Miskolc Mathematical Notes
Vol. 24 (2023), No. 3, pp. 1469–1476
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.3941

$\oplus$-g-RADICAL SUPPLEMENTED MODULES

CELIL NEBIYEV AND HILAL BAŞAK ÖZDEMIR

Received 03 October, 2021

Abstract. In this work $\oplus$-g-radical supplemented modules are defined and some properties of these modules are investigated. It is proved that the finite direct sum of $\oplus$-g-radical supplemented modules is also $\oplus$-g-radical supplemented. Let M be a $\oplus$-g-radical supplemented R-module. If $\mathrm{Rad}_g(M) \ll_g M$, then M is $\oplus$-g-supplemented. If $\mathrm{Rad}_g(M) \ll M$, then M is $\oplus$-supplemented.

2010 *Mathematics Subject Classification:* 16D10; 16D80

Keywords: essential submodules, small submodules, g-supplemented modules, g-radical supplemented modules

1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R-module. We will denote a submodule N of M by $N \leq M$. Let M be an R-module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small submodule* of M and denoted by $N \ll M$. Let M be an R-module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a *direct summand* of M and it is denoted by $M = N \oplus K$. For any R-module M, we have $M = M \oplus 0$. The intersection of all maximal submodules of M is called the *radical* of M and denoted by $\mathrm{Rad}(M)$. If M have no maximal submodules, then it is defined $\mathrm{Rad}(M) = M$. M is said to be *semilocal* if $M/\mathrm{Rad}(M)$ is semisimple. A submodule N of an R-module M is called an *essential submodule* of M and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $K = 0$ for every $K \leq M$ with $N \cap K = 0$. Let M be an R-module and K be a submodule of M. K is called a *generalized small* (or briefly, *g-small*) submodule of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, then we write $K \ll_g M$ (in [13], it is called an *e-small submodule* of M and denoted by $K \ll_e M$). It is clear that every small submodule is a generalized small submodule but the converse is not true in general. Let M be an R-module. M is called a *hollow module* if every proper submodule of M is small in M. M is called a *generalized hollow* (or briefly, *g-hollow*) *module* if every proper submodule of M is g-small in M. Here it is clear that every hollow module is generalized hollow. The converse of this statement is not always true. M is called a *local module* if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. M is called a *generalized local* (briefly, *g-local*) if M has a large proper essential submodule which contain all proper essential submodules of M or M have no proper essential submodules. Let U and V be submodules of M. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. If every submodule of M has a supplement that is a direct summand in M, then M is called a $\oplus$*-supplemented module*. Let M be an R-module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. M is said to be $\oplus$*-g-supplemented* if every submodule of M has a g-supplement that is a direct summand in M (see [10]). A module M is said to have the *Summand Sum Property* (*SSP*) if the sum of two direct summands of M is again a direct summand of M (see [12, Exercise 39.17(3)]). We say that a module M has $(D3)$ property if $M_1 \cap M_2$ is a direct summand of M for every direct summands M_1 and M_2 of M with $M = M_1 + M_2$ (see [8]). Let M be an R-module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq \mathrm{Rad}(V)$, then V is called a *generalized* (*radical*) *supplement* (briefly, *Rad-supplement*) of U in M. M is said to be *generalized* (*radical*) *supplemented* (briefly, *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. M is said to be *generalized* (*radical*) $\oplus$*-supplemented* (briefly, *Rad-*$\oplus$*-supplemented*) if every submodule of M has a Rad-supplement that is a direct summand in M. The intersection of all essential maximal submodules of an R-module M is called the *generalized radical* of M and denoted by $\mathrm{Rad}_g(M)$ (in [13], it is denoted by $\mathrm{Rad}_e(M)$). If M have no essential maximal submodules, then we denote $\mathrm{Rad}_g(M) = M$. An R-module M is said to be *g-semilocal* if $M/\mathrm{Rad}_g(M)$ is semisimple (see [9]). Let M be an R-module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq \mathrm{Rad}_g(V)$, then V is called a *generalized radical supplement* (or briefly, *g-radical supplement*) of U in M. M is said to be *generalized radical supplemented* (briefly, *g-radical supplemented*) if every submodule of M has a g-radical supplement in M.

More informations about supplemented modules are in [2, 8, 12]. More results about $\oplus$-supplemented modules are in [4, 5, 8]. More details about generalized (radical) supplemented modules are in [11]. More details about generalized (radical) $\oplus$-supplemented modules are in [1, 3]. More informations about g-supplemented modules are in [6]. More informations about g-radical supplemented modules are in [7].

Now we will give some important properties of the generalized radical of any module.

Lemma 1. *Let M be an R-module. The following conditions hold.*

(1) $\mathrm{Rad}_g(M) = \sum_{L \ll_g M} L$.
(2) *$Rm \ll_g M$ for every $m \in \mathrm{Rad}_g(M)$.*
(3) *If $N \leq M$, then $\mathrm{Rad}_g(N) \leq \mathrm{Rad}_g(M)$.*
(4) *If $K, L \leq M$, then $\mathrm{Rad}_g(K) + \mathrm{Rad}_g(L) \leq \mathrm{Rad}_g(K+L)$.*
(5) *Let N be an R-module and $f : M \to N$ be an R-module homomorphism. Then $f(\mathrm{Rad}_g(M)) \leq \mathrm{Rad}_g(N)$.*
(6) *If $K, L \leq M$, then $\frac{\mathrm{Rad}_g(K)+L}{L} \leq \mathrm{Rad}_g\left(\frac{K+L}{L}\right)$.*
(7) *If $M = \bigoplus_{i \in I} M_i$, then $\mathrm{Rad}_g(M) = \bigoplus_{i \in I} \mathrm{Rad}_g(M_i)$.*

Proof. See [7, Lemma 2, Lemma 3 and Lemma 4]. □

2. $\oplus$-g-RADICAL SUPPLEMENTED MODULES

Definition 1. Let M be an R-module. If every submodule of M has a g-radical supplement that is a direct summand of M, then M is called a $\oplus$-generalized radical supplemented (briefly, $\oplus$-g-radical supplemented) module.

Clearly we can see that every $\oplus$-g-supplemented module is $\oplus$-g-radical supplemented, but the converse is not true in general (see Example 1 and Example 2). We also clearly can see that every $\oplus$-supplemented and every g-hollow modules are $\oplus$-g-radical supplemented. Let M be $\oplus$-g-radical supplemented R-module. Then M is g-radical supplemented and by [7, Theorem 1] M is g-semilocal.

Lemma 2. *Let M be an R-module and $M = M_1 \oplus M_2$. If M_1 and M_2 are $\oplus$-g-radical supplemented, then M is also $\oplus$-g-radical supplemented.*

Proof. Let U be any submodule of M. Since M_2 is $\oplus$-g-radical supplemented, $(M_1+U) \cap M_2$ has a g-radical supplement X that is a direct summand of M_2. Since X is a g-radical supplement of $(M_1+U) \cap M_2$ in M_2, $M_2 = (M_1+U) \cap M_2 + X$ and $(M_1+U) \cap X = (M_1+U) \cap M_2 \cap X \leq \mathrm{Rad}_g(X)$. By $M_2 = (M_1+U) \cap M_2 + X$, $M = M_1 \oplus M_2 = M_1 + (M_1+U) \cap M_2 + X = M_1 + U + X$. Since M_1 is $\oplus$-g-radical supplemented, $(U+X) \cap M_1$ has a g-radical supplement Y that is a direct summand of M_1. Since Y is a g-radical supplement of $(U+X) \cap M_1$ in M_1, $M_1 = (U+X) \cap M_1 + Y$ and $(U+X) \cap Y = (U+X) \cap M_1 \cap Y \leq \mathrm{Rad}_g(Y)$. By $M_1 = (U+X) \cap M_1 + Y$, $M = M_1 + U + X = (U+X) \cap M_1 + Y + U + X = U + X + Y$. Since $(M_1+U) \cap X \leq \mathrm{Rad}_g(X)$ and $(U+X) \cap Y \leq \mathrm{Rad}_g(Y)$, by Lemma 1, $U \cap (X+Y) \leq (U+Y) \cap X + (U+X) \cap Y \leq (M_1+U) \cap X + (U+X) \cap Y \leq \mathrm{Rad}_g(X) + \mathrm{Rad}_g(Y) \leq \mathrm{Rad}_g(X+Y)$. Hence $X+Y$ is a g-radical supplement of U in M. Since X is a direct summand of M_2 and Y is a direct summand of M_1, $X+Y$ is a direct summand of $M = M_1 \oplus M_2$. Hence M is $\oplus$-g-radical supplemented. □

Corollary 1. *Let M be an R-module and $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$. If M_i is $\oplus$-g-radical supplemented for every $i = 1, 2, \ldots, n$, then M is also $\oplus$-g-radical supplemented.*

Proof. Clear from Lemma 2. □

Proposition 1. *Let M be a $\oplus$-g-radical supplemented R-module. If $\mathrm{Rad}_g(M) \ll_g M$, then M is $\oplus$-g-supplemented.*

Proof. Let $U \leq M$. Since M is $\oplus$-g-radical supplemented, U has a g-radical supplement V that is a direct summand in M. Here $M = U + V$ and $U \cap V \leq \mathrm{Rad}_g(V)$. By Lemma 1, $\mathrm{Rad}_g(V) \leq \mathrm{Rad}_g(M) \ll_g M$. Since V is a direct summand of M, we can see that $\mathrm{Rad}_g(V) \ll_g V$. Hence $U \cap V \ll_g V$ and V is a g-supplement of U in M. Therefore, M is $\oplus$-g-supplemented. □

Proposition 2. *Let M be a $\oplus$-g-radical supplemented R-module. If $\mathrm{Rad}_g(M) \ll M$, then M is $\oplus$-supplemented.*

Proof. Similar to proof of Proposition 1. □

Lemma 3. *Let $M = M_1 \oplus M_2$ and $X, Y \leq M_2$. Then Y is a g-radical supplement of X in M_2 if and only if Y is a g-radical supplement of $M_1 + X$ in M.*

Proof.

$\Longrightarrow$: Since Y is a g-radical supplement of X in M_2, $M_2 = X + Y$ and $X \cap Y \leq \mathrm{Rad}_g(Y)$. Then $M = M_1 + M_2 = M_1 + X + Y$ and by Modular Law, $(M_1 + X) \cap Y = (M_1 + X) \cap M_2 \cap Y = (M_1 \cap M_2 + X) \cap Y = (0 + X) \cap Y = X \cap Y \leq \mathrm{Rad}_g(Y)$. Hence Y is a g-radical supplement of $M_1 + X$ in M.

$\Longleftarrow$: Since Y is a g-radical supplement of $M_1 + X$ in M, $M = M_1 + X + Y$ and $(M_1 + X) \cap Y \leq \mathrm{Rad}_g(Y)$. Then by Modular Law, $M_2 = M_2 \cap M = M_2 \cap (M_1 + X + Y) = M_1 \cap M_2 + X + Y = 0 + X + Y = X + Y$ and $X \cap Y \leq (M_1 + X) \cap Y \leq \mathrm{Rad}_g(Y)$. Hence Y is a g-radical supplement of X in M_2.

□

Proposition 3. *Let M be a $\oplus$-g-radical supplemented module. If every g-radical supplement submodule in M is a direct summand of M, then every direct summand of M is $\oplus$-g-radical supplemented.*

Proof. Let N be a direct summand of M and $M = N \oplus K$ with $K \leq M$. Since M is g-radical supplemented, by [7, Lemma 9], M/K is g-radical supplemented. By $\frac{M}{K} = \frac{N \oplus K}{K} \cong \frac{N}{N \cap K} = \frac{N}{0} \cong N$, N is also g-radical supplemented. Let $X \leq N$ and Y be a g-radical supplement of X in N. Since $M = N \oplus K$, by Lemma 3, Y is a g-radical supplement of $K + X$ in M. Since every g-radical supplement submodule in M is a direct summand of M, Y is a direct summand of M. By $Y \leq N$, Y is also a direct summand of N. Hence N is $\oplus$-g-radical supplemented. □

Lemma 4. *Let M be a $\oplus$-g-radical supplemented R-module and $K \leq M$. If $\frac{X+K}{K}$ is a direct summand of $\frac{M}{K}$ for every direct summand X of M, then $\frac{M}{K}$ is $\oplus$-g-radical supplemented.*

Proof. Let U/K be any submodule of M/K. Since M is $\oplus$-g-radical supplemented, U has a g-radical supplement X in M that is a direct summand in M. Since X is a g-radical supplement of U in M and $K \leq U$, by [7, Lemma 8], $\frac{X+K}{K}$ is a g-radical supplement of U/K in M/K. Since X is a direct summand of M, by hypothesis, $\frac{X+K}{K}$ is a direct summand of M/K. Hence M/K is $\oplus$-g-radical supplemented. □

Lemma 5. *Let M be a distributive and $\oplus$-g-radical supplemented R-module. Then every factor module of M is $\oplus$-g-radical supplemented.*

Proof. Let $K \leq M$ and X be a direct summand of M. Since X is a direct summand of M, there exists $Y \leq M$ such that $M = X \oplus Y$. Since $M = X \oplus Y$, $\frac{M}{K} = \frac{X+K}{K} + \frac{Y+K}{K}$. Since M is distributive, $(X+K) \cap (Y+K) = K$ and $\frac{X+K}{K} \cap \frac{Y+K}{K} = \frac{(X+K)\cap(Y+K)}{K} = \frac{K}{K} = 0$. Hence $\frac{M}{K} = \frac{X+K}{K} \oplus \frac{Y+K}{K}$ and by Lemma 4, M/K is $\oplus$-g-radical supplemented. □

Corollary 2. *Let M be a distributive and $\oplus$-g-radical supplemented R-module. Then every homomorphic image of M is $\oplus$-g-radical supplemented.*

Proof. Clear from Lemma 5. □

Lemma 6. *Let M be an R-module and K be a direct summand of M. Then* $\mathrm{Rad}_g(K) = K \cap \mathrm{Rad}_g(M)$.

Proof. Since K is a direct summand of M, there exists $T \leq M$ such that $M = K \oplus T$. By Lemma 1, $\mathrm{Rad}_g(M) = \mathrm{Rad}_g(K) \oplus \mathrm{Rad}_g(T)$. Then by Modular Law, $K \cap \mathrm{Rad}_g(M) = K \cap (\mathrm{Rad}_g(K) \oplus \mathrm{Rad}_g(T)) = \mathrm{Rad}_g(K) \oplus K \cap \mathrm{Rad}_g(T) = \mathrm{Rad}_g(K) + 0 = \mathrm{Rad}_g(K)$, as desired. □

Lemma 7. *Let M be a $\oplus$-g-radical supplemented R-module with $(D3)$ property. Then every direct summand of M is $\oplus$-g-radical supplemented.*

Proof. Let K be any direct summand of M. Then there exists $T \leq M$ such that $M = K \oplus T$. Let $U \leq K$. Since M is $\oplus$-g-radical supplemented, U has a g-radical supplement X that is a direct summand in M. Here $M = U + X$ and $U \cap X \leq \mathrm{Rad}_g(X)$. Since $U \leq K$, $M = U + X = K + X$ and since M has $(D3)$ property, $K \cap X$ is a direct summand of M. Then there exists $Y \leq M$ such that $M = (K \cap X) \oplus Y$. Here $K = (K \cap X) \oplus (K \cap Y)$. Since $M = U + X$ and $U \leq K$, by Modular Law, $K = U + (K \cap X)$. Since $U \cap X \leq \mathrm{Rad}_g(X) \leq \mathrm{Rad}_g(M)$, $U \cap K \cap X \leq K \cap X \cap \mathrm{Rad}_g(M)$. Since $K \cap X$ is a direct summand of M, by Lemma 6, $U \cap K \cap X \leq K \cap X \cap \mathrm{Rad}_g(M) = \mathrm{Rad}_g(K \cap X)$. Hence $K \cap X$ is a g-radical supplement of U that is a direct summand in K. Therefore, K is $\oplus$-g-radical supplemented. □

Corollary 3. *Let M be a $\oplus$-g-radical supplemented R-module with $(D3)$ property. Then M/X is $\oplus$-g-radical supplemented for every direct summand X of M.*

Proof. Let X be any direct summand of M. Then there exists $Y \leq M$ such that $M = X \oplus Y$. By Lemma 7, Y is $\oplus$-g-radical supplemented. Then by $\frac{M}{X} = \frac{X+Y}{X} \cong \frac{Y}{X \cap Y} = \frac{Y}{0} \cong Y$, M/X is also $\oplus$-g-radical supplemented. □

Corollary 4. *Let M be a $\oplus$-g-radical supplemented R-module with $(D3)$ property and $f\colon M\to N$ be an R-module epimorphism with N is an R-module and $\operatorname{Ker}(f)$ is a direct summand of M. Then N is $\oplus$-g-radical supplemented.*

Proof. Clear from Corollary 3, since $M/\operatorname{Ker}(f)\cong \operatorname{Im}(f)=N$. □

Lemma 8. *Let M be a $\oplus$-g-radical supplemented R-module, $K\leq M$ and $K=(K\cap M_1)\oplus(K\cap M_2)$ for every $M_1,M_2\leq M$ with $M=M_1\oplus M_2$. Then M/K is $\oplus$-g-radical supplemented.*

Proof. Let $U/K\leq M/K$. Since M is $\oplus$-g-radical supplemented, U has a g-radical supplement V that is a direct summand in M. Here there exists $X\leq M$ such that $M=V\oplus X$. By hypothesis, $K=(K\cap V)\oplus(K\cap X)$. Since V is a g-radical supplement of U in M and $K\leq U$, by [7, Lemma 8], $\frac{V+K}{K}$ is a g-radical supplement of U/K in M/K. Since $M=V\oplus X$, $\frac{M}{K}=\frac{V+K}{K}+\frac{X+K}{K}$. Here $\frac{V+K}{K}\cap\frac{X+K}{K}=\frac{(V+K)\cap(X+K)}{K}=\frac{(V+K)\cap X+K}{K}=\frac{(V+K\cap V+K\cap X)\cap X+K}{K}=\frac{(V+K\cap X)\cap X+K}{K}=\frac{V\cap X+K\cap X+K}{K}=\frac{0+K}{K}=\frac{K}{K}=0$. Hence $\frac{M}{K}=\frac{V+K}{K}\oplus\frac{X+K}{K}$. Thus M/K is $\oplus$-g-radical supplemented. □

Corollary 5. *Let M be a $\oplus$-g-radical supplemented R-module, $f\colon M\to N$ be an R-module epimorphism with N be an R-module and $\operatorname{Ker}(f)=(\operatorname{Ker}(f)\cap M_1)\oplus(\operatorname{Ker}(f)\cap M_2)$ for every $M_1,M_2\leq M$ with $M=M_1\oplus M_2$. Then N is $\oplus$-g-radical supplemented.*

Proof. Clear from Lemma 8, since $M/\operatorname{Ker}(f)\cong\operatorname{Im}(f)=N$. □

Lemma 9. *Let M be a $\oplus$-g-radical supplemented R-module with SSP property. Then M/K is $\oplus$-g-radical supplemented for every direct summand K of M.*

Proof. Let K be any direct summand of M and $U/K\leq M/K$. Since M is $\oplus$-g-radical supplemented, U has a g-radical supplement V in M such that V is a direct summand of M. By [7, Lemma 8], $\frac{V+K}{K}$ is a g-radical supplement of U/K in M/K. Since K and V are direct summands of M and M has *SSP* property, $K+V$ is also a direct summand of M. Hence there exists $T\leq M$ such that $M=(K+V)\oplus T$. Since $M=(K+V)\oplus T$, $\frac{M}{K}=\frac{K+V+T}{K}=\frac{V+K}{K}+\frac{T+K}{K}$. Since $(V+K)\cap T=0$, $\frac{V+K}{K}\cap\frac{T+K}{K}=\frac{(V+K)\cap(T+K)}{K}=\frac{(V+K)\cap T+K}{K}=\frac{0+K}{K}=0$. Hence $\frac{M}{K}=\frac{V+K}{K}\oplus\frac{T+K}{K}$ and M/K is $\oplus$-g-radical supplemented. □

Corollary 6. *Let M be a $\oplus$-g-radical supplemented R-module with SSP property. Then every direct summand of M is $\oplus$-g-radical supplemented.*

Proof. Let T be any direct summand of M. Then there exists a submodule K of M such that $M=T\oplus K$. By Lemma 9, M/K is $\oplus$-g-radical supplemented. Since $\frac{M}{K}=\frac{T+K}{K}\cong\frac{T}{T\cap K}=\frac{T}{0}\cong T$, T is also $\oplus$-g-radical supplemented. □

Remark 1. Let M be an R-module which has only four proper submodules 0, A, B, C with $C\leq A$, $C\leq B$, $A\nleq B$ and $B\nleq A$. Then M is g-radical supplemented but not $\oplus$-g-radical supplemented.

Example 1. Consider the $\mathbb{Z}$-module $\mathbb{Q}$. Since $\text{Rad}_g(\mathbb{Q}) = \text{Rad}(\mathbb{Q}) = \mathbb{Q}$, ${}_{\mathbb{Z}}\mathbb{Q}$ is $\oplus$-g-radical supplemented. But, since ${}_{\mathbb{Z}}\mathbb{Q}$ is not supplemented and every nonzero submodule of ${}_{\mathbb{Z}}\mathbb{Q}$ is essential in ${}_{\mathbb{Z}}\mathbb{Q}$, ${}_{\mathbb{Z}}\mathbb{Q}$ is not g-supplemented and hence ${}_{\mathbb{Z}}\mathbb{Q}$ is not $\oplus$-g-supplemented.

Example 2. Consider the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ for a prime p. It is easy to check that $\text{Rad}_g(\mathbb{Z}_{p^2}) \neq \mathbb{Z}_{p^2}$. By Lemma 1, $\text{Rad}_g\left(\mathbb{Q} \oplus \mathbb{Z}_{p^2}\right) = \text{Rad}_g(\mathbb{Q}) \oplus \text{Rad}_g(\mathbb{Z}_{p^2}) \neq \mathbb{Q} \oplus \mathbb{Z}_{p^2}$. Since $\mathbb{Q}$ and $\mathbb{Z}_{p^2}$ are $\oplus$-g-radical supplemented, by Lemma 2, $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ is $\oplus$-g-radical supplemented. But $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ is not $\oplus$-g-supplemented.

REFERENCES

[1] H. Çalışıcı and E. Türkmen, "Generalized $\oplus$-supplemented modules." *Algebra Discrete Math.*, vol. 10, no. 2, pp. 10–18, 2010.
[2] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting modules. Supplements and projectivity in module theory.* Basel: Birkhäuser, 2006.
[3] Şule Ecevit, M. T. Koşan, and R. Tribak, "Rad-$\oplus$-supplemented modules and cofinitely Rad-$\oplus$-supplemented modules." *Algebra Colloq.*, vol. 19, no. 4, pp. 637–648, 2012, doi: 10.1142/S1005386712000508.
[4] A. Harmanci, D. Keskin, and P. F. Smith, "On $\oplus$-supplemented modules," *Acta Math. Hung.*, vol. 83, no. 1-2, pp. 161–169, 1999, doi: 10.1023/A:1006627906283.
[5] D. Keskin, P. F. Smith, and W. Xue, "Rings whose modules are $\oplus$-supplemented," *J. Algebra*, vol. 218, no. 2, pp. 470–487, 1999, doi: 10.1006/jabr.1998.7830.
[6] B. Koşar, C. Nebiyev, and N. Sökmez, "g-supplemented modules," *Ukr. Math. J.*, vol. 67, no. 6, pp. 975–980, 2015, doi: 10.1007/s11253-015-1127-8.
[7] B. Koşar, C. Nebiyev, and A. Pekin, "A generalization of g-supplemented modules," *Miskolc Math. Notes*, vol. 20, no. 1, pp. 345–352, 2019, doi: 10.18514/MMN.2019.2586.
[8] S. H. Mohamed and B. J. Müller, *Continuous and discrete modules.* Cambridge etc.: Cambridge University Press, 1990, vol. 147.
[9] C. Nebiyev and H. H. Ökten, "Weakly g-supplemented modules," *Eur. J. Pure Appl. Math.*, vol. 10, no. 3, pp. 521–528, 2017.
[10] C. Nebiyev and H. H. Ökten, "$\oplus$-g-supplemented modules," in *The International Symposium: New Trends in Rings and Modules I.* Gebze, Kocaeli, Turkey, 7 2018.
[11] Y. Wang and N. Ding, "Generalized supplemented modules." *Taiwanese J. Math.*, vol. 10, no. 6, pp. 1589–1601, 2006, doi: 10.11650/twjm/1500404577.
[12] R. Wisbauer, *Foundations of module and ring theory. A handbook for study and research.* Philadelphia etc.: Gordon and Breach Science Publishers, 1991, vol. 3.
[13] D. X. Zhou and X. R. Zhang, "Small-essential submodules and Morita duality." *Southeast Asian Bull. Math.*, vol. 35, no. 6, pp. 1051–1062, 2011.

Authors' addresses

Celil Nebiyev
(**Corresponding author**) Ondokuz Mayıs University, Department of Mathematics, Kurupelit, Atakum Samsun, Turkey
E-mail address: celilnebiyev@hotmail.com

Hilal Başak Özdemir
Ondokuz Mayıs University, Department of Mathematics, Kurupelit, Atakum Samsun, Turkey
E-mail address: hilal-basak@windowslive.com