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NON-EXISTENCE OF SOLUTION OF HARAUX-WEISSLER EQUATION ON A STRICTLY STARSHPED DOMAIN

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Abstract. Here, we prove the Haraux-Weissler equation has no solution on a domain which is strictly starshped with respect to 0. Finally, we present some questions.

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1. INTRODUCTION

A Porous Medium Equation (in short PME) is the equation

$$u_t = \Delta(u^m),$$

where $u = u(x,t)$ is a scalar function and m is a constant larger than 1. The space variable x takes values in $R^d, d \geq 1$, while $t \in R$. Physical considerations lead to the restriction $u \geq 0$, which is mathematically convenient and currently followed, but not essential.

The PME is an example of a nonlinear evolution equation, formally of parabolic type and there are a number application in boundary layer theory mathematical biology, lubrication, water infiltration, heat radiation in plasmas, processes involving diffusion or heat transfer (such as description of the flow of an isentropic gas through a porous medium) and other fields (see [1, 2, 5, 6, 14–17, 19, 20]).

Consider a quasilinear PME with a source term

$$u_t - \Delta(|u|^{m-1}u) = |u|^{p-1}u \ \text{ in } \mathbb{R}^N \times (0,T),$$

where the parameters are taken as $m > 0$ and $p > 1$. In studying this equation, the self-similar solution $u(x,t) = t^{-\alpha}U(r)$, where $r = |x|t^{-\beta}$, are of interest, as usually describe the large time behaviour of solutions of the Cauchy problem with general initial data (see [12]). Concerning this structure, Haraxu and Weissler [10] proposed

$$(|U|^{m-1}U)'' + \frac{N-1}{r}(|U|^{m-1}U)' + \beta rU' + \alpha U + (|U|^{p-1}U) = 0, \text{ in } \mathbb{R}^+, \qquad (1.1)$$

for the semilinear case $m = 1$, where $\alpha, \beta > 0, \alpha(m-1) + 2\beta = 1, (|U|^{m-1}U)(0) = 0, U(0) = a > 0$. In fact, if the equation originates from (PME), the parameters α, β have to fulfill the additional condition $\alpha p = p+1$ and are thus given by

$$\alpha = \frac{1}{p-1}, \ \beta = \frac{p-m}{2(p-1)}.$$

Another aspect of problem (1.1) is that it can be regarded as the equation satisfied by radial solutions of a quasilinear elliptic equation with a gradient term

$$\Delta(|u|^{m-1}u) + \beta x \cdot \nabla u + \alpha u + |u|^{p-1}u = 0 \ \text{ in } \mathbb{R}^N.$$

This equation was studied by Hirose [11], Chipot et al. [4] and Serrin et al. [18].

Haraxu and Weissler [10] studied the equation

$$\Delta u + \frac{1}{2} x \cdot \nabla u + \frac{1}{p-1} u + |u|^{p-1}u = 0 qquad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $m = 1$ for the self-similar solutions of the semilinear heat equation

$$\partial_t w = \Delta w + w^p. \tag{1.3}$$

Equation (1.3) has a special scaling invariance in the sense that w is a solution if and only if w_λ, defined by

$$w_\lambda(x,t) = \lambda^{\frac{2}{p-1}} w(\lambda x, \lambda^2 t),$$

is a solution for some (equivalently, all) $\lambda > 0$.

Definition 1. A solution w is said to self-similar if $w_\lambda = w$ for all $\lambda > 0$.

Remark 1. Notice that w is a self-similar solution to (1.3) iff $w(x,t) = t^{-\frac{1}{p-1}} u(\frac{x}{\sqrt{t}})$, where u satisfies (1.2).

Also, Kavian et al. [13] study the self-similar solutions to the nonlinear Schrödinger equation

$$iu_t + \Delta u + \varepsilon |u|^\alpha u = 0, \tag{1.4}$$

where $u = u(t,x)$ is a complex-valued function of $t \in \mathbb{R}$ (or a subset of $\mathbb{R}$) and $x \in \mathbb{R}^N$, α is a positive real number, and $\varepsilon := \pm 1$. Self-similar solution of (1.4) have played an important role in the study of the blow-up behavior of nonglobal solutions to (1.4) with $\varepsilon = 1$.

Also, the nonlinear Schrödinger equation (1.4) with the critical power $\alpha = \frac{4}{N}$, that is,

$$iu_t + \Delta u + \varepsilon |u|^{\frac{4}{N}} u = 0, \tag{1.5}$$

admits a special conservation law, called the pseudo-conformal conservation law, and is invariant under a corresponding transformation.

Finally, in [21], the existence of rapidly decaying radial solution of equation (1.2) when $(N-2)p \leq N$ is proved. Also, Fukuizumi et al. [9] gave a sufficient condition that non-radial H^1-solutions to the Haraux-Weissler equation should belong to the weighted Sobolev space $H^1_\rho(\mathbb{R}^N)$, where ρ is the weight function $exp(\frac{|x|^2}{4})$.

The main statement of this paper is to prove the existence of positive singular solutions of Haraux-Weissler equation

$$\begin{cases} \Delta u + \frac{2-N}{2} x \cdot \nabla u = 0 & \text{in} \quad \Omega, \\ x \cdot \nabla u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.6}$$

where Ω is a unite ball in $\mathbb{R}^N$ which is strictly starshaped with respect to zero, $N \geq 3$

2. Nonexistence result

Due to prove the nonexistence result, we need to recall some definitions and facts ([3, 8]) as follows.

Definition 2. We say that an open subset Ω of $\mathbb{R}^N$ is starshaped with respect to a point $y \in \Omega$ if for every $x \in \overline{\Omega}$, the segment joining y to x, namely the set

$$\{\lambda x + (1-\lambda)y : \lambda \in [0,1]\}$$

is entirely contained in $\overline{\Omega}$.

Definition 3. We say that an open subset Ω of $\mathbb{R}^N$ is strictly starshaped with respect to a point $y \in \Omega$ if for every $x \in \overline{\Omega}$, the segment joining y to x, namely the set

$$\{\lambda x + (1-\lambda)y : \lambda \in [0,1)\}$$

is entirely contained in $\overline{\Omega}$.

Remark 2. If $x \in \overline{\Omega}$ then $\lambda x \in \Omega$ for $\lambda \in [0,1)$, when $0 \in \Omega$.

Lemma 1. *([3, 8]) Let $\Omega \subset \mathbb{R}^N$ be smooth and strictly starshaped with respect to 0 and $\nu(x)$ denote the outward normal to $\partial\Omega$ at x. Then $\nu(x) \cdot x > 0$ for $x \in \partial\Omega$.*

Now, we can present the main result by the the following theorem.

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be unit ball which is smooth and strictly starshaped with respect to 0 and $N \geq 3$. Then the problem (1.6) has no solutions in $H^1_0(\Omega)$.*

Proof. We prove the theorem by contradiction. Assume the problem (1.6) has a solution u in $H^1_0(\Omega)$. This mean that $x \cdot \nabla u > 0$ in Ω. Now we multiply the equation in (1.6) by $x \cdot \nabla u$.

$$-\int_\Omega \Delta u x \cdot \nabla u \, dx = \frac{2-N}{2} \int_\Omega |x|^2 |\nabla u|^2 \, dx. \tag{2.1}$$

Notice that the left hand side by the Green's formula is

$$-\int_\Omega \Delta u x \cdot \nabla u \, dx = \int_\Omega \nabla u \cdot (x \cdot \nabla u) \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \nabla u \cdot x \, d\sigma. \tag{2.2}$$

Since

$$\begin{aligned}\frac{\partial}{\partial x_j}(\nabla u\cdot x)=&\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{N}\frac{\partial u}{\partial x_i}x_i\right)=\sum_{i=1}^{N}\left(\frac{\partial^2 u}{\partial x_i\partial x_j}x_i+\frac{\partial u}{\partial x_i}\delta_{ij}\right)\\=&\sum_{i=1}^{N}\frac{\partial^2 u}{\partial x_i\partial x_j}x_i+\frac{\partial u}{\partial x_j}.\end{aligned}$$

This means that

$$\begin{aligned}\nabla u\cdot\nabla(x\cdot\nabla u)=&\frac{1}{2}\nabla(|\nabla u|^2)\cdot x+|\nabla u|^2\\=&\frac{1}{4}\nabla(|\nabla u|^2)\cdot\nabla\left(|x|^2\right)+|\nabla u|^2.\end{aligned}$$

By the Green's formula we get

$$\begin{aligned}&\int_\Omega\nabla u\cdot\nabla(x\cdot\nabla u)\,dx\\&=\int_\Omega\frac{1}{4}\nabla(|\nabla u|^2)\cdot\nabla\left(|x|^2\right)\,dx+\int_\Omega|\nabla u|^2\,dx\\&=\frac{1}{4}\int_\Omega|\nabla u|^2\frac{\partial}{\partial\nu}\left(|x|^2\right)\,d\sigma-\frac{N}{2}\int_\Omega|\nabla u|^2\,dx+\int_\Omega|\nabla u|^2\,dx\\&=\frac{2-N}{2}\int_\Omega|\nabla u|^2\,dx+\frac{1}{2}\int_{\partial\Omega}|\nabla u|^2\nu(x)\cdot x\,d\sigma.\end{aligned}$$

Thus by (2.2) we get

$$\begin{aligned}&-\int_\Omega\Delta u\,x\cdot\nabla u\,dx\\&=\frac{2-N}{2}\int_\Omega|\nabla u|^2\,dx+\frac{1}{2}\int_{\partial\Omega}|\nabla u|^2\nu(x)\cdot x\,d\sigma-\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\nabla u(x)\cdot x\,d\sigma\\&=\frac{2-N}{2}\int_\Omega|\nabla u|^2\,dx-\frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)^2\nu(x)\cdot x\,d\sigma.\end{aligned}\tag{2.3}$$

Now considering (2.3) and (2.1) we have

$$\frac{2-N}{2}\int_\Omega|\nabla u|^2\,dx-\frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)^2\nu(x)\cdot x\,d\sigma=\frac{2-N}{2}\int_\Omega|x|^2\,|\nabla u|^2\,dx$$

or

$$\frac{2-N}{2}\int_\Omega(1-|x|^2)|\nabla u|^2\,dx=\frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)^2\nu(x)\cdot x\,d\sigma.$$

Since $x\in\Omega$ which is the unit ball, thus the left hand side of the above equation is negative for $N\geq 3$ and the right hand side is positive (Since Ω is strictly starshaped with respect to 0, we have $\nu(x)\cdot x > 0$ everywhere on $\partial\Omega$). This means that

$$\frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u}{\partial \nu}\right)^2 \nu(x)\cdot x\, d\sigma = 0$$

which implies that

$$\frac{2-N}{2}\int_{\Omega} x\cdot\nabla u\, dx = -\int_{\Omega}\Delta u\, dx = -\int_{\partial\Omega}\frac{\partial u}{\partial v}\, d\sigma = 0.$$

Since $x\cdot\nabla u$ is positive in Ω, this is impossible. □

Remark 3. If Ω is a unit ball in $\mathbb{R}^N$ which is strictly convex, Theorem 1 remains true. Because every strictly convex set is strictly starshaped set.

3. Some questions

Here, we present some questions as follow.

Question 1. Is it possible to prove that the equation

$$\begin{cases} \Delta u + \frac{1}{2}x\cdot\nabla u + \frac{1}{p-1}u + u^p = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

has no solution, where $\Omega\subset\mathbb{R}^N$ is a strictly starshaped with respect to 0?

Question 2. Is it possible to prove that the equation

$$\begin{cases} \Delta u + a(x)\cdot\nabla u + V(x)u + u^p = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.2}$$

has no solution, where $\Omega\subset\mathbb{R}^N$ is a strictly starshaped with respect to 0, $a(x)$ is a smooth vector field and $V(x)$ is a smooth potential?

Question 3. Is it possible to prove that the equation

$$\begin{cases} \Delta u + a(x)\cdot\nabla u + V(x)u + u^{p(x)} = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.3}$$

has no solution, where $\Omega\subset\mathbb{R}^N$ is a strictly starshaped with respect to 0, $a(x)$ is a smooth vector field, $V(x)$ is a smooth potential and $p(x)$ is a variable exponent?

Question 4. Is it possible to prove that the equation

$$\begin{cases} \Delta u + \frac{2-N}{2}x\cdot\nabla v = 0 & \text{in} \quad \Omega, \\ \Delta v + \frac{2-N}{2}x\cdot\nabla u = 0 & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.4}$$

has no solution, where $\Omega\subset\mathbb{R}^N$ is a strictly starshaped with respect to 0?

The Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ\ )$ is the space $\mathbb{R}^{2n+1}$ with the noncommutative law of product

$$(x,y,z)\circ(x',y',z') = (x+x', y+y', z+z'+2(\langle y,x'\rangle - \langle x,y'\rangle)),$$

where $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$, $z_1, z_2 \in \mathbb{R}$ and $\langle\ ,\ \rangle$ denotes the standard inner product in $\mathbb{R}^n$. This operation endows $\mathbb{H}^n$ with the structure of a Lie group. The Lie algebra of $\mathbb{H}^n$ is generated by the left-invariant vector fields

$$Z = \frac{\partial}{\partial z},\ \ X_i = \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial z},\ \ Y_i = \frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial z},\ \ i = 1,2,3,\cdots,n.$$

These generators satisfy the noncommutative formula

$$[X_i, Y_j] = -4\delta_{ij}Z,\quad [X_i, X_j] = [Y_i, Y_j] = [X_i, Z] = [Y_i, Z] = 0.$$

The Heisenberg gradient and the Kohn-Laplacian (the Heisenberg Laplacian) operator on $\mathbb{H}^n$ are given by

$$\nabla_{\mathbb{H}^n} = (X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n),$$

and

$$\Delta_{\mathbb{H}^n} = \sum_{i=1}^{n} X_i^2 + Y_i^2,$$

respectively (see [7] for more details).

Question 5. The quasilinear PME with a source term in the Heisenberg group $\mathbb{H}^n$ is

$$u_t - \Delta_{\mathbb{H}^n}(|u|^{m-1}u) = |u|^{p-1}u \qquad \text{in } \mathbb{R}^{2n+1}\times(0,T), \tag{3.5}$$

where the parameters are taken as $m > 0$ and $p > 1$.

Does the equation (3.5) has a self-similar solution in the Heisenberg group $\mathbb{H}^n$?

In the case $m = 1$, i.e. the equation

$$u_t - \Delta_{\mathbb{H}^n} u = |u|^{p-1}u \qquad \text{in } \mathbb{R}^{2n+1}\times(0,T), \tag{3.6}$$

may be regarded as a Haraxu-Weissler equatuion in the Heisenberg group setting. The existence of self-similar solutions of the parabolic equation (3.6) can be studied.

REFERENCES

[1] A. M. Alghamdi, S. Gala, C. Qian, and M. A. Ragusa, "The anisotropic integrability logarithmic regularity criterion for the 3d MHD equations," *Electron Res. Arch.*, vol. 28, no. 1, pp. 183–193, 2020, doi: 10.3934/era.2020012.

[2] F. Atkinson and L. Peletier, "Sur les solutions radiales de l'equation $\Delta u + \frac{1}{2}x\cdot\nabla u + \frac{\lambda}{2}u + |u^{p-1}u = 0$." *Comptes Rendus de l'Acadèmie des Sciences - Series I - Mathematics*, vol. 302, pp. 99–101, 1986.

[3] M. Badiale and E. Serra, *Semilinear elliptic equations for beginners. Existence results via the variational approach*, ser. Universitext. London: Springer, 2011. doi: 10.1007/978-0-85729-227-8.

[4] M. Chipot and F. Weissler, *On the elliptic problem $\Delta u - |\nabla u|^q + \lambda u^p = 0$, in: Nonl. Diff. Eq. and Their Equilibrium States, W. M. Ni, L. A. Peletier, J. Serrin (eds.).* New York: Springer Verlag, 1998.

[5] C. Dohmen, "Existence of fast decaying solutions to a Haraux–Weissler equation with a prescribed number of zeros." *Hokkaido University Preprint Series in Mathematic*, vol. 240, pp. 1–12, 1994.

[6] C. Dohmen and M. Hirose, "Structure of positive radial solutions to the Haraux-Weissler equation," *Nonlinear Anal., Theory Methods Appl.*, vol. 33, no. 1, pp. 51–69, 1998, doi: 10.1016/S0362-546X(97)00542-7.

[7] A. Domokos, "On the regularity of p-harmonic functions in the Heisenberg group." *Doctoral Dissertation, University of Pittsburgh*, 2004.

[8] L. C. Evans, *Partial Differential Equations, Graduate Studies in Mathematics. Vol 19*. American Mathematical Society, 1998.

[9] R. Fukuizumi and T. Ozawa, "On a decay property of solutions to the Haraux-Weissler equation," *J. Differ. Equations*, vol. 221, no. 1, pp. 134–142, 2006, doi: 10.1016/j.jde.2005.02.014.

[10] A. Haraux and F. B. Weissler, "Non-uniqueness for a semilinear initial value problem," *Indiana Univ. Math. J.*, vol. 31, pp. 167–189, 1982, doi: 10.1512/iumj.1982.31.31016.

[11] M. Hirose, "Structure of positive radial solutions to semilinear elliptic PDE with a gradient term." *Funkcial. Ekvac.*, vol. 39, pp. 323–345, 1996.

[12] S. Kamin and J. L. Vazquez, "Asymptotic behaviour of solutions of the porous medium equation with changing sign," *SIAM J. Math. Anal.*, vol. 22, no. 1, pp. 34–45, 1991, doi: 10.1137/0522003.

[13] O. Kavian and F. B. Weissler, "Self-similar solutions of the pseudo-conformally invariant nonlinear Schrödinger equation," *Mich. Math. J.*, vol. 41, no. 1, pp. 151–173, 1994, doi: 10.1307/mmj/1029004922.

[14] G. Mingione and V. Rădulescu, "Recent developments in problems with nonstandard growth and nonuniform ellipticity," *J. Math. Anal. Appl.*, vol. 501, no. 1, p. 41, 2021, id/No 125197, doi: 10.1016/j.jmaa.2021.125197.

[15] M. A. Ragusa, "On weak solutions of ultraparabolic equations." *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 47, no. 1, pp. 503–511, 2001, doi: 10.1016/S0362-546X(01)00195-X.

[16] M. A. Ragusa and A. Tachikawa, "Boundary regularity of minimizers of $p(x)$-energy functionals," *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, vol. 33, no. 2, pp. 451–476, 2016, doi: 10.1016/j.anihpc.2014.11.003.

[17] M. A. Ragusa and A. Tachikawa, "Correction and addendum to "Boundary regularity of minimizers of $p(x)$-energy functionals"," *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, vol. 34, no. 6, pp. 1633–1637, 2017, doi: 10.1016/j.anihpc.2017.09.004.

[18] J. Serrin and H. Zou, "Existence and non-existence results for ground states of quasilinear elliptic equations," *Arch. Ration. Mech. Anal.*, vol. 121, no. 2, pp. 101–130, 1992, doi: 10.1007/BF00375415.

[19] N. Shioji and K. Watanabe, "Uniqueness and nondegeneracy of positive radial solutions of $\mathrm{div}(\rho\nabla u) + \rho(-gu + hu^p) = 0$," *Calc. Var. Partial Differ. Equ.*, vol. 55, no. 2, p. 42, 2016, id/No 32, doi: 10.1007/s00526-016-0970-2.

[20] J. L. Vazquez, *An introduction to the mathematical theory of the porous medium equation*. Dordrecht: Kluwer Academic Publishers, 1992.

[21] E. Yanagida, "Uniqueness of rapidly decaying solutions to the Haraux-Weissler equation," *J. Differ. Equations*, vol. 127, no. 2, pp. 561–570, 1996, doi: 10.1006/jdeq.1996.0083.

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