi\left(\frac{1}{2}\right),$$

hold. Then the following statements are true:

(i) *Each classical solution* $\{u(x,t),a(t)\}$ *of problem* (1.1)–(1.5) *is the solution of problem* (1.1)–(1.3), (1.6), (1.7), *as well*

(ii) *each solution* $\{u(x,t),a(t)\}$ *of problem* (1.1)–(1.3), (1.6), (1.7) *is a classical solution to the problem* (1.1)–(1.5), *if*

$$\frac{(1+2\delta_1+3\delta_2+\delta_1\delta_2)T^2}{2(1+\delta_1)(1+\delta_2)}\|a(t)\|_{C[0,T]}<1.$$

2. Auxiliary facts and denotations

It is known that sequences of functions [14]

$$X_0(x)=px+q,\ X_{2k-1}(x)=(px+q)\cos\lambda_k x,\ X_{2k}(x)=\sin\lambda_k x,\ k=1,2,\ldots, \tag{2.1}$$

$$Y_0(x)=2,\ Y_{2k-1}(x)=4\sin\lambda_k x,\ Y_{2k}(x)=q(1-q-px)\cos\lambda_k x,\ k=1,2,\ldots, \tag{2.2}$$

form a biorthogonal system and system (2.1) forms a Riesz basis in $L_2(0,1)$ for $\lambda_k=2k\pi\ (k=1,2,\ldots)$. Here p and q denotes, in turn, the numbers

$$p=\frac{1-\beta}{1+\beta}\neq 0,\ q=\frac{\beta}{1+\beta}.$$

We state the following lemmas without proof.

Lemma 1. *(see [4, 11]) For any function* $v(x)$ *with the properties:*

$$v(x)\in C^{2i-1}[0,1],\ v^{(2i)}(x)\in L_2(0,1),$$
$$v^{(2s)}(0)=\beta v^{(2s)}(1),\ v^{(2s+1)}(0)=v^{(2s+1)}(1)\ \ (i\ge 1;\ s=\overline{0,i}),$$

the estimates are valid

$$\left(\sum_{k=0}^{\infty}(\lambda_k^{2i}v_{2k-1})^2\right)^{\frac{1}{2}}\le 2\sqrt{2}\left\|v^{(2i)}(x)\right\|_{L_2(0,1)},$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^{2i}v_{2k})^2\right)^{\frac{1}{2}} \leq 2\sqrt{2}\left\|v^{(2i)}(x)(1-q-px)-2ipv^{(2i-1)}(x)\right\|_{L_2(0,1)}, \qquad (2.3)$$

where

$$v_k=\int_0^1 v(x)Y_k(x)dx,\ k=0,1,\ldots.$$

Lemma 2. *(see [4, 11]) Under the assumptions:*

$$v(x)\in C^{2i}[0,1],\ \ v^{(2i+1)}(x)\in L_2(0,1),$$
$$v^{(2s)}(0)=\beta v^{(2s)}(1),\ \ v^{(2s+1)}(0)=v^{(2s+1)}(1)\ \ (i\geq 1;\ \ s=\overline{0,i}),$$

we establish the validity of the estimates

$$\left(\sum_{k=1}^{\infty}(\lambda_k^{2i+1}v_{2k-1})^2\right)^{\frac{1}{2}} \leq 2\sqrt{2}\left\|v^{(2i+1)}(x)\right\|_{L_2(0,1)},$$
$$\left(\sum_{k=1}^{\infty}(\lambda_k^{2i+1}v_{2k-1})^2\right)^{\frac{1}{2}} \leq 2\sqrt{2}\left\|v^{(2i+1)}(x)(1-q-px)-(2i+1)pv^{(2i)}(x)\right\|_{L_2(0,1)}. \qquad (2.4)$$

We now look at the following functional spaces:
$B^3_{2,T}$ [11] denotes a set of all functions of the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)X_k(x),$$

considered in D_T, where $u_k(t)\in C[0,T]$ and

$$J(u)\equiv\|u_0(t)\|_{C[0,T]}+\left\{\sum_{k=1}^{\infty}(\lambda_k^3\|u_{2k-1}(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}}$$
$$+\left\{\sum_{k=1}^{\infty}(\lambda_k^3\|u_{2k}(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}}<+\infty.$$

Actually, the functions $X_k(x)$ $(k=0,1,\ldots)$ defined by the relation (2.1). The norm on the set $J(u)$ is established as follows

$$\|u(x,t)\|_{B^3_{2,T}}=J(u).$$

Let E^3_T denote the space consisting of the topological product $B^3_{2,T}\times C[0,T]$, which is the norm of the element $z=\{u,a\}$ defined by the formula

$$\|z(x,t)\|_{E^3_T}=\|u(x,t)\|_{B^3_{2,T}}+\|a(t)\|_{C[0,T]}.$$

3. Classical solvability of inverse boundary value problem

Suppose that the data of problem (1.1)–(1.3),(1.6),(1.7) satisfies the following conditions:

C_1. $\delta_1, \delta_2 \geq 0,\ \ 1+\delta_1\delta_2 \geq \delta_1+\delta_2$;
C_2. $\varphi(x) \in C^2[0,1],\ \varphi'''(x) \in L_2(0,1),\ \varphi(0) = \beta\varphi(1),$
$\varphi'(0) = \varphi'(1),\ \varphi''(0) = \beta\varphi''(1)$;
C_3. $\psi(x) \in C^1[0,1],\ \psi''(x) \in L_2(0,1),\ \psi(0) = \beta\psi(1),\ \psi'(0) = \psi'(1)$;
C_4. $f(x,t), f_x(x,t) \in C(D_T),\ f_{xx}(x,t) \in L_2(D_T),\ f(0,t) = \beta f(1,t),$
$f_x(0,t) = f_x(1,t),\ 0 \leq t \leq T$;
C_5. $h(t) \in C^2[0,T],\ h(t) \neq 0,\ 0 \leq t \leq T$.

Since the system (2.1) forms a Riesz basis in $L_2(0,1)$. Then each solution to problem (1.1)–(1.3),(1.6),(1.7) can be sought in the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t)X_k(x), \tag{3.1}$$

where

$$u_k(t) = \int_0^1 u(x,t)Y_k(x)dx,\ k = 0,1,\ldots.$$

Moreover, $X_k(x)$ $(k = 0,1,\ldots)$ and $Y_k(x)$ $(k = 0,1,\ldots)$ are defined by relations (2.1) and (2.2), respectively.

To determine of the desired functions $u_k(t)$ $(k = 0,1,2,\ldots)$, using separation of variables and by (1.1) and (1.2), we get

$$u_0''(t) = F_0(t;u,a),\ 0 \leq t \leq T, \tag{3.2}$$

$$u_{2k-1}''(t) + \lambda_k^2 u_{2k-1}(t) = F_{2k-1}(t;u,a),\ k = 1,2,\ldots;\ 0 \leq t \leq T, \tag{3.3}$$

$$u_{2k}''(t) + \lambda_k^2 u_{2k}(t) = F_{2k}(t;u,a) - 2p\lambda_k u_{2k-1}(t),\ k = 1,2,\ldots;\ 0 \leq t \leq T, \tag{3.4}$$

$$u_k(0) + \delta_1 u_k(T) = \varphi_k,\ \ u_k'(0) + \delta_2 u_k'(T) = \psi_k,\ k = 0,1,2,\ldots, \tag{3.5}$$

where

$$F_k(t;u,a) = a(t)u_k(t) + f_k(t),\ \ f_k(t) = \int_0^1 f(x,t)Y_k(x)dx,\ k = 0,1,\ldots.$$

$$\varphi_k = \int_0^1 \varphi(x)Y_k(x)dx,\ \ \psi_k = \int_0^1 \psi(x)Y_k(x)dx,\ k = 0,1,2\ldots.$$

Solving problem (3.2)–(3.5), we find

$$u_0(t) = \frac{\varphi_0}{1+\delta_1} + \frac{t-\delta_1(T-t)}{(1+\delta_1)(1+\delta_2)}\psi_0 + \int_0^T G_0(t,\tau)F_0(\tau;u,a)d\tau, \tag{3.6}$$

$$u_{2k-1}(t)=\frac{1}{\rho_k(T)}\left[\varphi_{2k-1}(\cos\lambda_k t+\delta_2\cos\lambda_k(T-t))\right.$$

$$\left.+\frac{\psi_{2k-1}}{\lambda_k}(\sin\lambda_k t-\delta_1\sin\lambda_k(T-t))\right]+\int_0^T G_k(t,\tau)F_{2k-1}(\tau;u,a)d\tau, \tag{3.7}$$

$$u_{2k}(t)=\frac{1}{\rho_k(T)}\left[\varphi_{2k}(\cos\lambda_k t+\delta_2\cos\lambda_k(T-t))+\frac{\psi_{2k}}{\lambda_k}(\sin\lambda_k t-\delta_1\sin\lambda_k(T-t))\right]$$

$$+\int_0^T G_k(t,\tau)(F_{2k}(\tau;u,a))d\tau$$

$$-\varphi_{2k-1}\left\{-\frac{1}{\rho_k^2(T)}\frac{1}{\lambda_k}\left[\delta_1\cos\lambda_k t\left(\frac{T}{2}\sin\lambda_k T+\delta_2\frac{1}{4}(1-\cos 2\lambda_k T)\right)\right.\right.$$

$$+\delta_2\sin\lambda_k t\left(\frac{1}{2\lambda_k}\sin\lambda_k T+\frac{T}{2}\cos\lambda_k T+\delta_2\left(\frac{T}{2}+\frac{1}{4\lambda_k}\sin 2\lambda_k T\right)\right)$$

$$+\delta_1\delta_2\left(\frac{1}{4\lambda_k}(\cos\lambda_k(2T-t)-\cos\lambda_k t)+\frac{T}{2}\sin\lambda_k t\right.$$

$$\left.\left.+\delta_2\left(-\frac{T}{2}\sin\lambda_k(T-t)+\frac{1}{4\lambda_k}(\cos\lambda_k(T-t)-\cos\lambda_k(T+t))\right)\right)\right]$$

$$+\frac{t}{2}\sin\lambda_k t+\delta_2\left(-\frac{t}{2}\sin\lambda_k(T-t)\right.$$

$$\left.\left.+\frac{1}{4\lambda_k}(\cos\lambda_k(T-t)-\cos\lambda_k(T+t))\right)\right\}$$

$$+\frac{\psi_{2k-1}}{\lambda_k}\left\{-\frac{1}{\rho_k^2(T)}\cdot\frac{1}{\lambda_k}\left[\delta_1\cos\lambda_k t\left(\frac{1}{2\lambda_k}\sin\lambda_k T-\frac{T}{2}\cos\lambda_k T\right)\right.\right.$$

$$-\delta_1\left(\frac{T}{2}-\frac{1}{4\lambda_k}\sin 2\lambda_k T\right)+\delta_2\sin\lambda_k t\left(\frac{T}{2}\sin\lambda_k T-\frac{\delta_1}{4\lambda_k}(1-\cos 2\lambda_k T)\right)$$

$$+\delta_1\delta_2\left(\frac{1}{4\lambda_k}(\sin\lambda_k(2T-t)+\sin\lambda_k t)-\frac{T}{2}\cos\lambda_k t\right.$$

$$\left.\left.-\delta_1\left(\frac{T}{2}\cos\lambda_k(T-t)-\frac{1}{4\lambda_k}(\sin\lambda_k(T-t)+\sin\lambda_k(T+t))\right)\right)\right]$$

$$+\left(\frac{1}{2\lambda_k}\sin\lambda_k t-\frac{t}{2}\cos\lambda_k t-\delta_1\left(\frac{t}{2}\cos\lambda_k(T-t)+\frac{1}{4\lambda_k}(\sin\lambda_k(T-t)\right.\right.$$

$$\left.\left.-\sin\lambda_k(T+t))\right)\right\}+\int_0^T G_k(t,\tau)\left(\int_0^T G_k(\tau,\xi)F_{2k-1}(\xi;u,a)d\xi\right)d\tau, \tag{3.8}$$

where

$$G_0(t,\tau)=\begin{cases}-\frac{\delta_2 t+\delta_1(T-\tau)+\delta_1\delta_2(t-\tau)}{(1+\delta_1)(1+\delta_2)}, & t\in[0,\tau],\\ -\frac{\delta_2 t+\delta_1(T-\tau)-(1+\delta_1+\delta_2)(t-\tau)}{(1+\delta_1)(1+\delta_2)}, & t\in[\tau,T],\end{cases}$$

$$G_k(t,\tau)=\begin{cases} -\frac{1}{\rho_k(T)}\cdot\frac{1}{\lambda_k}[\delta_1\sin\lambda_k(T-\tau)\cos\lambda_k t+\delta_2\cos\lambda_k(T-\tau)\sin\lambda_k t \\ \quad +\delta_1\delta_2\sin\lambda_k(t-\tau)],\ t\in[0,\tau], \\ -\frac{1}{\rho_k(T)}\cdot\frac{1}{\lambda_k}[\delta_1\sin\lambda_k(T-\tau)\cos\lambda_k t+\delta_2\cos\lambda_k(T-\tau)\sin\lambda_k t \\ \quad +\delta_1\delta_2\sin\lambda_k(t-\tau)]+\frac{1}{\lambda_k}\sin\lambda_k(t-\tau),\ t\in[\tau,T], \end{cases}$$

and

$$\rho_k(T)=1+(\delta_1+\delta_2)\cos\lambda_k T+\delta_1\delta_2.$$

Substituting the expressions of (3.6), (3.7), and (3.8) into (3.1), we find the component $u(x,t)$ of the classical solution to problem (1.1)–(1.3), (1.6), (1.7) to be

$$\begin{aligned} u(x,t)=&\left\{\frac{\varphi_0}{1+\delta_1}+\frac{t-\delta_1(T-t)}{(1+\delta_1)(1+\delta_2)}\psi_0+\int_0^T G_0(t,\tau)F_0(\tau;u,a)d\tau\right\}X_0(x) \\ &+\sum_{k=1}^{\infty}\left\{\frac{1}{\rho_k(T)}\left[\varphi_{2k-1}(\cos\lambda_k t+\delta_2\cos\lambda_k(T-t))\right.\right. \\ &\left.\left.+\frac{\psi_{2k-1}}{\lambda_k}(\sin\lambda_k t-\delta_1\sin\lambda_k(T-t))\right]+\int_0^T G_k(t,\tau)F_{2k-1}(\tau;u,a)d\tau\right\}X_{2k-1}(x) \\ &+\sum_{k=1}^{\infty}\left\{\frac{1}{\rho_k(T)}\left[\varphi_{2k}(\cos\lambda_k t+\delta_2\cos\lambda_k(T-t))+\right.\right. \\ &\left.+\frac{\psi_{2k}}{\lambda_k}(\sin\lambda_k t-\delta_1\sin\lambda_k(T-t))\right]+\int_0^T G_k(t,\tau)F_{2k}(\tau;u,a)d\tau \\ &-\varphi_{2k-1}\left\{-\frac{1}{\rho_k^2(T)}\cdot\frac{1}{\lambda_k}\left[\delta_1\cos\lambda_k t\left(\frac{T}{2}\sin\lambda_k T+\delta_2\frac{1}{4}(1-\cos 2\lambda_k T)\right)\right.\right. \\ &+\delta_2\sin\lambda_k t\left(\frac{1}{2\lambda_k}\sin\lambda_k T+\frac{T}{2}\cos\lambda_k T+\delta_2\left(\frac{T}{2}+\frac{1}{4\lambda_k}\sin 2\lambda_k T\right)\right) \\ &+\delta_1\delta_2\left(\frac{1}{4\lambda_k}(\cos\lambda_k(2T-t)-\cos\lambda_k t)+\frac{T}{2}\sin\lambda_k t\right. \\ &\left.\left.+\delta_2\left(-\frac{T}{2}\sin\lambda_k(T-t)+\frac{1}{4\lambda_k}(\cos\lambda_k(T-t)-\cos\lambda_k(T+t))\right)\right)\right] \\ &\left.+\frac{t}{2}\sin\lambda_k t+\delta_2\left(-\frac{t}{2}\sin\lambda_k(T-t)+\frac{1}{4\lambda_k}(\cos\lambda_k(T-t)-\cos\lambda_k(T+t))\right)\right\} \\ &+\frac{\psi_{2k-1}}{\lambda_k}\left\{-\frac{1}{\rho_k^2(T)}\cdot\frac{1}{\lambda_k}\left[\delta_1\cos\lambda_k t\left(\frac{1}{2\lambda_k}\sin\lambda_k T-\frac{T}{2}\cos\lambda_k T\right)\right.\right. \\ &-\delta_1\left(\frac{T}{2}-\frac{1}{4\lambda_k}\sin 2\lambda_k T\right)+\delta_2\sin\lambda_k t\left(\frac{T}{2}\sin\lambda_k T-\frac{\delta_1}{4\lambda_k}(1-\cos 2\lambda_k T)\right) \\ &+\delta_1\delta_2\left(\frac{1}{4\lambda_k}(\sin\lambda_k(2T-t)+\sin\lambda_k t)-\frac{T}{2}\cos\lambda_k t\right. \end{aligned}$$

$$-\delta_1\left(\frac{T}{2}\cos\lambda_k(T-t)-\frac{1}{4\lambda_k}(\sin\lambda_k(T-t)+\sin\lambda_k(T+t))\right)\Big)\Big]$$
$$+\left(\frac{1}{2\lambda_k}\sin\lambda_k t-\frac{t}{2}\cos\lambda_k t-\delta_1\left(\frac{t}{2}\cos\lambda_k(T-t)+\frac{1}{4\lambda_k}(\sin\lambda_k(T-t)\right.\right.$$
$$\left.\left.-\sin\lambda_k(T+t)\right)\right)\Big\}+\int_0^T G_k(t,\tau)\left(\int_0^T G_k(\tau,\xi)F_{2k-1}(\xi;u,a)d\xi\right)d\tau\Big\}X_{2k}(x). \tag{3.9}$$

Now, using (3.1) from (1.7) we have

$$a(t)=[h(t)]^{-1}\left\{h''(t)-f\left(\frac{1}{2},t\right)-\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k\lambda_k^2u_{2k-1}(t)\right\}. \tag{3.10}$$

Substituting expressions $u_{2k-1}(t)$ $(k=1,2,\ldots)$ from (3.8) into (3.10), immediately yields:

$$a(t)=[h(t)]^{-1}$$
$$\times\left\{h''(t)-f\left(\frac{1}{2},t\right)-\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k\lambda_k^2\left\{\frac{1}{\rho_k(T)}[\varphi_{2k-1}(\cos\lambda_k t+\delta_2\cos\lambda_k(T-t))\right.\right.$$
$$\left.\left.+\frac{\psi_{2k-1}}{\lambda_k}(\sin\lambda_k t-\delta_1\sin\lambda_k(T-t))\right]+\int_0^T G_k(t,\tau)F_{2k-1}(\tau;u,a)d\tau\right\}\Bigg\}. \tag{3.11}$$

Thus, the solution of problem (1.1)–(1.3), (1.6), (1.7) is reduced to the solution of system (3.9), (3.11) with respect to unknown functions $u(x,t)$ and $a(t)$.

Using the same discussion in [11], one can prove

Lemma 3. *If $\{u(x,t),a(t)\}$ is any solution to problem (1.1)–(1.3), (1.6), (1.7), then the functions*

$$u_k(t)=\int_0^1 u(x,t)Y_k(x)dx,\quad k=0,1,\ldots,$$

satisfy the countable system (3.6), (3.7) and (3.8) on an interval $[0,T]$.

Obviously, if $u_k(t)=\int_0^1 u(x,t)Y_k(x)dx$, $k=0,1,\ldots$ is a solution to system (3.6), (3.7) and (3.8), then the pair $\{u(x,t),a(t)\}$ of functions $u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x)$ and $a(t)$ is also a solution to system (3.9), (3.11).

It follows from the Lemma 3 that

Corollary 1. *Let system (3.9), (3.11) have a unique solution. Then problem (1.1)–(1.3), (1.6), (1.7) cannot have more than one solution, i.e. if the problem (1.1)–(1.3), (1.6), (1.7) has a solution, then it is unique.*

Now consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a)\},$$

in the space E_T^3, where

$$\Phi_1(u,a) = \tilde{u}(x,t) \equiv \tilde{u}_0(t)X_0(x) + \sum_{k=0}^{\infty} \tilde{u}_{2k-1}(t)X_{2k-1}(x) + \sum_{k=1}^{\infty} \tilde{u}_{2k}(t)X_{2k}(x),$$
$$\Phi_2(u,a) = \tilde{a}(t),$$

and the functions $\tilde{u}_0(t), \tilde{u}_{2k-1}(t), \tilde{u}_{2k}(t)$ $(k = 1,2,...)$, and $\tilde{a}(t)$ are equal correspondingly to the right sides of (3.6), (3.7), (3.8), and (3.11).

It is clear that under conditions $\delta_1, \delta_2 \geq 0$, $\ 1+\delta_1\delta_2 > \delta_1+\delta_2$ and using the inequality

$$\rho_k(T) \geq 1-(\delta_1+\delta_2)+\delta_1\delta_2$$

we can write

$$\frac{1}{\rho_k(T)} \leq \frac{1}{1-(\delta_1+\delta_2)+\delta_1\delta_2} \equiv \rho > 0.$$

Using this relation, from (3.6), (3.7), (3.8), and (3.11) we obtain

$$\|\tilde{u}_0(t)\|_{C[0,T]} \leq \frac{1}{1+\delta_1}|\varphi_0| + \frac{T}{1+\delta_2}\psi_0 + \frac{1+3\delta_1+3\delta_2}{(1+\delta_1)(1+\delta_2)}T$$
$$\times \left(\sqrt{T} \left(\int_0^T |f_0(\tau)|^2 d\tau \right)^{\frac{1}{2}} + T\|a(t)\|_{C[0,T]} \|u_0(t)\|_{C[0,T]} \right), \tag{3.12}$$

$$\left\{ \sum_{k=1}^{\infty} (\lambda_k^3 \|\tilde{u}_{2k-1}(t)\|_{C[0,T]})^2 \right\}^{\frac{1}{2}} \leq 2\rho(1+\delta_2) \left(\sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_{2k-1}|)^2 \right)^{\frac{1}{2}}$$
$$+2\rho(1+\delta_1) \left(\sum_{k=1}^{\infty} (\lambda_k^2 |\psi_{2k-1}|)^2 \right)^{\frac{1}{2}}$$
$$+2(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{T} \left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^2 |f_{2k-1}(\tau)|)^2 d\tau \right)^{\frac{1}{2}}$$
$$+2(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T\|a(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_{2k-1}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}}, \tag{3.13}$$

$$\left\{ \sum_{k=1}^{\infty} (\lambda_k^3 \|\tilde{u}_{2k}(t)\|_{C[0,T]})^2 \right\}^{\frac{1}{2}} \leq 2\sqrt{2}\rho(1+\delta_2) \left(\sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_{2k}|)^2 \right)^{\frac{1}{2}}$$
$$+2\sqrt{2}\rho(1+\delta_1) \left(\sum_{k=1}^{\infty} (\lambda_k^2 |\psi_{2k}|)^2 \right)^{\frac{1}{2}}$$

$$+2\sqrt{2}(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|f_{2k}(\tau)|)^2d\tau\right)^{\frac{1}{2}}$$

$$+2\sqrt{2}(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{2}\rho_1\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_{2k-1}|)^2\right)^{\frac{1}{2}}+2\sqrt{2}\rho_2\left(\sum_{k=1}^{\infty}(\lambda_k^2\,|\psi_{2k-1}|)^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{2}(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))^2T\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|f_{2k-1}(\tau)|)^2d\tau\right)^{\frac{1}{2}}\right.$$

$$\left.+T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right], \tag{3.14}$$

$$\|\tilde{a}(t)\|_{C[0,T]}\le\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-f\left(\frac{1}{2},t\right)\right\|_{C[0,T]}+\frac{1}{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\right.$$

$$\times\left[\rho(1+\delta_2)\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_{2k-1}|)^2\right)^{\frac{1}{2}}+\rho(1+\delta_1)\left(\sum_{k=1}^{\infty}(\lambda_k^2\,|\psi_{2k-1}|)^2\right)^{\frac{1}{2}}\right.$$

$$+(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|f_{2k-1}(\tau)|)^2d\tau\right)^{\frac{1}{2}}$$

$$\left.\left.+(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}, \tag{3.15}$$

where

$$\rho_1=\rho^2\left[\delta_1\left(\frac{\delta_2}{2}+\frac{T}{2}\right)+\delta_2\left(\frac{1}{2}+\frac{T}{2}+\delta_2\left(\frac{1}{4}+\frac{T}{2}\right)\right)\right.$$
$$\left.+\delta_1\delta_2\left(\frac{1}{2}+\frac{T}{2}+\delta_2\left(\frac{1}{2}+\frac{T}{2}\right)\right)\right]+\frac{T}{2}+\delta_2\left(\frac{1}{2}+\frac{T}{2}\right),$$
$$\rho_2=\rho^2\left[\delta_1\left(\frac{1}{2}+\frac{T}{2}\right)+\delta_1\left(\frac{1}{4}+\frac{T}{2}\right)+\delta_2\left(\frac{\delta_1}{2}+\frac{T}{2}\right)\right.$$
$$\left.+\delta_1\delta_2\left(\frac{1}{2}+\frac{T}{2}+\delta_1\left(\frac{1}{2}+\frac{T}{2}\right)\right)\right]+\frac{1}{2}+\frac{T}{2}+\delta_1\left(\frac{1}{2}+\frac{T}{2}\right).$$

Then from (3.12), (3.13), (3.14), and (3.15), taking into account (2.3), (2.4), respectively, we find:

$$\|\tilde{u}_0(t)\|_{C[0,T]} \le \frac{2}{1+\delta_1}\|\varphi(x)\|_{L_2(0,1)} + \frac{2T}{1+\delta_2}\|\psi(x)\|_{L_2(0,1)} + \frac{1+3\delta_1+3\delta_2}{(1+\delta_1)(1+\delta_2)}T$$
$$\times\left(2\sqrt{T}\|f(x,t)\|_{L_2(D_T)} + T\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}}\right),$$

$$\left\{\sum_{k=1}^{\infty}(\lambda_k^3\|\tilde{u}_{2k-1}(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}} \le 4\sqrt{2}\rho(1+\delta_2)\left\|\varphi'''(x)\right\|_{L_2(0,1)}$$
$$+4\sqrt{2}\rho(1+\delta_1)\left\|\psi''(x)\right\|_{L_2(0,1)}$$
$$+4(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{2T}\|f_{xx}(x,t)\|_{L_2(D_T)}$$
$$+2(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}},$$

$$\left\{\sum_{k=1}^{\infty}(\lambda_k^3\|\tilde{u}_{2k}(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}} \le 8\rho(1+\delta_2)\left\|\varphi'''(x)(1-q-px)-3p\varphi''(x)\right\|_{L_2(0,1)}$$
$$+8\rho(1+\delta_1)\left\|\psi''(x)(1-q-px)-2p\psi'(x)\right\|_{L_2(0,1)}$$
$$+8(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{T}\|f_{xx}(x,t)(1-q-px)-2pf_x(x,t)\|_{L_2(D_T)}$$
$$+2\sqrt{2}(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}}$$
$$+8\rho_1\left\|\varphi'''(x)\right\|_{L_2(0,1)}+8\rho_2\left\|\psi''(x)\right\|_{L_2(0,1)}$$
$$+2\sqrt{2}(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))^2T[2\sqrt{2T}\|f_{xx}(x,t)\|_{L_2(D_T)}$$
$$+T\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}}],$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le \left\|[h(t)]^{-1}\right\|_{C[0,T]}$$
$$\times\left\{\left\|h''(t)-f\left(\frac{1}{2},t\right)\right\|_{C[0,T]}+\frac{1}{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}[2\sqrt{2}\rho(1+\delta_2)\left\|\varphi'''(x)\right\|_{L_2(0,1)}\right.$$
$$+2\sqrt{2}\rho(1+\delta_1)\left\|\psi''(x)\right\|_{L_2(0,1)}$$
$$+(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))2\sqrt{2T}\|f_{xx}(x,t)\|_{L_2(D_T)}$$
$$\left.+(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}}]\right\},$$

or

$$\|\tilde{u}_0(t)\|_{C[0,T]} \le A_1(T)+B_1(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}}, \quad (3.16)$$

$$\left\{\sum_{k=1}^{\infty}(\lambda_k^3\|\tilde{u}_{2k-1}(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}} \le A_2(T)+B_2(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}}, \quad (3.17)$$

$$\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|\tilde{u}_{2k}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} \leq A_3(T)+B_3(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B_{2,T}^3}, \quad (3.18)$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]} \leq A_4(T)+B_4(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B_{2,T}^3}, \quad (3.19)$$

where

$$\begin{aligned}
A_1(T)=&\frac{2}{1+\delta_1}\left\|\varphi(x)\right\|_{L_2(0,1)}\\
&+\frac{2T}{1+\delta_2}\left\|\psi(x)\right\|_{L_2(0,1)}+\frac{2(1+3\delta_1+3\delta_2)}{(1+\delta_1)(1+\delta_2)}T\sqrt{T}\left\|f(x,t)\right\|_{L_2(D_T)},\\
B_1(T)=&\frac{1+3\delta_1+3\delta_2}{(1+\delta_1)(1+\delta_2)}T^2,\\
A_2(T)=&4\sqrt{2}\rho(1+\delta_2)\left\|\varphi'''(x)\right\|_{L_2(0,1)}+4\sqrt{2}\rho(1+\delta_1)\left\|\psi''(x)\right\|_{L_2(0,1)}\\
&+4(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{2T}\left\|f_{xx}(x,t)\right\|_{L_2(D_T)},\\
B_2(T)=&2(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T,\\
A_3(T)=&8\rho(1+\delta_2)\left\|\varphi'''(x)(1-q-px)-3p\varphi''(x)\right\|_{L_2(0,1)}\\
&+8\rho(1+\delta_1)\left\|\psi''(x)(1-q-px)-2p\psi'(x)\right\|_{L_2(0,1)}\\
&+8(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{T}\left\|f_{xx}(x,t)(1-q-px)-2pf_x(x,t)\right\|_{L_2(D_T)}\\
&+8\rho_1\left\|\varphi'''(x)\right\|_{L_2(0,1)}+8\rho_2\left\|\psi''(x)\right\|_{L_2(0,1)}\\
&+8(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))^2T\sqrt{T}\left\|f_{xx}(x,t)\right\|_{L_2(D_T)},\\
B_3(T)=&2\sqrt{2}(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T+2\sqrt{2}(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))^2T^2,\\
A_4(T)=&\left\|[h(t)]^{-1}\right\|_{C[0,T]}\\
&\times\left\{\left\|h''(t)-f\left(\frac{1}{2},t\right)\right\|_{C[0,T]}+\frac{1}{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[2\sqrt{2}\rho(1+\delta_2)\left\|\varphi'''(x)\right\|_{L_2(0,1)}\right.\right.\\
&+2\sqrt{2}\rho(1+\delta_1)\left\|\psi''(x)\right\|_{L_2(0,1)}\\
&\left.\left.+(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))2\sqrt{2T}\left\|f_{xx}(x,t)\right\|_{L_2(D_T)}\right]\right\},\\
B_4(T)=&\frac{1}{2}\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T.
\end{aligned}$$

Now, from (3.16)–(3.18) we obtain:

$$\left\|\tilde{u}(x,t)\right\|_{B_{2,T}^3} \leq A_5(T)+B_5(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B_{2,T}^3}, \quad (3.20)$$

where

$$A_5(T) = A_1(T) + A_2(T) + A_3(T), \;\; B_5(T) = B_1(T) + B_2(T) + B_3(T).$$

Finally, from (3.19) and (3.20) we conclude:

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \le A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}, \tag{3.21}$$

where

$$A(T) = A_4(T) + A_5(T), \;\; B(T) = B_4(T) + B_5(T).$$

So, let us prove the following theorem.

Theorem 2. *Let conditions C_1 - C_5 be satisfied and the inequality*

$$B(T)(A(T)+2)^2 < 1 \tag{3.22}$$

holds. Then problem (1.1)–(1.3), (1.6), (1.7) has a unique solution in the ball $K = K_R(\|z\|_{E^3_T} \le R \le A(T)+2)$ of the space E^3_T.

Remark 1. Inequality (3.22) is satisfied for sufficiently small values of T.

Proof. Let us denote $z = [u(x,t), a(t)]^T$ and rewrite the system of equations (3.9) and (3.11) in the following operator equation

$$z = \Phi z, \tag{3.23}$$

where $\Phi = [\varphi_1, \varphi_2]^T$ and $\varphi_1(z), \varphi_2(z)$ defined by the right sides of (3.9) and (3.11), respectively.

Consider the operator $\Phi(u,a)$ in the ball $K = K_R$ of the space E^3_T. Similar to (3.21) we obtain that for any $z_1, z_2, z \in K_R$ the following estimates hold:

$$\|\Phi z\|_{E^3_T} \le A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} \le A(T) + B(T)(A(T)+2)^2, \tag{3.24}$$

$$\|\Phi z_1 - \Phi z_2\|_{E^3_T} \le B(T)R(\|a_1(t) - a_2(t)\|_{C[0,T]} + \|u_1(x,t) - u_2(x,t)\|_{B^3_{2,T}}). \tag{3.25}$$

Then taking into account (3.22) in (3.24) and (3.25), it follows that the operator Φ acts in the ball $K = K_R$ and is contractive. Therefore, in the ball $K = K_R$ the operator Φ has a unique fixed point $\{z\} = \{u, a\}$ that is a unique solution of (3.23) in the ball $K = K_R$; i.e. it is a unique solution of system (3.9), (3.11) in the ball $K = K_R$. □

Thus, we obtain that the function $u(x,t)$ as an element of the space $B^3_{2,T}$ is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in D_T.

Analogous to [14] one can show that the function $u_{tt}(x,t)$ is continuous in the region D_T.

It is easy to verify that Eq.(1.1) and conditions (1.2), (1.3), (1.6), and (1.7) are satisfied in the ordinary sense. Consequently, $\{u(x,t), a(t)\}$ is a solution of problem (1.1)–(1.3), (1.6), (1.7) and by Remark 3.1 this solution is unique in the ball $K = K_R$. The theorem is proved.

From Theorem 1 and Theorem 2 immediately imply that the original problem (1.1)–(1.5) has a unique classical solution.

Theorem 3. *Suppose that all the conditions of Theorem 2 are satisfied and*

$$\int_0^1 f(x,t)dx = 0,\ 0 \le t \le T,\ \ \int_0^1 \varphi(x)dx = 0,\ \ \int_0^1 \psi(x)dx = 0,$$

$$h(0) + \delta_1 h(T) = \varphi\left(\frac{1}{2}\right),\ \ h'(0) + \delta_2 h'(T) = \psi\left(\frac{1}{2}\right),$$

$$\frac{(1+2\delta_1+3\delta_2+\delta_1\delta_2)T^2(A(T)+2)}{2(1+\delta_1)(1+\delta_2)} < 1.$$

Then problem (1.1)–(1.5) *has a unique classical solution in the ball* $K = K_R(\|z\|_{E_T^3} \le R \le A(T)+2)$ *of the space* E_T^3.

4. Conclusion

In the work, the classical solvability of a nonlinear inverse boundary value problem for a second-order hyperbolic equation with nonlocal conditions was studied. The considered problem was reduced to an auxiliary inverse boundary value problem in a certain sense and its equivalence to the original problem is shown. Then using the Fourier method and contraction mappings principle, the existence and uniqueness theorem for auxiliary problem is proved. Further, on the basis of the equivalency of these problems, the existence and uniqueness theorem for the classical solution of the original inverse coefficient problem is established for the smaller value of time.

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