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NUMERICAL RADIUS INEQUALITIES OF OPERATOR MATRICES FROM A NEW NORM ON $\mathcal{B}(\mathcal{H})$

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Abstract. This paper is a continuation of a recent work on a new norm, christened the (α,β)-norm, on the space of bounded linear operators on a Hilbert space. We obtain some upper bounds for the said norm of $n\times n$ operator matrices. As an application of the present study, we estimate bounds for the numerical radius and the usual operator norm of $n\times n$ operator matrices, which generalize the existing ones.

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1. Introduction

The purpose of the present article is to study the bounds for the newly introduced [15] (α,β)-norm of $n\times n$ operator matrices, from which we obtain bounds for the numerical radius of $n\times n$ operator matrices. Let us first introduce the following notations and terminologies to be used throughout the article.

Let $\mathcal{H}_i,\mathcal{H}_j$ be two complex Hilbert spaces with usual inner product $\langle .,.\rangle$ and let $\mathcal{B}(\mathcal{H}_i,\mathcal{H}_j)$ denote the space of all bounded linear operators from $\mathcal{H}_i$ to $\mathcal{H}_j$. If $\mathcal{H}_i=\mathcal{H}_j=\mathcal{H}$ then we write $\mathcal{B}(\mathcal{H},\mathcal{H})=\mathcal{B}(\mathcal{H})$. For $T\in\mathcal{B}(\mathcal{H})$, we write $Re(T)$ and $Im(T)$ for the real part of T and the imaginary part of T, respectively, i.e., $Re(T)=\frac{T+T^*}{2}$ and $Im(T)=\frac{T-T^*}{2i}$. Let T^* denote the adjoint of T and let $|T|$ be the positive operator $(T^*T)^{\frac{1}{2}}$. Let $\sigma(T)$ denote the spectrum of T. The spectral radius of T, denoted by $r(T)$, is defined by $r(T)=\sup\{|\lambda|:\lambda\in\sigma(T)\}$. The numerical range of T, denoted by $W(T)$, is defined as $W(T)=\{\langle Tx,x\rangle : x\in\mathcal{H},\|x\|=1\}$. The usual operator norm and the numerical radius of T, denoted by $\|T\|$ and $w(T)$, respectively, are defined as $\|T\|=\sup\{\|Tx\|:x\in\mathcal{H},\|x\|=1\}$ and $w(T)=\sup\{|c|:c\in W(T)\}$. Let M_T denote the usual operator norm attainment set of T, i.e., $M_T=\{x\in\mathcal{H}:\|Tx\|=\|T\|,\|x\|=1\}$.

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It is well-known that the numerical radius defines a norm on $\mathcal{B}(\mathcal{H})$ and is equivalent to the usual operator norm, satisfying that for $T \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|.$$

The study of the numerical range of an operator and the associated numerical radius inequalities are an important area of research in operator theory and it has attracted many mathematicians [1, 3, 5–7, 12] over the years. Recently, some generalizations for the concept of numerical radius have been introduced in [2, 4, 14, 16, 18]. One of these generalizations is the A-numerical radius of an operator $T \in \mathcal{B}(\mathcal{H})$ defined by $w_A(T) = \sup\{|\langle ATx,x\rangle| : x \in \mathcal{H}, \langle Ax,x\rangle = 1\}$, see, e.g., [8, 10, 17]. Here, A is a positive bounded linear operator on $\mathcal{H}$. With an aim to develop better upper and lower bounds for the numerical radius, a new norm named as the (α,β)-norm, was introduced on $\mathcal{B}(\mathcal{H})$ in [15]. For $T \in \mathcal{B}(\mathcal{H})$, the (α,β)-norm of T, denoted by $\|T\|_{\alpha,\beta}$, is defined as:

$$\|T\|_{\alpha,\beta} = \sup\left\{\sqrt{\alpha|\langle Tx,x\rangle|^2 + \beta\|Tx\|^2} : x \in \mathcal{H}, \|x\| = 1\right\},$$

where α,β are real positive constants with $(\alpha,\beta) \neq (0,0)$. We note that if $\alpha = 1, \beta = 0$ then $\|T\|_{\alpha,\beta} = w(T)$, and if $\alpha = 0, \beta = 1$ then $\|T\|_{\alpha,\beta} = \|T\|$. Also, if we consider $\alpha = \beta = 1$, then we have the modified Davis-Wielandt radius of T, that is, $\|T\|_{\alpha,\beta} = dw^*(T)$, (see [9]). In this article, we consider $\alpha+\beta = 1$, i.e., $\beta = 1-\alpha$ and explore the α-norm of $n \times n$ operator matrices, where the α-norm of T is defined as:

$$\|T\|_{\alpha} = \sup\left\{\sqrt{\alpha|\langle Tx,x\rangle|^2 + (1-\alpha)\|Tx\|^2} : x \in \mathcal{H}, \|x\| = 1\right\}.$$

We compute the exact value of the α-norm of 2×2 operator matrices in $\mathcal{B}(\mathcal{H}\oplus\mathcal{H})$ of the form $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$, where $X \in \mathcal{B}(\mathcal{H})$. We obtain some upper bounds for the α-norm of $n \times n$ operator matrices, which generalize the existing numerical radius inequalities and the usual operator norm inequalities of $n \times n$ operator matrices. As an application of our results, we estimate new upper bounds for the numerical radius and the usual operator norm of $n \times n$ operator matrices.

2. Main results

We begin this section with the following proposition, the proof of which follows from the weakly unitarily invariant property of the α-norm, i.e., for $T \in \mathcal{B}(\mathcal{H})$, $\|U^*TU\|_\alpha = \|T\|_\alpha$ for every unitary operator $U \in \mathcal{B}(\mathcal{H})$ (see [15, Prop. 2.6]).

Proposition 1. *Let $A,B \in \mathcal{B}(\mathcal{H})$. Then the following results hold:*

(a) $\left\|\begin{pmatrix} 0 & A \\ e^{i\theta}B & 0 \end{pmatrix}\right\|_\alpha = \left\|\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right\|_\alpha$ *for every* $\theta \in \mathbb{R}$.

(b) $\left\| \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \right\|_\alpha = \left\| \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \right\|_\alpha.$

(c) $\left\| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\|_\alpha = \left\| \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \right\|_\alpha.$

(d) $\left\| \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\|_\alpha = \left\| \begin{pmatrix} A-B & 0 \\ 0 & A+B \end{pmatrix} \right\|_\alpha.$

Next, we estimate upper and lower bounds for the α-norm of 2×2 operator matrices in $\mathcal{B}(\mathcal{H}\oplus\mathcal{H})$ of the form $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$, where $X,Y\in\mathcal{B}(\mathcal{H})$. Let us first note the following inequality for $X\in\mathcal{B}(\mathcal{H})$,

$$\alpha|\langle Xx,x\rangle|^2+(1-\alpha)\|Xx\|^2 \le \|X\|_\alpha^2\|x\|^2 \text{ for all } x\in\mathcal{H} \text{ with } \|x\|\le 1.$$

Theorem 1. *Let $X,Y\in\mathcal{B}(\mathcal{H})$. Then the following inequalities hold:*

$$(i)\ \max\{\|X\|_\alpha,\|Y\|_\alpha\} \le \left\| \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right\|_\alpha \le \max\left\{\sqrt{\|X\|_\alpha^2+\alpha w^2(X)},\sqrt{\|Y\|_\alpha^2+\alpha w^2(Y)}\right\} \le \sqrt{2}\max\{\|X\|_\alpha,\|Y\|_\alpha\}.$$

$$(ii)\quad \left\| \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right\|_\alpha \le \sqrt{\max\{\|X\|_\alpha^2,\|Y\|_\alpha^2\}+\alpha w(X)w(Y)}.$$

$$(iii)\quad \left\| \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right\|_\alpha \le \|X\|_\alpha+\|Y\|_\alpha.$$

Proof. (i) Let $T=\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$. Let $x\in\mathcal{H}$ with $\|x\|=1$ and let $\tilde{x}=\begin{pmatrix} x \\ 0 \end{pmatrix}\in\mathcal{H}\oplus\mathcal{H}$. Clearly, $\|\tilde{x}\|=1$. Therefore, we have,

$$\sqrt{\alpha|\langle Xx,x\rangle|^2+(1-\alpha)\|Xx\|^2}=\sqrt{\alpha|\langle T\tilde{x},\tilde{x}\rangle|^2+(1-\alpha)\|T\tilde{x}\|^2}\le\|T\|_\alpha.$$

Taking supremum over all unit vector in $\mathcal{H}$, we get,

$$\|X\|_\alpha\le\|T\|_\alpha.$$

Similarly, it can be proved that

$$\|Y\|_\alpha\le\|T\|_\alpha.$$

Combining the above two inequalities, we get the first inequality in (i). Let us now prove the second inequality in (i). Let $z=\begin{pmatrix} x \\ y \end{pmatrix}\in\mathcal{H}\oplus\mathcal{H}$ with $\|z\|=1$, i.e., $\|x\|^2+\|y\|^2=1$. Then we have,

$$\alpha|\langle Tz,z\rangle|^2+(1-\alpha)\|Tz\|^2$$

$$
\begin{aligned}
&= \alpha\,|\langle Xx,x\rangle + \langle Yy,y\rangle|^2 + (1-\alpha)(\|Xx\|^2+\|Yy\|^2)\\
&\leq \alpha\,(|\langle Xx,x\rangle| + |\langle Yy,y\rangle|)^2 + (1-\alpha)\left(\|Xx\|^2+\|Yy\|^2\right)\\
&\leq \alpha|\langle Xx,x\rangle|^2 + (1-\alpha)\|Xx\|^2 + \alpha|\langle Yy,y\rangle|^2 + (1-\alpha)\|Yy\|^2\\
&\qquad + \alpha\left(|\langle Xx,x\rangle|^2 + |\langle Yy,y\rangle|^2\right)\\
&\leq \|X\|_\alpha^2\|x\|^2 + \|Y\|_\alpha^2\|y\|^2\\
&\qquad + \alpha\left(w^2(X)\|x\|^2 + w^2(Y)\|y\|^2\right) \quad (\text{since } \ \|x\|\leq 1, \|y\|\leq 1)\\
&= \left(\|X\|_\alpha^2 + \alpha w^2(X)\right)\|x\|^2 + \left(\|Y\|_\alpha^2 + \alpha w^2(Y)\right)\|y\|^2\\
&\leq \max\left\{\|X\|_\alpha^2 + \alpha w^2(X), \|Y\|_\alpha^2 + \alpha w^2(Y)\right\}.
\end{aligned}
$$

Therefore, taking supremum over all unit vectors in $\mathcal{H}\oplus\mathcal{H}$, we get the second inequality in (i). The remaining inequality in (i) follows from the inequalities $\alpha w^2(X) \leq \|X\|_\alpha^2$ and $\alpha w^2(Y) \leq \|Y\|_\alpha^2$. This completes the proof of (i).

(ii) From

$$
\alpha|\langle Tz,z\rangle|^2 + (1-\alpha)\|Tz\|^2 \leq \alpha\,(|\langle Xx,x\rangle| + |\langle Yy,y\rangle|)^2 + (1-\alpha)\left(\|Xx\|^2+\|Yy\|^2\right),
$$

we get

$$
\begin{aligned}
&\alpha|\langle Tz,z\rangle|^2 + (1-\alpha)\|Tz\|^2\\
&\qquad \leq \alpha|\langle Xx,x\rangle|^2 + (1-\alpha)\|Xx\|^2 + \alpha|\langle Yy,y\rangle|^2 + (1-\alpha)\|Yy\|^2\\
&\qquad\qquad + 2\alpha|\langle Xx,x\rangle| \ |\langle Yy,y\rangle|\\
&\qquad \leq \alpha|\langle Xx,x\rangle|^2 + (1-\alpha)\|Xx\|^2 + \alpha|\langle Yy,y\rangle|^2 + (1-\alpha)\|Yy\|^2\\
&\qquad\qquad + 2\alpha w(X)w(Y)\|x\|^2\|y\|^2\\
&\qquad \leq \|X\|_\alpha^2\|x\|^2 + \|Y\|_\alpha^2\|y\|^2\\
&\qquad\qquad + 2\alpha w(X)w(Y)\|x\|\|y\| \quad (\text{since } \ \|x\|\leq 1, \|y\|\leq 1)\\
&\qquad \leq \max\left\{\|X\|_\alpha^2, \|Y\|_\alpha^2\right\} + \alpha w(X)w(Y).
\end{aligned}
$$

Taking supremum over all unit vectors in $\mathcal{H}\oplus\mathcal{H}$, we get the inequality in (ii).

(iii) The inequality in (iii) follows from the triangle inequality of the α-norm, and by using the inequality in (ii). □

In the following theorem, we obtain the exact value of the α-norm of 2×2 operator matrices in $\mathcal{B}(\mathcal{H}\oplus\mathcal{H})$ of the form $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$, where $X\in\mathcal{B}(\mathcal{H})$.

Theorem 2. *Let $X\in\mathcal{B}(\mathcal{H})$. Then*

$$
\left\|\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\right\|_\alpha = \begin{cases} \frac{1}{2\sqrt{\alpha}}\,\|X\| & \text{if } \ \alpha > \frac{1}{2}\\ \sqrt{1-\alpha}\,\|X\| & \text{if } \ \alpha \leq \frac{1}{2}. \end{cases}
$$

Proof. Let $T = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$. Let $z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H}$ with $\|z\| = 1$, i.e., $\|x\|^2 + \|y\|^2 = 1$. Then $\langle Tz, z\rangle = \langle Xy, x\rangle$ and $\|Tz\| = \|Xy\|$. Now we have,

$$\begin{aligned} \|T\|_\alpha^2 &= \sup_{\|z\|=1} (\alpha|\langle Tz,z\rangle|^2 + (1-\alpha)\|Tz\|^2) \\ &= \sup_{\|x\|^2+\|y\|^2=1} (\alpha|\langle Xy,x\rangle|^2 + (1-\alpha)\|Xy\|^2) \\ &\leq \sup_{\|x\|^2+\|y\|^2=1} (\alpha\|X\|^2\|y\|^2\|x\|^2 + (1-\alpha)\|X\|^2\|y\|^2) \\ &= \sup_{\theta\in[0,\frac{\pi}{2}]} \|X\|^2 \sin^2\theta(\alpha\cos^2\theta + (1-\alpha)). \end{aligned}$$

First we consider the case $\alpha > \frac{1}{2}$. Then

$$\sup_{\theta\in[0,\frac{\pi}{2}]} \|X\|^2 \sin^2\theta(\alpha\cos^2\theta + (1-\alpha)) = \frac{1}{4\alpha}\|X\|^2.$$

Therefore, $\|T\|_\alpha^2 \leq \frac{1}{4\alpha}\|X\|^2$. We claim that there exists a sequence $\{z_n\}$ in $\mathcal{H}\oplus\mathcal{H}$ with $\|z_n\| = 1$ such that

$$\lim_{n\to\infty} (\alpha|\langle Tz_n, z_n\rangle|^2 + (1-\alpha)\|Tz_n\|^2) = \frac{1}{4\alpha}\|X\|^2.$$

Clearly, there exists a sequence $\{y_n\}$ in $\mathcal{H}$ with $\|y_n\| = 1$ such that $\lim_{n\to\infty} \|Xy_n\| = \|X\|$. Let $z_n = \frac{1}{\sqrt{\|Xy_n\|^2+k^2}} \begin{pmatrix} Xy_n \\ ky_n \end{pmatrix}$, where $k = \sqrt{\frac{1}{2\alpha-1}}\|X\|$. Then

$$\lim_{n\to\infty} \alpha|\langle Tz_n, z_n\rangle|^2 + (1-\alpha)\|Tz_n\|^2 = \frac{1}{4\alpha}\|X\|^2.$$

Therefore, $\|T\|_\alpha = \frac{1}{2\sqrt{\alpha}}\|X\|$ if $\alpha > \frac{1}{2}$.

Next we consider the case $\alpha \leq \frac{1}{2}$. Then

$$\sup_{\theta\in[0,\frac{\pi}{2}]} \|X\|^2 \sin^2\theta(\alpha\cos^2\theta + (1-\alpha)) = (1-\alpha)\|X\|^2$$

Therefore, $\|T\|_\alpha^2 \leq (1-\alpha)\|X\|^2$. Proceeding as before, we can show that there exists a sequence $\{z_n\}, \|z_n\| = 1$ such that $\lim_{n\to\infty}(\alpha|\langle Tz_n, z_n\rangle|^2 + (1-\alpha)\|Tz_n\|^2) = (1-\alpha)\|X\|^2$. Therefore, $\|T\|_\alpha = \sqrt{(1-\alpha)}\|X\|$ if $\alpha \leq \frac{1}{2}$. □

Remark 1. It follows from Proposition 1 (b) that $\left\|\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\right\|_\alpha = \left\|\begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}\right\|_\alpha$. Also, it follows from Theorem 1 that $\left\|\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}\right\|_\alpha = \left\|\begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}\right\|_\alpha = \|X\|_\alpha$.

Our next goal is to obtain upper bounds for the α-norm of $n\times n$ operator matrices in $\mathcal{B}(\oplus_{i=1}^n \mathcal{H}_i)$. We require the following lemmas for our purpose.

Lemma 1. *([11, p. 44]) Let $T=(t_{ij})\in M_n(\mathbb{C})$ with $t_{ij}\geq 0$ for all i,j. Then*
$$w(T)=r(Re(T))=\|Re(T)\|.$$

Lemma 2. *([13]) Let $T\in\mathcal{B}(\mathcal{H})$ be self-adjoint and let $x\in\mathcal{H}$. Then*
$$|\langle Tx,x\rangle|\leq\langle|T|x,x\rangle.$$

Lemma 3. *([13]) Let $T\in\mathcal{B}(\mathcal{H})$ with $T\geq 0$ and let $x\in\mathcal{H}$ with $\|x\|=1$. Then*
$$\langle Tx,x\rangle^p\leq\langle T^px,x\rangle \;\text{ for all } p\geq 1.$$

Lemma 4. *([15, Th. 2.1]) Let $T\in\mathcal{B}(\mathcal{H})$. Then the following inequalities hold:*
$$w(T)\leq\|T\|_\alpha\leq\sqrt{4-3\alpha}\,w(T),$$
$$\max\left\{\frac{1}{2},\sqrt{(1-\alpha)}\right\}\|T\|\leq\|T\|_\alpha\leq\|T\|.$$

Now we are in a position to prove the following inequality.

Theorem 3. *Let $\mathcal{H}_1,\mathcal{H}_2,\ldots,\mathcal{H}_n$ be Hilbert spaces. Let $T=(T_{ij})$ be an $n\times n$ operator matrix, where $T_{ij}\in\mathcal{B}(\mathcal{H}_j,\mathcal{H}_i)$. Then*
$$\|T\|_\alpha\leq\sqrt{\|\alpha|R|^2+(1-\alpha)|S|^2\|},$$
where $R=(r_{ij})_{n\times n}$, $r_{ij}=\begin{cases}w(T_{ij}) & \text{if } i=j\\ \frac{1}{2}(\|T_{ij}\|+\|T_{ji}\|) & \text{if } i\neq j\end{cases}$
and $S=(s_{ij})_{n\times n}$, $s_{ij}=\|T_{ij}\|$.

Proof. Let $x=(x_1,x_2,\ldots,x_n)\in\oplus_{i=1}^n\mathcal{H}_i$ with $\|x\|=1$ and let $\tilde{x}=(\|x_1\|,\|x_2\|,\ldots,\|x_n\|)$. Clearly, $\tilde{x}$ is a unit vector in $\mathbb{C}^n$. Now,
$$\begin{aligned}|\langle Tx,x\rangle| &= \left|\sum_{i,j=1}^n\langle T_{ij}x_j,x_i\rangle\right|\leq\sum_{i,j=1}^n|\langle T_{ij}x_j,x_i\rangle|\\ &\leq\sum_{i=1}^n|\langle T_{ii}x_i,x_i\rangle|+\sum_{i,j=1;i\neq j}^n|\langle T_{ij}x_j,x_i\rangle|\\ &\leq\sum_{i=1}^n w(T_{ii})\|x_i\|^2+\sum_{i,j=1;i\neq j}^n\|T_{ij}\|\|x_j\|\|x_i\|\\ &=\sum_{i,j=1}^n\tilde{t_{ij}}\|x_j\|\|x_i\|=\langle\tilde{T}\tilde{x},\tilde{x}\rangle=\langle Re(\tilde{T})\tilde{x},\tilde{x}\rangle+i\langle Im(\tilde{T})\tilde{x},\tilde{x}\rangle,\end{aligned}$$
where $\tilde{T}=(\tilde{t_{ij}})$, $\tilde{t_{ij}}=\begin{cases}w(T_{ij}) & \text{if } i=j\\ \|T_{ij}\| & \text{if } i\neq j.\end{cases}$

Clearly, $\langle Im(\tilde{T})\tilde{x},\tilde{x}\rangle=0$. So by using Lemma 2 and Lemma 3, we get
$$|\langle Tx,x\rangle|\leq\langle Re(\tilde{T})\tilde{x},\tilde{x}\rangle\leq\langle|Re(\tilde{T})|\tilde{x},\tilde{x}\rangle$$

$$\Rightarrow |\langle Tx,x\rangle|^2 \le \langle |Re(\tilde{T})|\tilde{x},\tilde{x}\rangle^2 \le \langle |Re(\tilde{T})|^2\tilde{x},\tilde{x}\rangle = \langle |R|^2\tilde{x},\tilde{x}\rangle.$$

Also,

$$\begin{aligned}\|Tx\|^2 = |\langle Tx,Tx\rangle| &= \left|\sum_{i,j,k=1}^{n}\langle T_{kj}x_j, T_{ki}x_i\rangle\right| \\ &\le \sum_{i,j,k=1}^{n}|\langle T_{kj}x_j, T_{ki}x_i\rangle| \le \sum_{i,j,k=1}^{n}|\langle T_{ki}^{*}T_{kj}x_j, x_i\rangle| \\ &\le \sum_{i,j,k=1}^{n}\|T_{ki}\|\|T_{kj}\|\|x_j\|\|x_i\| = \langle |S|^2\tilde{x},\tilde{x}\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha|\langle Tx,x\rangle|^2 + (1-\alpha)\|Tx\|^2 &\le \alpha\langle |R|^2\tilde{x},\tilde{x}\rangle + (1-\alpha)\langle |S|^2\tilde{x},\tilde{x}\rangle \\ &= \langle \left(\alpha|R|^2 + (1-\alpha)|S|^2\right)\tilde{x},\tilde{x}\rangle \le \left\|\alpha|R|^2 + (1-\alpha)|S|^2\right\|.\end{aligned}$$

Taking supremum over all unit vectors in $\oplus_{i=1}^{n}\mathcal{H}_i$, we get the desired inequality. □

As a consequence of Theorem 3, the following numerical radius inequality and the usual operator norm inequality can be proved quite easily.

Corollary 1. *Let $\mathcal{H}_1,\mathcal{H}_2,\ldots,\mathcal{H}_n$ be Hilbert spaces. Let $T=(T_{ij})$ be an $n\times n$ operator matrix, where $T_{ij}\in\mathcal{B}(\mathcal{H}_j,\mathcal{H}_i)$. Then*

$$(i)\ \ w(T) \le \min_{0\le\alpha\le 1}\sqrt{\|\alpha|R|^2 + (1-\alpha)|S|^2\|} \le w(\tilde{T}),$$

$$(ii)\ \ \|T\| \le \min_{0\le\alpha\le 1}\frac{1}{\max\left\{\frac{1}{2},\sqrt{1-\alpha}\right\}}\sqrt{\|\alpha|R|^2 + (1-\alpha)|S|^2\|} \le \|S\|,$$

where $\tilde{T}=(\tilde{t_{ij}})_{n\times n}$, $\tilde{t_{ij}} = \begin{cases} w(T_{ij}) & \text{if } i=j \\ \|T_{ij}\| & \text{if } i\neq j\end{cases}$ and R,S are same as described in Theorem 3.

We would like to note that the inequalities in [1, Th. 1] and [12, Th. 1.1] follow from (i) and (ii) of Corollary 1, respectively.

In our next result, we obtain an upper bound for the α-norm of $n\times n$ operator matrices in terms of non-negative continuous functions on $[0,\infty)$. First we need the following lemma.

Lemma 5. *([13, Th. 5]) Let $T\in\mathcal{B}(\mathcal{H})$ and let f and g be two non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t)=t,\ \forall\ t\in[0,\infty)$. Then*

$$|\langle Tx,y\rangle| \le \|f(|T|)x\|\|g(|T^*|)y\|,\ \ \forall\ x,y\in\mathcal{H}.$$

Theorem 4. *Let $T=(T_{ij})$ be an $n\times n$ operator matrix, where $T_{ij}\in\mathcal{B}(\mathcal{H})$. Let f and g be two non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t)=t$, $\forall\, t\geq 0$. Then*

$$\|T\|_\alpha \leq \sqrt{\|\alpha|R|^2+(1-\alpha)|S|^2\|},$$

where $R=(r_{ij})_{n\times n}$, $r_{ij}=\frac{1}{2}\left(\|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}}+\|f^2(|T_{ji}|)\|^{\frac{1}{2}}\|g^2(|T_{ji}^*|)\|^{\frac{1}{2}}\right)$ and $S=(s_{ij})_{n\times n}$, $s_{ij}=\|T_{ij}\|$.

Proof. Let $x=(x_1,x_2,...,x_n)\in\oplus_{i=1}^n\mathcal{H}$ with $\|x\|=1$ and let $\tilde{x}=(\|x_1\|,\|x_2\|,...,\|x_n\|)$. Clearly, $\tilde{x}$ is a unit vector in $\mathbb{C}^n$. Using Lemma 5, we get that

$$\begin{aligned}
|\langle Tx,x\rangle| &= \left|\sum_{i,j=1}^n\langle T_{ij}x_j,x_i\rangle\right| \leq \sum_{i,j=1}^n|\langle T_{ij}x_j,x_i\rangle| \\
&\leq \sum_{i,j=1}^n\|f(|T_{ij}|)x_j\|\|g(|T_{ij}^*|)x_i\| = \sum_{i,j=1}^n\langle f^2(|T_{ij}|)x_j,x_j\rangle^{\frac{1}{2}}\langle g^2(|T_{ij}^*|)x_i,x_i\rangle^{\frac{1}{2}} \\
&\leq \sum_{i,j=1}^n\|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}}\|x_i\|\|x_j\| = \sum_{i,j=1}^n\tilde{t_{ij}}\|x_j\|\|x_i\| \\
&= \langle\tilde{T}\tilde{x},\tilde{x}\rangle = \langle Re(\tilde{T})\tilde{x},\tilde{x}\rangle + i\langle Im(\tilde{T})\tilde{x},\tilde{x}\rangle,
\end{aligned}$$

where $\tilde{T}=(\tilde{t_{ij}})$, $\tilde{t_{ij}}=\|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}}$.

Proceeding similarly as in the proof of Theorem 3, we get

$$|\langle Tx,x\rangle|^2\leq\langle|R|^2\tilde{x},\tilde{x}\rangle \ \text{ and } \ \|Tx\|^2\leq\langle|S|^2\tilde{x},\tilde{x}\rangle.$$

Therefore,

$$\alpha|\langle Tx,x\rangle|^2+(1-\alpha)\|Tx\|^2\leq\left\|\alpha|R|^2+(1-\alpha)|S|^2\right\|.$$

Taking supremum over all unit vectors in $\oplus_{i=1}^n\mathcal{H}$, we get the desired inequality. □

The following numerical radius inequality is an easy consequence of Theorem 4 .

Corollary 2. *Let $T=(T_{ij})$ be an $n\times n$ operator matrix, where $T_{ij}\in\mathcal{B}(\mathcal{H})$. Let f and g be non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t)=t$, $\forall\, t\geq 0$. Then*

$$w(T)\leq\min_{0\leq\alpha\leq 1}\sqrt{\|\alpha|R|^2+(1-\alpha)|S|^2\|}\leq w(\tilde{T}),$$

where $\tilde{T}=(\tilde{t_{ij}})_{n\times n}$, $\tilde{t_{ij}}=\|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}}$ and R,S are same as described in Theorem 4.

We would like to note that the inequality in [7, Th. 3.1] follows from Corollary 2.

In our next theorem, we obtain an upper bound for the α-norm of $n\times n$ operator matrices.

Theorem 5. *Let $T = (T_{ij})$ be an $n \times n$ operator matrix, where $T_{ij} \in \mathcal{B}(\mathcal{H})$. Let f and g be two non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t) = t$, $\forall\, t \geq 0$. If $p \geq 1$, then*

$$\|T\|_\alpha^p \leq \sqrt{\|\alpha|R|^{2p} + (1-\alpha)|S|^{2p}\|},$$

where $R = (r_{ij})_{n\times n}$,

$$r_{ij} = \begin{cases} \frac{1}{2}\left\|f^2(|T_{ii}|) + g^2(|T_{ii}^*|)\right\| & \text{if } i = j \\ \frac{1}{2}\left(\|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}} + \|f^2(|T_{ji}|)\|^{\frac{1}{2}}\|g^2(|T_{ji}^*|)\|^{\frac{1}{2}}\right) & \text{if } i \neq j \end{cases}$$

and $S = (s_{ij})_{n\times n}$, $s_{ij} = \|T_{ij}\|$.

Proof. Let $x = (x_1, x_2, \ldots, x_n) \in \oplus_{i=1}^n \mathcal{H}$ with $\|x\| = 1$ and let $\tilde{x} = (\|x_1\|, \|x_2\|, \ldots, \|x_n\|)$. Clearly, $\tilde{x}$ is a unit vector in $\mathbb{C}^n$. Using Lemma 5, we get that

$$\begin{aligned}
|\langle Tx, x\rangle| &= \left|\sum_{i,j=1}^n \langle T_{ij}x_j, x_i\rangle\right| \leq \sum_{i,j=1}^n |\langle T_{ij}x_j, x_i\rangle| \\
&\leq \sum_{i,j=1}^n \|f(|T_{ij}|)x_j\|\|g(|T_{ij}^*|)x_i\| = \sum_{i,j=1}^n \langle f^2(|T_{ij}|)x_j, x_j\rangle^{\frac{1}{2}}\langle g^2(|T_{ij}^*|)x_i, x_i\rangle^{\frac{1}{2}} \\
&\leq \sum_{i=1}^n \frac{1}{2}\left(\langle f^2(|T_{ii}|)x_i, x_i\rangle + \langle g^2(|T_{ii}^*|)x_i, x_i\rangle\right) \\
&\quad + \sum_{i,j=1, i\neq j}^n \langle f^2(|T_{ij}|)x_j, x_j\rangle^{\frac{1}{2}}\langle g^2(|T_{ij}^*|)x_i, x_i\rangle^{\frac{1}{2}} \\
&= \sum_{i=1}^n \frac{1}{2}\langle\left(f^2(|T_{ii}|) + g^2(|T_{ii}^*|)\right)x_i, x_i\rangle \\
&\quad + \sum_{i,j=1, i\neq j}^n \langle f^2(|T_{ij}|)x_j, x_j\rangle^{\frac{1}{2}}\langle g^2(|T_{ij}^*|)x_i, x_i\rangle^{\frac{1}{2}} \\
&\leq \sum_{i=1}^n \frac{1}{2}\left\|f^2(|T_{ii}|) + g^2(|T_{ii}^*|)\right\|\,\|x_i\|^2 \\
&\quad + \sum_{i,j=1, i\neq j}^n \|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}}\|x_i\|\|x_j\| \\
&= \sum_{i,j=1}^n \tilde{t_{ij}}\|x_j\|\|x_i\| = \langle\tilde{T}\tilde{x}, \tilde{x}\rangle = \langle Re(\tilde{T})\tilde{x}, \tilde{x}\rangle + i\langle Im(\tilde{T})\tilde{x}, \tilde{x}\rangle,
\end{aligned}$$

where $\tilde{T} = (\tilde{t_{ij}})_{n\times n}$,

$$\tilde{t_{ij}} = \begin{cases} \frac{1}{2}\left\|f^2(|T_{ii}|) + g^2(|T_{ii}^*|)\right\| & \text{if } i = j \\ \|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}} & \text{if } i \neq j. \end{cases}$$

Clearly, $\langle Im(\tilde{T})\tilde{x},\tilde{x}\rangle = 0$, and so using Lemma 2 and Lemma 3, we get that

$$\begin{aligned}|\langle Tx,x\rangle| \leq \langle Re(\tilde{T})\tilde{x},\tilde{x}\rangle &\Rightarrow |\langle Tx,x\rangle| \leq \langle |Re(\tilde{T})|\tilde{x},\tilde{x}\rangle\\ &\Rightarrow |\langle Tx,x\rangle|^{2p} \leq \langle |Re(\tilde{T})|\tilde{x},\tilde{x}\rangle^{2p} \Rightarrow |\langle Tx,x\rangle|^{2p} \leq \langle |Re(\tilde{T})|^{2p}\tilde{x},\tilde{x}\rangle\\ &\Rightarrow |\langle Tx,x\rangle|^{2p} \leq \langle |R|^{2p}\tilde{x},\tilde{x}\rangle.\end{aligned}$$

Now proceeding similarly as in the proof of Theorem 3 and using Lemma 3, we obtain

$$\|Tx\|^{2p} \leq \langle |S|^2\tilde{x},\tilde{x}\rangle^p \leq \langle |S|^{2p}\tilde{x},\tilde{x}\rangle.$$

By convexity of $t^p, p \geq 1$, it follows that

$$\begin{aligned}\left(\alpha|\langle Tx,x\rangle|^2 + (1-\alpha)\|Tx\|^2\right)^p &\leq \left(\alpha|\langle Tx,x\rangle|^{2p} + (1-\alpha)\|Tx\|^{2p}\right)\\ &\leq \left(\alpha\langle |R|^{2p}\tilde{x},\tilde{x}\rangle + (1-\alpha)\langle |S|^{2p}\tilde{x},\tilde{x}\rangle\right)\\ &= \langle \left(\alpha|R|^{2p} + (1-\alpha)|S|^{2p}\right)\tilde{x},\tilde{x}\rangle\\ &\leq \left\|\alpha|R|^{2p} + (1-\alpha)|S|^{2p}\right\|.\end{aligned}$$

Therefore, taking supremum over all unit vectors in $\oplus_{i=1}^n \mathcal{H}$, we get the desired inequality. □

We simply state the following result and omit its proof, as it can be completed using similar arguments as given in the proof of Theorem 5.

Theorem 6. *Let $T = (T_{ij})$ be an $n \times n$ operator matrix, where $T_{ij} \in \mathcal{B}(\mathcal{H})$. Let f and g be two non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t) = t$, $\forall\, t \geq 0$. Then*

$$\|T\|_\alpha \leq \sqrt{\|\alpha|R|^2 + (1-\alpha)|S|^2\|},$$

where $R = (r_{ij})_{n\times n}$,

$$r_{ij} = \begin{cases} \frac{1}{2}\left\|f^2(|T_{ii}|) + g^2(|T_{ii}^*|)\right\| & \text{if } i = j\\ \frac{1}{2}\left(\|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}} + \|f^2(|T_{ji}|)\|^{\frac{1}{2}}\|g^2(|T_{ji}^*|)\|^{\frac{1}{2}}\right) & \text{if } i \neq j\end{cases}$$

and $S = (s_{ij})_{n\times n}$, $s_{ij} = \|T_{ij}\|$.

The following numerical radius inequality follows easily from Theorem 6 by using Lemma 4.

Corollary 3. *Let $T = (T_{ij})$ be an $n \times n$ operator matrix, where $T_{ij} \in \mathcal{B}(\mathcal{H})$. Let f and g be two non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t) = t$, $\forall\, t \geq 0$. Then*

$$w(T) \leq \min_{0\leq\alpha\leq 1} \sqrt{\|\alpha|R|^2 + (1-\alpha)|S|^2\|},$$

where R, S are same as described in Theorem 6.

Remark 2. In particular, if we consider $\alpha = 1$ in Corollary 3 then using Lemma 1, we get

$$w(T) \leq \min_{0\leq\alpha\leq 1} \sqrt{\| \alpha |R|^2 + (1-\alpha)|S|^2 \|} \leq w(\tilde{T}),$$

where $\tilde{T} = (\tilde{t_{ij}})_{n\times n}$, $\tilde{t_{ij}} = \begin{cases} \frac{1}{2} \left\| f^2(|T_{ii}|) + g^2(|T_{ii}^*|) \right\| & \text{if } i = j \\ \| f^2(|T_{ij}|)\|^{\frac{1}{2}} \| g^2(|T_{ij}^*|)\|^{\frac{1}{2}} & \text{if } i \neq j. \end{cases}$ Note that the existing inequality in [7, Th. 3.3] follows from Corollary 3.

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