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BIVARIATE k-MITTAG-LEFFLER FUNCTIONS WITH 2D-k-LAGUERRE-KONHAUSER POLYNOMIALS AND CORRESPONDING k-FRACTIONAL OPERATORS

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Abstract. In this paper, we first introduce new class of 2D-k-Laguerre-Konhauser polynomials, ${}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)$, which generalizes the 2D-Laguerre-Konhauser polynomials (see [18]). Then, we define a new family of bivariate k-Mittag-Leffler functions $E_{k,\alpha,\beta,\delta}^{(\gamma)}(x,y)$ and establish the k-Riemann-Liouville double fractional integral and derivative of the functions $E_{k,\alpha,\beta,\delta}^{(\gamma)}(x,y)$. Moreover, we introduce an integral operator ${}_{k}\varepsilon_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}^{(\gamma)}$ which contains the bivariate k-Mittag-Leffler functions $E_{k,\alpha,\beta,\delta}^{(\gamma)}(x,y)$ in the kernel and investigate the semigroup property of this operator. Finally, the left inverse operator of the integral operator ${}_{k}\varepsilon_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}^{(\gamma)}$ is constructed.

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1. INTRODUCTION

In recent years, fractional calculus and its wide application have been paid to an ever increasing extent attentions. In mathematical analysis, the fractional calculus is a very important device to achive differentiation and integration with the real number or complex number of the differential or intergral operator (see [2, 9, 10]). Miller and Rose [13] and Kiryakova [12] defined a well known number of different fractional calculus operators along with their properties and applications.

In recent decades, the fractional integral and derivative operators in terms of special functions have started the use of important applications in different areas of mathematical analysis. The Mittag-Leffler function which includes the Gamma function is one of the most famous special funtion which is used in the solution of fractional order integral equations or fractional order differential equations.

Apart from fractional calculus, the Mittag-Leffler function also plays a significant role in various braches of applied mathematics and engineering sciences, such as chemistry, biology, statistics, thermodynamics, mechanics, quantum physics, informatics, signal processing and others.

In this paper, we define and study a new class of 2D-k-Laguerre Konhauser polynomials ${}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)$ which generalizes the 2D-Laguerre-Konhauser polynomials (see [3]). Also, we introduce a new family of k-Mittag-Leffler functions, $E_{k,\alpha,\beta,\delta}^{(\gamma)}(x,y)$. We calculate k-Riemann-Liouville double fractional integral and derivative of the functions $E_{k,\alpha,\beta,\delta}^{(\gamma)}(x,y)$. Moreover, we introduce an integral operator ${}_{k}\varepsilon_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}^{(\gamma)}$ with respect to two variable k-Mittag-Leffler function, $E_{k,\alpha,\beta,\delta}^{(\gamma)}(x,y)$ in the kernel. We are also able to define fractional derivative operators, making use of the semigroup property.

In 2006, Diaz and Pariguan [6, 17] defined the Pochhammer k-symbol and k-gamma function as

$$(\gamma)_{n,k} := \begin{cases} \frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)} & \text{if } k \in \mathbb{R};\ \gamma \in \mathbb{C}\setminus\{0\}; \\ \gamma(\gamma+k)\ldots(\gamma+(n-1)k) & \text{if } n \in \mathbb{N};\ \gamma \in \mathbb{C}; \end{cases} \tag{1.1}$$

where k-Gamma function Γ_k has a relation with the classical Euler's gamma function as:

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right) \qquad (\mathrm{Re}(x) > 0,\ k \in \mathbb{R}^+). \tag{1.2}$$

Knowing that

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \qquad (\mathrm{Re}(x) > 0).$$

When $k = 1$, (1.2) reduces to the classical Pochhammer symbol and Euler's gamma function, respectively. Integral representation of k-Gamma function [6, 17] is given by

$$\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{k}}t^{x-1}dt \qquad (\mathrm{Re}(x) > 0,\ k \in \mathbb{R}^+).$$

From (1.1) and (1.2), we have

$$(x)_{rq,k} = k^{rq}\left(\frac{x}{k}\right)_{rq} \qquad (k \in \mathbb{R}^+,\ x,r,q \in \mathbb{C}).$$

As particular case

$$(x)_{r,k} = k^{r}\left(\frac{x}{k}\right)_{r}.$$

The k-Beta function B_k [6] is defined by

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt \qquad (k \in \mathbb{R}^+,\ \mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0)$$

and we have

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x,y)} = \frac{1}{k}B\left(\frac{m}{k},\frac{n}{k}\right).$$

The generalized k-Mittag-Leffler function is introduced by Chand et al. [5] as

$$E^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(\alpha n+\beta)} \frac{z^n}{n!} \tag{1.3}$$
$$(k\in\mathbb{R},\ \alpha,\beta,\gamma\in\mathbb{C},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0,\ q\in\mathbb{R}^+).$$

Special Cases of $E^{\gamma,q}_{k,\alpha,\beta}(z)$:

(1) For $q=1$, (1.3) reduces to k-Mittag-Leffler function (see [8]) such that

$$E^{\gamma,1}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)} \frac{z^n}{n!} = E^{\gamma}_{k,\alpha,\beta}(z).$$

(2) For $k=1$, (1.3) yields Mittag-Leffler function (see [4]) defined as

$$E^{\gamma,q}_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} = E^{\gamma,q}_{\alpha,\beta}(z).$$

(3) For $q=1$ and $k=1$, (1.3) gives Mittag-Leffler function presented by Prabhakar [19]

$$E^{\gamma,1}_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} = E^{\gamma}_{\alpha,\beta}(z).$$

(4) For $q=1$, $k=1$ and $\gamma=1$, (1.3) reduces the following Mittag-Leffler function (see [21])

$$E^{1,1}_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} = E_{\alpha,\beta}(z).$$

(5) For $q=1$, $k=1$, $\gamma=1$ and $\beta=1$, (1.3) gives Mittag-Leffler function introduced by Gösta Mittag-Leffler (see [14, 15])

$$E^{1,1}_{1,\alpha,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} = E_{\alpha}(z).$$

As an initial motivation, we consider the following class of generalized Laguerre-Konhauser polynomials were introduced by Bin-Saad [3]

$${}_{\delta}L^{(\alpha,\beta)}_{n}(x,y) = n!\sum_{s=0}^{n}\sum_{r=0}^{n-s} \frac{(-1)^{s+r}x^{r+\alpha}y^{\delta s+\beta}}{s!r!(n-s-r)!\Gamma(\alpha+r+1)\Gamma(\delta s+\beta+1)} \tag{1.4}$$
$$(\alpha,\beta\in\mathbb{R},\ \delta=1,2,\cdots).$$

Recently, Özarslan and Kürt (see [18]) have considered Laguerre-Konhauser polynomials ${}_{\delta}L_n^{(\alpha,\beta)}(x,y)$ and defined a bivariate Mittag-Leffler functions $E_{\alpha,\beta,\delta}^{(\gamma)}(x,y)$ by

$$E_{\alpha,\beta,\delta}^{(\gamma)}(x,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}}{\Gamma(\alpha+r)\Gamma(\beta+\delta s)}\frac{x^r}{r!}\frac{y^{\delta s}}{s!}$$
$$(\alpha,\beta,\gamma\in\mathbb{C},\ \mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta)>0). \tag{1.5}$$

Remark 1. According to the convergence conditions investigated by Srivastava and Daoust ([20], pp. 155) for the generalized Lauricella series in two variables, the series in (1.5) converges absolutely for $\mathrm{Re}(\delta)>0$.

From (1.4) and (1.5), we see

$${}_{\delta}L_n^{(\alpha,\beta)}(x,y)=x^{\alpha}y^{\beta}E_{\alpha+1,\beta+1,\delta}^{(-n)}(x,y).$$

In a view of the above definitions and motivations, in this paper, we introduce a new class of 2D-k-Laguerre-Konhauser polynomials by

$$\begin{aligned}{}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)&=\sum_{s=0}^{n}\sum_{r=0}^{n-s}\frac{(-nk)_{k,r+s}x^{r+\alpha}y^{\delta s+\beta}}{\Gamma_k(\alpha+r+1)\Gamma_k(\delta s+\beta+1)}\frac{x^{r+\alpha}}{r!}\frac{y^{\delta s+\beta}}{s!}\\&=n!\sum_{s=0}^{n}\sum_{r=0}^{n-s}\frac{(-1)^{r+s}k^{r+s}}{(n-s-r)!\Gamma_k(\alpha+r+1)\Gamma_k(\delta s+\beta+1)}\frac{x^{r+\alpha}}{r!}\frac{y^{\delta s+\beta}}{s!}\end{aligned} \tag{1.6}$$

In the case $k=1$, relation (1.6) coincides with the 2D-Laguerre-Konhauser polynomials (1.4).

According to the inverse operator $\hat{D}_x^{-\alpha}$ i.e.

$$\hat{D}_x^{-\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_0^x f(t)dt\,,\qquad \mathrm{Re}(\alpha)>0;$$

we can rewrite ${}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)$ in the operational representation as

$${}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)=\sum_{s=0}^{n}\sum_{r=0}^{n-s}(-nk)_{k,r+s}\,{}_k\hat{D}_x^{-r}\,{}_k\hat{D}_y^{-\delta s}\left\{\frac{x^{\frac{\alpha}{k}}y^{\frac{\beta}{k}}}{\Gamma_k(\alpha+1)\Gamma_k(\beta+1)}\right\}$$

which further yields the Rodrigues-type relation

$$E_{\alpha,\beta,\delta}^{(\gamma)}(x,y)=\left(1-\hat{D}_x^{-1}\hat{D}_y^{-\delta}\right)^{-\gamma}\left\{\frac{x^{\alpha-1}y^{\beta-1}}{\Gamma_k(\alpha)\Gamma_k(\beta)}\right\}.$$

We also motivate by the above results and define the following extended type k-Mittag-Leffler function of two variables:

$$E_{k,\alpha,\beta,\delta}^{(\gamma)}(x,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s,k}}{\Gamma_k(\alpha+r)\Gamma_k(\beta+\delta s)}\frac{x^r}{r!}\frac{y^{\delta s}}{s!}$$
$$\left(k\in\mathbb{R}^+,\ \alpha,\beta,\gamma\in\mathbb{C},\ \mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta)>0\right). \tag{1.7}$$

Remark 2. According to the convergence conditions investigated by Srivastava and Daoust ([20], pp. 155) for the generalized Lauricella series in two variables, the series in (1.5) converges absolutely for $\mathrm{Re}(\delta) > 0$.

Special Cases of $E^{(\gamma)}_{k,\alpha,\beta,\delta}(x,y)$:

(1) For $k = 1$, (1.7) is a particular case of (1.5).

(2) When $x \to 0$ and $r = 0$, we get a k-Mittag-Leffler function of one variable

$$E^{(\gamma)}_{k,\alpha,\beta,\delta}(0,y) = \frac{1}{\Gamma_k(\alpha)} E^{\gamma}_{k,\delta,\beta}(y^{\delta}).$$

Comparing (1.6) and (1.7), we see that

$$_{\delta}L^{(\alpha,\beta)}_{k,n}(x,y) = x^{\alpha} y^{\beta} E^{(-nk)}_{k,\alpha+1,\beta+1,\delta}(x,y). \tag{1.8}$$

This paper is organised as follows: In Section 2, we investigate the Riemann-Liouville double k-fractional integral and derivative of $E^{(\gamma)}_{k,\alpha,\beta,\delta}(x,y)$ and $_{\delta}L^{(\alpha,\beta)}_{k,n}(x,y)$. In Section 3, we obtain the double Laplace transform of $E^{(\gamma)}_{k,\alpha,\beta,\delta}(x,y)$ and $_{\delta}L^{(\alpha,\beta)}_{k,n}(x,y)$. Also, we find the linear generating function for the polynomials $_{\delta}L^{(\alpha,\beta)}_{k,n}(x,y)$ by means of $E^{(\gamma)}_{k,\alpha,\beta,\delta}(x,y)$. In Section 4, we introduce an integral operator and establish its fundamental properties.

2. Fractional calculus approach

In this section, we study the Riemann-Liouville double k-fractional integral and derivative properties of $E^{(\gamma)}_{k,\alpha,\beta,\delta}(x,y)$ and $_{\delta}L^{(\alpha,\beta)}_{k,n}(x,y)$.

Definition 1 (see [16]). Let $\alpha \in \mathbb{R}^{+}$ and $n \in \mathbb{N}$ such that $n-1 < \alpha < n$, $f \in L_1([0,\infty)$. Then the Riemann-Liouville integral of f is

$$I^{\alpha}_{k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt.$$

Clearly, when $k = 1$ then $I^{\alpha}_{k} f(x)$ reduces to the Riemann-Liouville fractional integration formula (see [11]).

Similarly, k-Riemann-Liouville double fractional integral of $f(x,y)$ is defined as follows:

$$_{k,y}I^{\lambda}_{b^+}\, _{k,x}I^{\mu}_{a^+} f(x,y) = \frac{1}{k^2\Gamma_k(\mu)\Gamma_k(\lambda)} \int_b^y \int_a^x (y-\tau)^{\frac{\lambda}{k}-1}(x-t)^{\frac{\mu}{k}-1} f(t,\tau)dtd\tau$$
$$(x > a,\ y > b,\ \mathrm{Re}\,(\lambda) > 0,\ \mathrm{Re}\,(\mu) > 0).$$

Theorem 1. *For* $\alpha,\beta,\gamma,\delta \in \mathbb{C}$, $\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\delta), \mathrm{Re}(\gamma), \mathrm{Re}(\lambda), \mathrm{Re}(\mu) > 0$, *we have*

$$_{k,x}I^{\lambda}_{a^+}\, _{k,y}I^{\mu}_{b^+} \left[(x-a)^{\frac{\alpha}{k}-1}(y-b)^{\frac{\beta}{k}-1} E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}})\right]$$

$$= \frac{(x-a)^{\frac{\alpha+\lambda}{k}-1}(y-b)^{\frac{\beta+\mu}{k}-1}}{k^2} E^{(\gamma)}_{k,\alpha+\lambda,\beta+\mu,\delta}(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}}).$$

Proof. Interchanging the order of series and fractional integral operators yields

$$ {}_{k,x}I^{\lambda}_{a^+}\, {}_{k,y}I^{\mu}_{b^+} \left[(x-a)^{\frac{\alpha}{k}-1}(y-b)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}})\right]$$

$$= \int_a^x \int_b^y (x-t)^{\frac{\lambda}{k}-1}(t-a)^{\frac{\alpha}{k}-1}(y-\tau)^{\frac{\mu}{k}-1}(\tau-b)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}$$

$$(\omega_1(t-a)^{\frac{1}{k}}, \omega_2(\tau-b)^{\frac{1}{k}})d\tau dt$$

$$= \frac{1}{k\Gamma_k(\lambda)} \frac{1}{k\Gamma_k(\mu)} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\gamma)_{r+s}}{\Gamma_k(\alpha+r)\Gamma_k(\beta+\delta s)} \frac{\omega_1^r}{r!}\frac{\omega_2^{\delta s}}{s!} \times$$

$$\int_a^x (x-t)^{\frac{\lambda}{k}-1}(t-a)^{\frac{\alpha}{k}+\frac{r}{k}-1}dt \int_b^y (y-\tau)^{\frac{\mu}{k}-1}(\tau-b)^{\frac{\beta}{k}+\frac{\delta s}{k}-1}d\tau$$

$$= \frac{(x-a)}{k}^{\frac{\alpha+\lambda}{k}-1} \frac{(y-b)}{k}^{\frac{\beta+\mu}{k}-1} \times$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\gamma)_{k,r+s}}{\Gamma_k(\alpha+\lambda+r)\Gamma_k(\beta+\mu+\delta s)} \frac{\left[\omega_1(x-a)^{\frac{1}{k}}\right]}{r!} \frac{\left[\omega_2(y-b)^{\frac{1}{k}}\right]^{\delta s}}{s!}$$

$$= \frac{(x-a)}{k}^{\frac{\alpha+\lambda}{k}-1} \frac{(y-b)}{k}^{\frac{\beta+\mu}{k}-1} E^{(\gamma)}_{k,\alpha+\lambda,\beta+\mu,\delta}(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}}),$$

which completes the desired proof. □

Corollary 1. *As a consequence of* (1.8) *and Theorem 1, we have*

$$ {}_{k,x}I^{\lambda}_{a^+}\, {}_{k,y}I^{\mu}_{b^+} \left[{}_{\delta}L^{(\alpha,\beta)}_{k,n}(\omega_1(x-a), \omega_2(y-b))\right] = {}_{\delta}L^{(\alpha+\lambda,\beta+\mu)}_{k,n}(\omega_1(x-a), \omega_2(y-b)),$$

where $\alpha,\beta,\gamma,\delta \in \mathbb{C}$, $\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\delta), \mathrm{Re}(\gamma), \mathrm{Re}(\lambda), \mathrm{Re}(\mu) > 0$.

Definition 2 (see [7])**.** Let $k,\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$ such that $n-1 < \alpha < n$, $f \in L_1([0,\infty)$. Then the Riemann-Liouville derivative of f is

$$D^{\alpha}_k f(x) = \left(\frac{d}{dx}\right)^n k^n\, {}_{k,x}I^{nk-\alpha}_{a^+} f(x).$$

Obviously, if $k = 1$ then $D^{\alpha}_k f(x)$ coincides with the Riemann-Liouville fractional derivative (see [11]).

Similarly, k-Riemann-Liouville double fractional derivative of $f(x,y)$ is defined as follows:

$$ {}_{k,y}D^{\lambda}_{b^+}\, {}_{k,x}D^{\mu}_{a^+} f(x,y) = \left(\frac{\partial}{\partial x}\right)^n \left(\frac{\partial}{\partial y}\right)^m k^n k^m\, {}_{k,x}I^{nk-\alpha}_{a^+}\, {}_{k,y}I^{mk-\mu}_{b^+} f(x,y)$$

$(x > a, qy > b,\ \mathrm{Re}(\lambda) > 0,\ \mathrm{Re}(\mu) > 0).$

Theorem 2. *For* $\alpha,\beta,\gamma,\delta \in \mathbb{C},\ \mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta),\mathrm{Re}(\gamma),\mathrm{Re}(\lambda),\mathrm{Re}(\mu) > 0$*, we have*

$$
\begin{aligned}
&{}_{k,x}D^{\lambda}_{a^+}\ {}_{k,y}D^{\mu}_{b^+}\left[(x-a)^{\frac{\alpha}{k}-1}(y-b)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}})\right]\\
&= (x-a)^{\frac{\alpha-\lambda}{k}-1}(y-b)^{\frac{\beta-\mu}{k}-1}E^{(\gamma)}_{k,\alpha-\lambda,\beta-\mu,\delta}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}}).
\end{aligned}
$$

Proof. Interchanging the order of series and fractional integral operators yields

$$
\begin{aligned}
&{}_{k,x}D^{\lambda}_{a^+}\ {}_{k,y}D^{\mu}_{b^+}\left[(x-a)^{\frac{\alpha}{k}-1}(y-b)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}})\right]\\
&= \left(\frac{\partial}{\partial x}\right)^n\left(\frac{\partial}{\partial y}\right)^m k^n k^m\\
&={}_{k,n}\, I^{nk-\lambda}_{a^+}\ {}_{k,m}I^{mk-\mu}_{b^+}\left[(x-a)^{\frac{\alpha}{k}-1}(y-b)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}})\right]\\
&= \left(\frac{\partial}{\partial x}\right)^n\left(\frac{\partial}{\partial y}\right)^m k^n k^m \frac{1}{k\Gamma_k(nk-\lambda)}\frac{1}{k\Gamma_k(mk-\mu)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{k,r+s}\omega_1^r\omega_2^{\delta s}}{r!s!\Gamma_k(\alpha+r)\Gamma_k(\beta+\delta s)}\\
&\quad\times\int_a^x (x-t)^{\frac{nk-\lambda}{k}-1}(t-a)^{\frac{\alpha+r}{k}-1}dt\int_b^x (y-\tau)^{\frac{mk-\mu}{k}-1}(\tau-b)^{\frac{\beta+\delta s}{k}-1}d\tau\\
&= \left(\frac{\partial}{\partial x}\right)^n\left(\frac{\partial}{\partial y}\right)^m k^n k^m \frac{1}{k\Gamma_k(nk-\lambda)}\frac{1}{k\Gamma_k(mk-\mu)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{k,r+s}\omega_1^r\omega_2^{\delta s}}{r!s!\Gamma_k(\alpha+r)\Gamma_k(\beta+\delta s)}\\
&\quad\times\frac{\Gamma_h(nk-\lambda)\Gamma_k(\alpha+r)}{\Gamma_k(nk-\lambda+\alpha+r)}\frac{\Gamma_k(mk-\mu)\Gamma_k(\beta+\delta s)}{\Gamma_k(mk-\mu+\beta+\delta s)}\\
&\quad\times(x-a)^{\frac{nk-\lambda+\alpha}{k}+\frac{r}{k}-1}(y-b)^{\frac{mk-\mu+\beta}{k}+\frac{\delta s}{k}-1}
\end{aligned}
$$

Since $\Gamma_k(nk-\lambda+\alpha+r) = k^{\frac{nk-\lambda+\alpha+r}{k}-1}\Gamma(\frac{nk-\lambda+\alpha+r}{k})$ and $\Gamma_k(mk-\mu+\beta+\delta s) = k^{\frac{mk-\mu+\beta+\delta s}{k}-1}\Gamma(\frac{mk-\mu+\beta+\delta s}{k})$, we have

$$
\begin{aligned}
&{}_{k,x}D^{\lambda}_{a^+}\ {}_{k,y}D^{\mu}_{b^+}\left[(x-a)^{\frac{\alpha}{k}-1}(y-b)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}})\right]\\
&= (x-a)^{\frac{\alpha-\lambda}{k}-1}(y-b)^{\frac{\beta-\mu}{k}-1}\\
&\quad\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{k,r+s}\omega_1^r\omega_2^{\delta s}}{\Gamma_k(\alpha-\lambda+r)\Gamma_k(\beta-\mu+\delta s)}\frac{(x-a)^{\frac{r}{k}-1}}{r!}\frac{(y-b)^{\frac{\delta s}{k}-1}}{s!}\\
&= (x-a)^{\frac{\alpha-\lambda}{k}-1}(y-b)^{\frac{\beta-\mu}{k}-1}E^{(\gamma)}_{k,\alpha-\lambda,\beta-\mu,\delta}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}}).
\end{aligned}
$$

Hence, the result is achieved. □

Corollary 2. *As a consequence of* (1.8) *and Theorem* 2*, we have*

$$\begin{aligned}&{}_{k,x}D^{\lambda}_{a^+}\,{}_{k,y}D^{\mu}_{b^+}\left[{}_{\delta}L^{(\alpha,\beta)}_{k,n}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}})\right]\\&={}_{\delta}L^{(\alpha-\lambda,\beta-\mu)}_{k,n}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}}),\end{aligned}$$

where $\alpha,\beta,\gamma,\delta\in\mathbb{C}$, $\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta),\mathrm{Re}(\gamma),\mathrm{Re}(\lambda),\mathrm{Re}(\mu)>0$.

3. Laplace transform and generating function

In this section, we obtain the double Laplace transform of $E^{(\gamma)}_{k,\alpha,\beta,\delta}(x,y)$ and ${}_{\delta}L^{(\alpha,\beta)}_{k,n}(x,y)$. Moreover, we derive the linear generating function for the polynomials ${}_{\delta}L^{(\alpha,\beta)}_{k,n}(x,y)$ in terms of $E^{(\gamma)}_{k,\alpha,\beta,\delta}(x,y)$. The double Laplace transform (see [1]) of $f(x,y)$ is defined as

$$\mathbb{L}_2[f(x,y)](p,q)=\int_0^{\infty}\int_0^{\infty}e^{-(px+qy)}f(x,y)dxdy \qquad (\mathrm{Re}(p)>0,\ \mathrm{Re}(q)>0).$$

Theorem 3. *Let* $\alpha,\beta,\gamma,\delta\in\mathbb{C}$, $\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta),\mathrm{Re}(\gamma),\mathrm{Re}(p),\mathrm{Re}(q)>0$, $\left|\frac{\omega_2^{\delta}}{q^{\delta}}\right|<1$ *and* $\left|\frac{\omega_1 q^{\delta}}{p\left(q^{\delta}-\omega_2^{\delta}\right)}\right|<1$*. Then we have*

$$\begin{aligned}&\mathbb{L}_2\left(x^{\frac{\alpha}{k}-1}y^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1x^{\frac{1}{k}},\omega_2y^{\frac{1}{k}})\right)(p,q)\\&=p^{-\frac{\alpha}{k}}q^{-\frac{\beta}{k}}k^2k^{-\frac{\alpha}{k}}k^{-\frac{\beta}{k}}\left(1-\frac{k\omega_2^{\delta}(pk)^{\frac{1}{k}}+\omega_1(qk)^{\frac{\delta}{k}}}{(pk)^{\frac{1}{k}}(qk)^{\frac{\delta}{k}}}\right)^{-\frac{\gamma}{k}}.\end{aligned}$$

Proof. Interchanging the order of series and fractional integral operators yields

$$\begin{aligned}&\mathbb{L}_2\left(x^{\frac{\alpha}{k}-1}y^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1x^{\frac{1}{k}},\omega_2y^{\frac{1}{k}})\right)(p,q)\\&=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{k,r+s}\omega_1^r\omega_2^{\delta s}}{\Gamma_k(\alpha+r)\Gamma_k(\beta+\delta s)r!s!}\int_0^{\infty}e^{-px}x^{\frac{\alpha}{k}+\frac{r}{k}-1}dx\int_0^{\infty}e^{-qy}y^{\frac{\beta}{h}+\frac{\delta s}{k}-1}dy\\&=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{k,r+s}}{\Gamma_k(\alpha+r)\Gamma_k(\beta+\delta s)r!s!}\frac{\omega_1^r}{p^{\frac{\alpha+r}{k}}}\frac{\omega_2^{\delta s}}{q^{\frac{\beta+\delta s}{k}}}\int_0^{\infty}e^{-u}u^{\frac{\alpha+r}{k}-1}du\int_0^{\infty}e^{-v}v^{\frac{\beta+\delta s}{k}-1}dv\\&=\frac{1}{p^{\frac{\alpha}{k}}}\frac{1}{q^{\frac{\beta}{h}}}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{k,r+s}}{\Gamma_h(\alpha+r)\Gamma_h(\beta+\delta s)r!s!}\Gamma\left(\frac{\alpha+r}{h}\right)\Gamma\left(\frac{\beta+\delta s}{k}\right)\left(\frac{\omega_1}{p}\right)^{\frac{r}{k}}\left(\frac{\omega_2}{q}\right)^{\frac{\delta s}{k}}\end{aligned}$$

Since $\frac{\Gamma\left(\frac{\alpha+r}{k}\right)}{\Gamma_k(\alpha+r)}=\frac{1}{k^{\frac{\alpha+r}{k}-1}}$ and $\frac{\Gamma\left(\frac{\beta+\delta s}{k}\right)}{\Gamma_k(\beta+\delta s)}=\frac{1}{k^{\frac{\beta+\delta s}{k}-1}}$, it follows

$$\mathbb{L}_2\left(x^{\frac{\alpha}{k}-1}y^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1x^{\frac{1}{k}},\omega_2y^{\frac{1}{k}})\right)(p,q)$$

$$= \frac{1}{p^{\frac{\alpha}{h}}}\frac{1}{q^{\frac{\beta}{k}}}k^2\frac{1}{k^{\frac{\alpha}{k}}}\frac{1}{k^{\frac{\beta}{k}}}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{k,r+s}\omega_1^r\omega_2^{\delta s}}{r!s!\left(k^{\frac{1}{k}}\right)^r\left(k^{\frac{\delta}{k}}\right)^s}\left(\frac{1}{p^{\frac{1}{k}}}\right)^r\left(\frac{1}{q^{\frac{\delta}{k}}}\right)^s$$

$$= \frac{1}{p^{\frac{\alpha}{k}}}\frac{1}{q^{\frac{\beta}{k}}}k^2\frac{1}{k^{\frac{\alpha}{k}}}\frac{1}{k^{\frac{\beta}{k}}}\sum_{r=0}^{\infty}\frac{(\gamma)_{k,r}\omega_1^r}{r!\left(k^{\frac{1}{k}}\right)^r}\left(\frac{1}{p^{\frac{1}{h}}}\right)^r\sum_{s=0}^{\infty}\frac{(\gamma+kr)_{h,s}\omega_2^{\delta s}}{s!\left(k^{\frac{\delta}{k}}\right)^s}\left(\frac{1}{q^{\frac{\delta}{k}}}\right)^s$$

$$= \frac{1}{p^{\frac{\alpha}{k}}}\frac{1}{q^{\frac{\beta}{k}}}k^2\frac{1}{k^{\frac{\alpha}{k}}}\frac{1}{k^{\frac{\beta}{k}}}\sum_{r=0}^{\infty}\frac{(\gamma)_{k,r}\omega_1^r}{r!\left(k^{\frac{1}{k}}\right)^r}\left(\frac{1}{p^{\frac{1}{k}}}\right)^r\left(1-\frac{k\omega_2^{\delta}}{(qk)^{\frac{\delta}{k}}}\right)^{-\left(\frac{\gamma+kr}{k}\right)}$$

$$= \frac{1}{p^{\frac{\alpha}{k}}}\frac{1}{q^{\frac{\beta}{k}}}k^2\frac{1}{k^{\frac{\alpha}{k}}}\frac{1}{k^{\frac{\beta}{k}}}\left(1-\frac{k\omega_2^{\delta}}{(qk)^{\frac{\delta}{k}}}\right)^{-\frac{\gamma}{k}}\sum_{r=0}^{\infty}\frac{(\gamma)_{k,r}\omega_1^r}{r!\left(k^{\frac{1}{k}}\right)^r}\left(1-\frac{k\omega_2^{\delta}}{(qk)^{\frac{\delta}{k}}}\right)^{-r}$$

$$= \frac{1}{p^{\frac{\alpha}{k}}}\frac{1}{q^{\frac{\beta}{k}}}k^2\frac{1}{k^{\frac{\alpha}{k}}}\frac{1}{k^{\frac{\beta}{k}}}\left(1-\frac{k\omega_2^{\delta}}{(qk)^{\frac{\delta}{k}}}\right)^{-\frac{\gamma}{k}}\left(1-\frac{\omega_1(qk)^{\frac{\delta}{k}}}{(pk)^{\frac{1}{k}}\left((qk)^{\frac{\delta}{k}}-k\omega_2^{\delta}\right)}\right)^{-\frac{\gamma}{k}}$$

$$= p^{-\frac{\alpha}{k}}q^{-\frac{\beta}{k}}k^2k^{-\frac{\alpha}{k}}k^{-\frac{\beta}{k}}\left(1-\frac{k\omega_2^{\delta}(pk)^{\frac{1}{k}}+\omega_1(qk)^{\frac{\delta}{k}}}{(pk)^{\frac{1}{k}}(qk)^{\frac{\delta}{k}}}\right)^{-\frac{\gamma}{k}}.$$

Whence the result. □

Corollary 3. *For the polynomials* ${}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)$*, we have*

$$\mathbb{L}_2\left({}_{\delta}L_{k,n}^{(\alpha,\beta)}(\omega_1 x,\omega_2 y)\right)(p,q) = \frac{\omega_1^{\alpha}}{p^{\alpha+1}}\frac{\omega_2^{\beta}}{q^{\beta+1}}\left(\frac{pq^{\delta}-(\omega_2^{\delta}p+\omega_1 q^{\delta})}{pq)^{\delta}}\right)^n.$$

In the next theorem, we give the linear generating function for the polynomials ${}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)$ in terms of the bivariate k-Mittag-Leffler function $E_{k,\alpha,\beta,\delta}^{(\gamma)}(x,y)$.

Theorem 4. *For* $\alpha,\beta,\sigma\in\mathbb{C}$ *and* $|kt|<1$*, we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\sigma)_{k,n}}{n!}\,{}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)t^n\\&= x^{\alpha}y^{\beta}(1-kt)^{-\frac{\sigma}{k}}E_{k,\alpha+1,\beta+1,\delta}^{(\sigma)}\left(\frac{xt}{kt-1},y\left(\frac{t}{kt-1}\right)^{\frac{1}{\delta}}\right).\end{aligned}\tag{3.1}$$

When $k=1$, (3.1) *reduces to Theorem 22 in [18].*

Proof. Using the Cauchy product of the series, we get

$$\sum_{n=0}^{\infty}\frac{(\sigma)_{k,n}}{n!}\,{}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)t^n$$

$$= \sum_{n=0}^{\infty}\sum_{s=0}^{n}\sum_{r=0}^{n-s}\frac{(\sigma)_{k,n}(-1)^{s+r}k^{s+r}x^{\alpha+r}y^{\beta+\delta s}}{s!r!(n-s-r)!\Gamma_k(\alpha+r+1)\Gamma_k(\beta+\delta s+1)}t^n$$
$$= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{n}\frac{(\sigma)_{k,n+s}(-1)^{s+r}k^{s+r}x^{\alpha+r}y^{\beta+\delta s}}{s!r!(n-r)!\Gamma_k(\alpha+r+1)\Gamma_k(\beta+\delta s+1)}t^{n+s}$$
$$= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\frac{(\sigma)_{k,n+r+s}(-1)^{s+r}k^{s+r}x^{\alpha+r}y^{\beta+\delta s}}{s!r!\Gamma_k(\alpha+r+1)\Gamma_k(\beta+\delta s+1)}\frac{t^{n+r+s}}{n!}.$$

From $(\sigma)_{k,n+r+s} = (\sigma)_{k,r+s}(\sigma+rk+k)_{k,n}$, we get

$$\sum_{n=0}^{\infty}\frac{(\sigma)_n}{n!}\ {}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)t^n$$
$$= \sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{s+r}k^{s+r}(\sigma)_{delta,r+s}x^{\alpha+r}y^{\beta+\delta s}t^{s+r}}{s!r!\Gamma_k(\alpha+r+1)\Gamma_k(\beta+\delta s+1)}\sum_{n=0}^{\infty}(\sigma+r+s)_{k,n}\frac{t^n}{n!}$$
$$= x^{\alpha}y^{\beta}(1-kt)^{-\frac{\sigma}{k}}$$
$$\times\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{s+r}k^{s+r}(\sigma)_{k,r+s}}{s!r!\Gamma_k(\alpha+r+1)\Gamma_k(\beta+\delta s+1)}\left(\left(\frac{xt}{kt-1}\right)^r\left(\frac{y^{\delta}t}{kt-1}\right)^s\right)$$
$$= x^{\alpha}y^{\beta}(1-kt)^{-\frac{\sigma}{k}}E_{k,\alpha+1,\beta+1,\delta}^{(\sigma)}\left(\frac{xt}{kt-1},y\left(\frac{t}{kt-1}\right)^{\frac{1}{\delta}}\right),$$

where we have interchanged the order of summations which is guaranteed because of the uniform convergence of the series under the condition $|kt|<1$. □

4. k-FRACTIONAL CALCULUS OPERATORS

In this section, we consider the following double (fractional) k-integral operator

$$\left({}_k\varepsilon_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}^{(\gamma)}\varphi\right)(x,y)$$
$$= \int_c^y\int_a^x\frac{(x-t)^{\frac{\alpha}{k}-1}}{k}\frac{(y-\tau)^{\frac{\beta}{k}-1}}{k}E_{k,\alpha,\beta,\delta}^{(\gamma)}[\omega_1(x-t),\omega_2(y-\tau)]\times\varphi(t,\tau)dtd\tau$$
$$(x>a,\ y>c).$$

In the case $\gamma=0$, the integral operator ${}_k\varepsilon_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}^{(\gamma)}$ reduces to the left-sided k-Riemann-Liouville double fractional integral operator, i.e.

$$\left({}_h\varepsilon_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}^{(0)}\varphi\right)(x,y) = \left({}_{h,y}I_{c^+}^{\beta}\ {}_{h,x}I_{a^+}^{\alpha}\varphi\right)(x,y).$$

In the following theorem, we show the integral operator ${}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}$ is bounded in the space $L((a,b)\times(c,d))$ of Lebesgue measurable functions where

$$L((a,b)\times(c,d)) = \left\{\|f\|_1 := \int_a^b\int_c^d |f(x,y)|\,dydx < \infty\right\}$$

Theorem 5. *In the space $L((a,b)\times(c,d))$ of Lebesgue measurable functions, we show the integral operator ${}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}$ is bounded, i.e.*

$$\left\|{}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}\varphi\right\|_1 \le A\,\|\varphi\|_1$$

where the constant A $(0<A<\infty)$ is given by

$$\begin{aligned} A = &\frac{(b-a)^{\mathrm{Re}(\frac{\alpha}{k})}(d-c)^{\mathrm{Re}(\frac{\beta}{k})}}{k^2}\\ &\times\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{|(\gamma)_{k,r+s}|}{\left\{\mathrm{Re}(\frac{\alpha}{k})+r\right\}|\Gamma_k(\alpha+r)|\left\{\mathrm{Re}(\frac{\beta}{k})+\delta s\right\}|\Gamma_k(\beta+\delta s)|}\\ &\times\frac{|\omega_1(b-a)|^r}{r!}\frac{|\omega_2(d-c)|^{\delta s}}{s!}<\infty. \end{aligned}$$

Proof. By using the Fubini's Theorem, we get

$$\begin{aligned} &\left\|{}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}\varphi\right\|_1 \le\\ &\int_a^b\int_c^d|\varphi(t,\tau)|\times\left(\int_t^b\int_\tau^d\frac{(x-t)^{\mathrm{Re}(\frac{\alpha}{k})-1}}{k}\frac{(y-\tau)^{\mathrm{Re}(\frac{\beta}{k})-1}}{k}\right.\\ &\left.\left|E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-t),\omega_2(y-\tau))\right|dydx\right)d\tau dt\\ &=\int_a^b\int_c^d|\varphi(t,\tau)|\left(\int_0^{b-t}\int_0^{d-\tau}\frac{u^{\mathrm{Re}(\frac{\alpha}{k})-1}}{k}\frac{v^{\mathrm{Re}(\frac{\beta}{k})-1}}{k}\left|E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1u,\omega_2v)\right|dudv\right)d\tau dt\\ &\le\int_a^b\int_c^d|\varphi(t,\tau)|\left(\int_0^{b-a}\int_0^{d-c}\frac{u^{\mathrm{Re}(\frac{\alpha}{k})-1}}{k}\frac{v^{\mathrm{Re}(\frac{\beta}{k})-1}}{k}\left|E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1u,\omega_2v)\right|dudv\right)d\tau dt\\ &\le\frac{1}{k^2}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{|(\gamma)_{k,r+s}|}{|\Gamma_k(\alpha+r)|\,|\Gamma_k(\beta+\delta s)|}\frac{|\omega_1|^r}{r!}\frac{|\omega_2|^{\delta s}}{s!}\\ &\int_0^{b-a}u^{\mathrm{Re}(\frac{\alpha}{h})+r-1}du\int_0^{d-c}v^{\mathrm{Re}(\frac{\beta}{k})+\delta s-1}dv\,\|\varphi\|_1 = A\,\|\varphi\|_1. \end{aligned}$$

Whence the result. □

In the next theorem, we calculate double integrals which consist the product of bivariate k-Mittag-Leffler functions $E^{(\gamma)}_{k,\alpha,\beta,\delta}(x,y)$ in the integrand.

Theorem 6. *Let* $\alpha,\beta,\delta,\zeta,\sigma,\gamma,\eta\in\mathbb{C},\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta),\mathrm{Re}(\gamma),\mathrm{Re}(\sigma)>0$. *Then*

$$\int_0^x\int_0^y (x-t)^{\frac{\alpha}{k}-1}(y-\tau)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-t)^{\frac{1}{k}},\omega_2(y-\tau)^{\frac{1}{k}})t^{\frac{\zeta}{k}-1}\tau^{\frac{\sigma}{k}-1}$$
$$\times E^{(\eta)}_{k,\zeta,\sigma,\delta}(\omega_1 t^{\frac{1}{k}},\omega_2\tau^{\frac{1}{k}})d\tau dt=\frac{1}{k^2}x^{\frac{\alpha+\zeta}{k}-1}y^{\frac{\beta+\sigma}{k}-1}E^{(\gamma+\eta)}_{k,\alpha+\zeta,\beta+\sigma,\delta}(\omega_1 x^{\frac{1}{k}},\omega_2 y^{\frac{1}{k}}).$$

Proof. Using the double convolution theorem for the double Laplace transform, we have,

$$\mathbb{L}_2\left\{\int_0^x\int_0^y (x-t)^{\frac{\alpha}{k}-1}(y-\tau)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-t)^{\frac{1}{k}},\omega_2(y-\tau)^{\frac{1}{k}})\times\right.$$
$$\left. t^{\frac{\zeta}{k}-1}\tau^{\frac{\sigma}{k}-1}E^{(\eta)}_{h,\zeta,\sigma,\delta}(\omega_1 t^{\frac{1}{k}},\omega_2\tau^{\frac{1}{k}})d\tau dt\right\}(p,q)$$
$$=\mathbb{L}_2\left\{x^{\frac{\alpha}{k}-1}y^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1 x^{\frac{1}{k}},\omega_2 y^{\frac{1}{k}})\right\}(p,q)$$
$$\times\mathbb{L}_2\left\{x^{\frac{\zeta}{k}-1}y^{\frac{\sigma}{k}-1}E^{(\eta)}_{k,\zeta,\sigma,\delta}(\omega_1 x^{\frac{1}{k}},\omega_2 y^{\frac{1}{k}})\right\}(p,q).$$

From Theorem 3, we have for $\mathrm{Re}(p),\mathrm{Re}(q)>0$, $\left|\frac{k\omega_2^{\delta}}{(qk)^{\frac{\delta}{k}}}\right|<1$, $\left|\frac{\omega_1(qk)^{\frac{\delta}{k}}}{(pk)^{\frac{1}{k}}\left((qk)^{\frac{\delta}{k}}-k\omega_2^{\delta}\right)}\right|<1$,

$$\mathbb{L}_2\left\{\int_0^x\int_0^y (x-t)^{\frac{\alpha}{k}-1}(y-\tau)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-t)^{\frac{1}{k}},\omega_2(y-\tau)^{\frac{1}{k}})\times\right.$$
$$\left. t^{\frac{\zeta}{k}-1}\tau^{\frac{\sigma}{k}-1}E^{(\eta)}_{k,\zeta,\sigma,\delta}(\omega_1 t^{\frac{1}{k}},\omega_2\tau^{\frac{1}{k}})d\tau dt\right\}(p,q)$$
$$=\mathbb{L}_2\left\{x^{\frac{\alpha}{k}-1}y^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1 x^{\frac{1}{k}},\omega_2 y^{\frac{1}{k}})\right\}(p,q)$$
$$\times\mathbb{L}_2\left\{x^{\frac{\zeta}{k}-1}y^{\frac{\sigma}{k}-1}E^{(\eta)}_{k,\zeta,\sigma,\delta}(\omega_1 x^{\frac{1}{k}},\omega_2 y^{\frac{1}{k}})\right\}(p,q).$$
$$=\frac{1}{p^{\frac{\alpha}{k}}}\frac{1}{q^{\frac{\beta}{k}}}k^2\frac{1}{k^{\frac{\alpha}{k}}}\frac{1}{k^{\frac{\beta}{k}}}\left(1-\frac{k\omega_2^{\delta}(pk)^{\frac{1}{k}}+\omega_1(qk)^{\frac{\delta}{k}}}{(pk)^{\frac{1}{k}}(qk)^{\frac{\delta}{k}}}\right)^{-\frac{\gamma}{k}} \tag{4.1}$$
$$\times\frac{1}{p^{\frac{\zeta}{k}}}\frac{1}{q^{\frac{\sigma}{k}}}k^2\frac{1}{k^{\frac{\zeta}{k}}}\frac{1}{k^{\frac{\sigma}{k}}}\left(1-\frac{k\omega_2^{\delta}(pk)^{\frac{1}{k}}+\omega_1(qk)^{\frac{\delta}{k}}}{(pk)^{\frac{1}{k}}(qk)^{\frac{\delta}{k}}}\right)^{-\frac{\eta}{k}}$$

$$= \frac{1}{p^{\frac{\alpha+\zeta}{k}}}\frac{1}{q^{\frac{\beta+\sigma}{k}}}k^4\frac{1}{k^{\frac{\alpha+\zeta}{k}}}\frac{1}{k^{\frac{\beta+\sigma}{k}}}\left(1-\frac{k\omega_2^{\delta}(pk)^{\frac{1}{k}}+\omega_1(qk)^{\frac{\delta}{k}}}{(pk)^{\frac{1}{k}}(qk)^{\frac{\delta}{k}}}\right)^{-(\frac{\gamma+\eta}{k})}$$
$$= \mathbb{L}_2\left\{x^{\frac{\alpha+\zeta}{k}-1}y^{\frac{\beta+\sigma}{k}-1}E^{(\gamma+\eta)}_{k,\alpha+\zeta,\beta+\sigma,\delta}(\omega_1 x^{\frac{1}{k}},\omega_2 y^{\frac{1}{k}})\right\}(p,q).$$

Taking inverse Laplace on both sides of (4.1), we get

$$\int_0^x\int_0^y (x-t)^{\frac{\alpha}{k}-1}(y-\tau)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-t)^{\frac{1}{k}},\omega_2(y-\tau)^{\frac{1}{k}})\times$$
$$t^{\frac{\zeta}{k}-1}\tau^{\frac{\sigma}{k}-1}E^{(\eta)}_{h,\zeta,\sigma,\delta}(\omega_1 t^{\frac{1}{k}},\omega_2\tau^{\frac{1}{k}})d\tau dt$$
$$= \frac{1}{k^2}x^{\frac{\alpha+\zeta}{k}-1}y^{\frac{\beta+\sigma}{k}-1}E^{(\gamma+\eta)}_{k,\alpha+\zeta,\beta+\sigma,\delta}(\omega_1 x^{\frac{1}{k}},\omega_2 y^{\frac{1}{k}}).$$

□

In the following theorem, we state the semigroup property of the operator ${}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}$.

Theorem 7. *Let* $\alpha,\beta,\delta,\gamma,\zeta,\eta,\sigma,\omega_1,\omega_2\in\mathbb{C}$, $\mathrm{Re}(\gamma),\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\sigma),\mathrm{Re}(\zeta)$, $\mathrm{Re}(\eta)$, $\mathrm{Re}(\delta)>0$, *then the relation*

$$\left({}_k\varepsilon^{(\gamma)}_{\alpha,\beta;\omega_1,\omega_2;a^+,c^+}\,{}_k\varepsilon^{(\sigma)}_{\zeta,\eta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y) = \left({}_k\varepsilon^{(\gamma+\sigma)}_{\alpha+\zeta,\beta+\eta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y)$$

is valid for any summable function $\varphi\in L((a,b)\times(c,d))$. *In particular*

$$\left({}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}\,{}_k\varepsilon^{(-\gamma)}_{\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y) = \left({}_{k,y}I^{\beta+\eta}_{c^+}\,{}_{k,x}I^{\alpha+\zeta}_{a^+}\varphi\right)(x,y).$$

Proof. Direct calculations yield

$$\left({}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}\,{}_k\varepsilon^{(\sigma)}_{\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y)$$
$$= \frac{1}{k^2}\int_c^y\int_a^x (x-t)^{\frac{\alpha}{k}-1}(y-\tau)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-t),\omega_2(y-\tau))$$
$$\times{}_k\,\varepsilon^{(\sigma)}_{(\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+)}\varphi(t,\tau)dtd\tau$$
$$= \frac{1}{k^4}\int_c^y\int_a^x\int_c^\tau\int_a^t (x-t)^{\frac{\alpha}{k}-1}(y-\tau)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-t),\omega_2(y-\tau))$$
$$\times(t-u)^{\frac{\zeta}{k}-1}(\tau-v)^{\frac{\eta}{k}-1}\times E^{(\sigma)}_{k,\zeta,\eta,\delta}(\omega_1(t-u),\omega_2(\tau-v))\varphi(u,v)du\,dv\,dt\,d\tau$$
$$= \frac{1}{k^4}\int_c^y\int_a^x\int_v^y\int_u^x (x-t)^{\frac{\alpha}{k}-1}(y-\tau)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-t),\omega_2(y-\tau))$$

$$\times (t-u)^{\frac{\zeta}{k}-1}(\tau-v)^{\frac{\eta}{k}-1}E^{(\sigma)}_{k,\zeta,\eta,\delta}(\omega_1(t-u),\omega_2(\tau-v))\varphi(u,v)dt\, d\tau\, du\, dv$$

$$= \int_c^y \int_a^x \int_0^{y-v} \int_0^{x-u} (x-m-u)^{\frac{\alpha}{k}-1}(y-n-v)^{\frac{\beta}{k}-1}$$

$$\times E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-m-u),\omega_2(y-n-v))k^{\frac{\zeta}{k}-1}l^{\frac{\eta}{k}-1}$$

$$\times E^{(\sigma)}_{k,\zeta,\eta,\delta}(\omega_1 m,\omega_2 n)\varphi(u,v)dm\, dn\, du\, dv.$$

Now, using Theorem 6, we obtain

$$\left({}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}\ {}_k\varepsilon^{(\sigma)}_{\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y)$$

$$= \frac{1}{k^2}\int_c^y \int_a^x (x-u)^{\frac{\alpha+\zeta}{k}-1}(y-v)^{\frac{\beta+\eta}{k}-1}E^{(\gamma+\sigma)}_{k,\alpha+\zeta,\beta+\eta,\delta}(\omega_1(x-u)^{\frac{1}{k}},\omega_2(y-v)^{\frac{1}{k}})\varphi(u,v)dudv$$

$$= \left({}_k\varepsilon^{(\gamma+\sigma)}_{\alpha+\zeta,\beta+\eta,\delta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y),$$

which is the desired result. When $\sigma = -\gamma$, the above theorem yields h-Riemann-Liouville double fractional integral of $\varphi(x,y)$. □

By using the above theorem, we can enable to define an inverse to the fractional integral operator and hence we give the following theorem to find the corresponding fractional derivative operator.

Theorem 8. *Let* $\alpha,\beta,\gamma,\delta,\omega_1,\omega_2 \in \mathbb{C}$ *and let* $\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta) > 0$. *For any* $\zeta,\eta \in \mathbb{C}$ *with* $\mathrm{Re}(\zeta),\mathrm{Re}(\eta) > 0$, *then the following left inverse operator of* ${}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}$ *is defined by*

$${}_k\mathcal{D}^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}\ \varphi(x,y) = {}_{k,y}D^{\beta+\eta}_{c^+}\ {}_{k,x}D^{\alpha+\zeta}_{a^+}\ {}_k\varepsilon^{(-\gamma)}_{\zeta,\eta;\omega_1,\omega_2;a^+,c^+}\varphi(x,y).$$

Proof. Consider $f \in L^1((a,b)\times(c,d))$. Then

$$\varphi(x,y) := \left({}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}f\right)(x,y). \tag{4.2}$$

By Theorem 5, the operator ${}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}$ is bounded on the space L^1space. It follows that $\varphi \in L^1$. Now, we contruct the operator ${}_k\varepsilon^{(-\gamma)}_{\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+}$ and we apply it to (4.2) and we get

$$\begin{aligned} {}_k\varepsilon^{(-\gamma)}_{\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+}\varphi(x,y) &= \left({}_k\varepsilon^{(-\gamma)}_{\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+}{}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}f\right)(x,y) \\ &= \left({}_{k,y}I^{\beta+\eta}_{c^+}\ {}_{k,x}I^{\alpha+\zeta}_{a^+}f\right)(x,y). \end{aligned} \tag{4.3}$$

Since $\mathrm{Re}(\alpha+\zeta)>0$ and $\mathrm{Re}(\beta+\eta)>0$, by the Theorem 5, ${}_k\varepsilon^{(-\gamma)}_{\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+}\in L^1$. Applying the double fractional $k-$derivative on both sides of (4.3), we get

$$\left({}_{k,y}D^{\beta+\eta}_{c^+}\,{}_{k,x}D^{\alpha+\zeta}_{a^+}\,{}_k\varepsilon^{(-\gamma)}_{\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y)=\left({}_{k,y}D^{\beta+\eta}_{c^+}\,{}_{k,x}D^{\alpha+\zeta}_{a^+}\,{}_{k,y}I^{\beta+\eta}_{c^+}\,{}_{k,x}I^{\alpha+\zeta}_{a^+}f\right)(x,y)$$
$$=f(x,y).$$

In other words,

$${}_k\mathcal{D}^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}\ \varphi(x,y)=f(x,y).$$

So, we have shown that, ${}_k\mathcal{D}^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}$ is the left-inverse of ${}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}$. □

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