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LYAPUNOV AND HARTMAN-TYPE INEQUALITIES FOR HIGHER-ORDER DISCRETE FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. By employing Green's function, we obtain new Lyapunov and Hartman-type inequalities for higher-order discrete fractional boundary value problems. Reported results essentially generalize some theorems existing in the literature. As an application, we discuss the corresponding eigenvalue problems.

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1. INTRODUCTION

Lyapunov inequality was first introduced by the Russian Mathematician Liapounoff [13] who has demonstrated significant effect in the qualitative theory of differential equations such as oscillation and Sturmian theory, asymptotic theory, disconjugacy, boundary value problems, eigenvalue problems and numerous other applications. Due to its importance, Lyapunov inequality has been improved and generalized in many forms, see for instance [2–4, 8, 9, 16–20]. For the sake of convenience, we recall the following some versions of Lyapunov inequality.

Consider the Hill equation

$$z''(\rho)+\omega(\rho)z(\rho)=0, \qquad \rho_1<\rho<\rho_2, \tag{1.1}$$

where $\omega(\rho)\in L^1[\rho_1,\rho_2]$ is a real-valued function. Let z be a non-trivial solution of (1.1) satisfying the boundary conditions:

$$z(\rho_1)=z(\rho_2)=0,$$

where $\rho_1,\rho_2\in\mathbb{R}$ ($\rho_1<\rho_2$). If ρ_1 and ρ_2 are consecutive zeros of z, then the inequality

$$\int_{\rho_1}^{\rho_2}|\omega(\rho)|d\rho>\frac{4}{\rho_2-\rho_1} \tag{1.2}$$

holds. In 1951, Wintner [21] (and then many authors) showed that the function "$|\omega(\rho)|$" in (1.2) can be replaced by the function "$\omega^+(\rho)$" so that (1.2) turns out to be the inequality

$$\int_{\rho_1}^{\rho_2} \omega^+(\rho)d\rho > \frac{4}{\rho_2-\rho_1}, \tag{1.3}$$

where $\omega^+(\rho) = \max\{\omega(\rho), 0\}$ and the constant "4" in the right-hand-side of inequalities (1.2) and (1.3) is the best possible largest number (see [10]). Since then, inequality (1.3) has been known as *Lyapunov inequality* in the literature.

In 1964, Hartman [10] obtained the inequality

$$\int_{\rho_1}^{\rho_2} (\rho_2-\rho)(\rho-\rho_1)\omega^+(\rho)d\rho > \rho_2-\rho_1$$

which is known as *Hartman inequality* in the literature and it is the best Lyapunov-type inequality for being stronger than both inequalities (1.2) and (1.3).

Although Lyapunov-type inequalities have been obtained by some authors [12, 14, 15] for continuous fractional boundary value problems, there are few relevant results, see for example the paper by Ferreira [5] for the right-focal discrete boundary value problem of the form

$$\begin{cases} \Delta^{\nu} z(\rho) + q(\rho+\nu-1)z(\rho+\nu-1) = 0, & \rho \in \mathbb{N}_0^{b+1}, \\ z(\nu-2) = 0 = \Delta z(\nu+b), \end{cases} \tag{1.4}$$

and the inequality

$$\sum_{s=0}^{b+1} |q(s+\nu-1)| > \frac{1}{\Gamma(\nu-1)(b+2)}$$

is obtained. However the estimation for the corresponding Green's function of Prb. (1.4) obtained in [5] is not true. The corrected result is published as an addendum version by the same author [6].

For the sake of improving and generalizing existing results, we derive new Lyapunov and Hartman-type inequalities for higher-order discrete fractional BVP of the form

$$\begin{cases} \Delta^{\nu}_{\nu-n_0} z(\rho) + q(\rho)z(\rho+\nu-1) = 0, & \rho \in \mathbb{N}_0^{b+m_0}, \\ \Delta^{i} z(\nu-n_0) = 0, & i \in \mathbb{N}_0^{n_0-2}, \\ \Delta^{\mu}_{\nu-n_0} z(b+m_0+\nu-\mu) = 0, \end{cases} \tag{1.5}$$

where $q(\rho)$ is real-valued function defined on $\mathbb{N}_0^{b+m_0}$, $b, n_0, m_0 \in \mathbb{N}$, $n_0-1 < \nu \le n_0$, $m_0-1 < \mu \le m_0$ and

- $1 \le \mu < \nu$ when $\nu \ge 2$;
- $\mu = 1$ when $1 < \nu < 2$.

2. PRELIMINARIES

In this section we assemble some basic definitions on delta fractional calculus. The terms and statements of this section are adopted for the remarkable monograph [7].

The functions that we consider in this paper is defined on a set

$$\mathbb{N}_c = \{c, c+1, c+2, ...\}, \qquad c \in \mathbb{R}$$

or a set

$$\mathbb{N}_c^d = \{c, c+1, c+2, ..., d\}, \qquad c, d \in \mathbb{R}.$$

It is well known that the forward difference operator Δ is defined by

$$\Delta w(\rho) = w(\rho+1) - w(\rho).$$

Further, the operator Δ^n is recursively defined by

$$\Delta^n w(\rho) = \Delta(\Delta^{n-1} w(\rho)), \qquad n = 1, 2, \ldots$$

and Δ^0 denotes the identity operator, that is, $\Delta^0 w(\rho) = w(\rho)$.

Definition 1. The (generalized) falling function is defined by

$$\rho^{\underline{r}} := \frac{\Gamma(\rho+1)}{\Gamma(\rho-r+1)},$$

where $\Gamma(*)$ is the *Gamma function*. Note that the definition is valid if and only if the right-hand side of this equation is well-defined.

Definition 1 can be extended by prevalent convention that $\rho^{\underline{r}} = 0$ when $\rho - r + 1$ and $\rho + 1$ are not non-positive integers.

Definition 2. The v-th fractional Taylor monomial based at s is defined by

$$h_v(\rho, s) := \frac{(\rho - s)^{\underline{v}}}{\Gamma(v+1)} \tag{2.1}$$

whenever the right-hand side is well-defined.

Now, we can define v-th fractional sum.

Definition 3. Assume $w : \mathbb{N}_c \to \mathbb{R}$ and $v > 0$. Then the v-th fractional sum of w (based at c) is defined by

$$\Delta_c^{-v} w(\rho) := \sum_{\tau=c}^{\rho-v} h_{v-1}(\rho, \sigma(\tau)) w(\tau), \qquad \rho \in \mathbb{N}_{c+v},$$

where the function h_* is defined in (2.1) and σ is forward jump operator defined by $\sigma(\rho) = \rho + 1$.

Next we define the fractional difference in terms of the fractional sum.

Definition 4. Assume $w:\mathbb{N}_c\to\mathbb{R}$ and $\nu>0$. Take a positive integer n_0 such that $n_0-1<\nu\leq n_0$. Then as known, ν-th fractional difference is defined by

$$\Delta_c^{\nu}w(\rho):=\Delta^{n_0}\Delta_c^{-(n_0-\nu)}w(\rho),\qquad \rho\in\mathbb{N}_{c+n_0-\nu}.$$

Note that

$$\Delta_c^{\nu}w(\rho):=\Delta^{n_0}\Delta_c^{-(n_0-\nu)}w(\rho)=\Delta^{n_0}\Delta_c^{0}w(\rho)=\Delta^{n_0}w(\rho)\qquad(\rho\in\mathbb{N}_c)$$

for any $\nu=n_0\in\mathbb{N}_0$.

Lemma 1. *Assume $w:\mathbb{N}_c\to\mathbb{R}$, $k\in\mathbb{N}_0$ and $\nu,\mu>0$ and $n_0-1<\mu\leq n_0$. Then*

$$\Delta_c^{-\nu}\Delta^k w(\rho)=\Delta_c^{k-\nu}w(\rho)-\sum_{j=0}^{k-1}h_{\nu-k+j}(\rho,c)\Delta^j w(c)$$

for $\rho\in\mathbb{N}_{c+\nu}$, and

$$\Delta_{c+n_0-\mu}^{-\nu}\Delta_c^{\mu}w(\rho)=\Delta_c^{\mu-\nu}w(\rho)-\sum_{j=0}^{n_0-1}h_{\nu-n_0+j}(\rho-n_0+\mu,c)\Delta_c^{j-(n_0-\mu)}w(c+n_0-\mu)$$

for $\rho\in\mathbb{N}_{c+n_0-\mu+\nu}$, where the function h_* is defined in (2.1).

3. Main Results

Before proceeding, we present some lemmas that will be used to prove our main results. Next, we obtain Lyapunov and Hartman-type inequalities for Prb. (1.5). Finally, an example is given.

To prove the main results, we will employ the following lemmas.

Lemma 2. *If g is defined on $\mathbb{N}_0^{b+m_0}$, then the solution of the BVP*

$$\begin{cases}-\Delta_{\nu-n_0}^{\nu}y(\rho)=g(\rho), & \rho\in\mathbb{N}_0^{b+m_0},\\ \Delta^i y(\nu-n_0)=0, & i\in\mathbb{N}_0^{n_0-2},\\ \Delta_{\nu-n_0}^{\mu}y(b+m_0+\nu-\mu)=0\end{cases}\tag{3.1}$$

can be represented by

$$y(\rho)=\sum_{s=0}^{b+m_0}\psi(\rho,s)g(s),\qquad \rho\in\mathbb{N}_{\nu-n_0}^{b+m_0+\nu},\tag{3.2}$$

where

$$\psi(\rho,s):=\frac{1}{\Gamma(\nu)}\begin{cases}u_1(\rho,s), & 0\leq\rho-\nu+1\leq s\leq b+m_0,\\ u_2(\rho,s), & 0\leq s\leq\rho-\nu\leq b+m_0,\\ 0, & \nu-n_0\leq\rho\leq\nu-2,\ 0\leq s\leq b+m_0\end{cases}\tag{3.3}$$

is the Green function for Prb. (3.1) with

$$u_1(\rho,s)=\frac{[b+m_0+\nu-\mu-\sigma(s)]^{\underline{\nu-\mu-1}}}{(b+m_0+\nu-\mu)^{\underline{\nu-\mu-1}}}\rho^{\underline{\nu-1}}$$

and

$$u_2(\rho,s)=\frac{[b+m_0+v-\mu-\sigma(s)]^{\underline{v-\mu-1}}}{(b+m_0+v-\mu)^{\underline{v-\mu-1}}}\rho^{\underline{v-1}}-[\rho-\sigma(s)]^{\underline{v-1}}.$$

The proof of Lemma 2 can be found in [11]. It is also shown in [11] that

$$\min_{\rho\in\mathbb{N}_{v-1}^{b+m_0+v}}\psi(\rho,s)>0$$

and

$$\max_{\rho\in\mathbb{N}_{v-n_0}^{b+m_0+v}}\psi(\rho,s)=\psi(b+m_0+v,s) \tag{3.4}$$

for $s\in\mathbb{N}_0^{b+m_0}$.

Lemma 3. *The inequality*

$$\psi(\rho,s)\le\frac{1}{\Gamma(v)}f_1(s)f_2(s);\qquad(\rho,s)\in\mathbb{N}_{v-n_0}^{b+m_0+v}\times\mathbb{N}_0^{b+m_0} \tag{3.5}$$

holds, where $\psi(\rho,s)$ is the Green function of Prb. (3.1)*, and the functions $f_1(s)$ and $f_2(s)$ are defined by*

$$f_1(s):=[b+m_0+v-\mu-\sigma(s)]^{\underline{v-\mu-1}} \tag{3.6}$$

and

$$f_2(s):=(b+m_0+v)^{\underline{\mu}}-[b+m_0+v-\sigma(s)]^{\underline{\mu}} \tag{3.7}$$

for $s\in\mathbb{N}_0^{b+m_0}$, respectively.

Proof. Let $\psi(\rho,s)$ be the Green's function of Prb. (3.1). Then equations (3.3) and (3.4) yield that

$$\begin{aligned}\psi(\rho,s)&\le\psi(b+m_0+v,s)\\&=\frac{1}{\Gamma(v)}\left\{\frac{[b+m_0+v-\mu-\sigma(s)]^{\underline{v-\mu-1}}}{(b+m_0+v-\mu)^{\underline{v-\mu-1}}}\times(b+m_0+v)^{\underline{v-1}}\right.\\&\qquad\left.-[b+m_0+v-\mu-\sigma(s)]^{\underline{v-1}}\right\}\end{aligned} \tag{3.8}$$

for $(\rho,s)\in\mathbb{N}_{v-1}^{b+m_0+v}\times\mathbb{N}_0^{b+m_0}$.

If we multiply both sides of (3.8) by $\Gamma(v)$, then it turns out to be

$$\begin{aligned}\Gamma(v)\psi(\rho,s)\le&\frac{[b+m_0+v-\mu-\sigma(s)]^{\underline{v-\mu-1}}}{(b+m_0+v-\mu)^{\underline{v-\mu-1}}}\\&\times(b+m_0+v)^{\underline{v-1}}-[b+m_0+v-\sigma(s)]^{\underline{v-1}}\\=&[b+m_0+v-\mu-\sigma(s)]^{\underline{v-\mu-1}}\times\frac{\Gamma(b+m_0+v+1)}{\Gamma(b+m_0+v-\mu+1)}\\&-[b+m_0+v-\sigma(s)]^{\underline{v-1}}\end{aligned}$$

$$
\begin{aligned}
&= [b+m_0+v-\mu-\sigma(s)]^{\underline{v-\mu-1}} \times (b+m_0+v)^{\underline{\mu}} - [b+m_0+v-\sigma(s)]^{\underline{v-1}} \\
&= \frac{\Gamma(b+m_0+v-\mu-s)}{\Gamma(b+m_0-s+1)} \times (b+m_0+v)^{\underline{\mu}} - \frac{\Gamma(b+m_0+v-s)}{\Gamma(b+m_0-s+1)} \\
&= \frac{\Gamma(b+m_0+v-\mu-s)}{\Gamma(b+m_0-s+1)} \times \left[(b+m_0+v)^{\underline{\mu}} - \frac{\Gamma(b+m_0+v-s)}{\Gamma(b+m_0+v-\mu-s)}\right] \\
&= [b+m_0+v-\mu-\sigma(s)]^{\underline{v-\mu-1}} \times \left\{(b+m_0+v)^{\underline{\mu}} - [b+m_0+v-\sigma(s)]^{\underline{\mu}}\right\} \\
&= f_1(s)f_2(s)
\end{aligned}
$$

which implies the desired inequality (3.5). □

The following is our first main result.

Theorem 1 (Hartman-type inequality)**.** *Let z be a non-trivial solution of Prb.* (1.5)*. If $z(\rho+v-1) \neq 0$ for $\rho \in \mathbb{N}_0^{b+m_0}$, then the inequality*

$$\sum_{s=0}^{b+m_0} f_1(s)f_2(s)|q(s)| \geq \Gamma(v) \tag{3.9}$$

holds, where the functions $f_1(s)$ and $f_2(s)$ are defined in (3.6) *and* (3.7)*, respectively.*

Proof. Let z be a non-trivial solution of Prb. (1.5) such that $z(\rho+v-1) \neq 0$ for $\rho \in \mathbb{N}_0^{b+m_0}$. Then the solution $z(\rho)$ can be explained as

$$z(\rho) = \sum_{s=0}^{b+m_0} \psi(\rho,s)q(s)z(s+v-1) \tag{3.10}$$

by (3.2) in Lemma 2, where $\psi(\rho,s)$ is the Green function of Prb. (3.1).

Let $|z(c)| = \max_{\rho\in\mathbb{N}_{v-n_0}^{b+m_0+v}} |z(\rho)|$. Then, taking into account Eq. (3.10) together with inequality (3.5), it turns out the inequality

$$
\begin{aligned}
|z(c)| &= \left|\sum_{s=0}^{b+m_0} \psi(c,s)q(s)z(s+v-1)\right| \\
&\leq \sum_{s=0}^{b+m_0} |\psi(c,s)||q(s)||z(s+v-1)| \\
&\leq |z(c)| \sum_{s=0}^{b+m_0} |\psi(c,s)||q(s)| \\
&\leq \frac{|z(c)|}{\Gamma(v)} \sum_{s=0}^{b+m_0} f_1(s)f_2(s)|q(s)| \qquad (3.11)
\end{aligned}
$$

for $c \in \mathbb{N}_{v-n_0}^{b+m_0+v}$, where the functions $f_1(s)$ and $f_2(s)$ are defined in (3.6) and (3.7), respectively. If we multiply the both sides of inequality (3.11) by $\Gamma(v)/|z(c)| > 0$, then we obtain the desired inequality (3.9). This completes the proof. □

Corollary 1 (Lyapunov-type inequality)**.** *Let z be a non-trivial solution of Prb. (1.5). If $z(\rho+\nu-1)\neq 0$ for $\rho\in\mathbb{N}_0^{b+m_0}$, then the inequality*

$$\sum_{s=0}^{b+m_0}|q(s)|\geq\frac{1}{M_0}\Gamma(\nu-\mu)\left[\frac{(b+m_0+\nu)^{\underline{b+m_0+1}}}{(b+m_0+\nu-\mu)^{\underline{b+m_0+1}}}-1\right]^{-1}\tag{3.12}$$

holds, where

$$M_0=\max\left\{\Gamma(\nu-\mu),\frac{\Gamma(b+m_0+\nu-\mu)}{\Gamma(b+m_0+1)}\right\}.\tag{3.13}$$

Proof. Let z be a non-trivial solution of Prb. (1.5) such that $z(\rho+\nu-1)\neq 0$ for $\rho\in\mathbb{N}_0^{b+m_0}$. Then by Theorem 2, we have the Hartman-type inequality (3.9) for Prb. (1.5).

Now, we will find the maximum value of the functions $f_1(s)$ and $f_2(s)$ on $[0,b+m_0]$, where the functions $f_1(s)$ and $f_2(s)$ are defined in (3.6) and (3.7), respectively. For that purpose, applying the difference operator to them, we find

$$\begin{aligned}\Delta f_1(s)&=-(\nu-\mu-1)\,[b+m_0+\nu-\mu-1-\sigma(s)]^{\underline{\nu-\mu-2}}\\&=-(\nu-\mu-1)\times\frac{\Gamma(b+m_0+\nu-\mu-s-1)}{\Gamma(b+m_0-s+1)}\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}\Delta f_2(s)&=\mu\times[b+m_0+\nu-1-\sigma(s)]^{\underline{\mu-1}}\\&=\mu\times\frac{\Gamma(b+m_0+\nu-s-1)}{\Gamma(b+m_0+\nu-\mu-s)}.\end{aligned}\tag{3.15}$$

Thus, considering Eq. (3.14), we find that if $\nu-\mu<1$, then the function $f_1(s)$ is increasing and if $\nu-\mu>1$, then it is decreasing on $[0,b+m_0]$, that is, if $\nu-\mu<1$, then

$$f_1(s)\leq f(b+m_0)=(\nu-\mu-1)^{\underline{\nu-\mu-1}}=\Gamma(\nu-\mu),\tag{3.16}$$

and that if $\nu-\mu>1$, then

$$f_1(s)\leq f(0)=(b+m_0+\nu-\mu-1)^{\underline{\nu-\mu-1}}=\frac{\Gamma(b+m_0+\nu-\mu)}{\Gamma(b+m_0+1)}.\tag{3.17}$$

on $[0,b+m_0]$. From inequalities (3.16) and (3.17), we deduce that

$$f_1(s)\leq\max\left\{\Gamma(\nu-\mu),\frac{\Gamma(b+m_0+\nu-\mu)}{\Gamma(b+m_0+1)}\right\}=M_0.\tag{3.18}$$

Similarly, noticing Eq. (3.15), we see that the function $f_2(s)$ is increasing on $[0,b+m_0]$ and hence we have

$$\begin{aligned}f_2(s)&\leq g(b+m_0)\\&=(b+m_0+\nu)^{\underline{\mu}}-(\nu-1)^{\underline{\mu}}\end{aligned}$$

$$\begin{aligned}
&= \frac{\Gamma(b+m_0+v+1)}{\Gamma(b+m_0+v-\mu+1)} - \frac{\Gamma(v)}{\Gamma(v-\mu)} \\
&= \frac{(b+m_0+v)\times\cdots\times v\times\Gamma(v)}{(b+m_0+v-\mu)\times\cdots\times(v-\mu)\times\Gamma(v-\mu)} - \frac{\Gamma(v)}{\Gamma(v-\mu)} \\
&= \frac{\Gamma(v)}{\Gamma(v-\mu)}\left[\frac{(b+m_0+v)^{\underline{b+m_0+1}}}{(b+m_0+v-\mu)^{\underline{b+m_0+1}}} - 1\right]. \qquad (3.19)
\end{aligned}$$

Now, using inequalities (3.18) and (3.19), inequality (3.9) turns out to be

$$\Gamma(v) \leq \frac{M_0\Gamma(v)}{\Gamma(v-\mu)} \times \left[\frac{(b+m_0+v)^{\underline{b+m_0+1}}}{(b+m_0+v-\mu)^{\underline{b+m_0+1}}} - 1\right] \times \sum_{s=0}^{b+m_0} |q(s)|.$$

Dividing both sides of the inequality above by $\Gamma(v) > 0$, we get the desired Lyapunov-type inequality (3.12) which completes the proof of Corollary 1. ☐

Corollary 2. *Let $v = n_0 \in \mathbb{N}_2$ and $\mu = 1$. If z is a non-trivial solution of Prb.* (1.5) *and $z(\rho+n_0-1) \neq 0$ for $\rho \in \mathbb{N}_0^{b+1}$, then the inequality*

$$\begin{aligned}
\sum_{s=0}^{b+1} |q(s)| \geq \frac{(n_0-1)!}{(b+2)(b+n_0-1)^{\underline{n_0-2}}} &= \frac{\Gamma(b+2)}{b+2} \times \frac{(n_0-1)!}{\Gamma(b+n_0)} \\
&= \frac{(n_0-1)!}{b+2} \times \prod_{p=2}^{n_0-1} (p+b)^{-1}
\end{aligned}$$

holds.

We remark that a lower bound for the eigenvalues of an eigenvalue problem can be obtained via Lyapunov-type inequality for the same problem. In this sense, we have the following example as an application of the Lyapunov-type inequality.

Example 1. Consider the eigenvalue problem

$$\begin{cases}
\Delta_{v-n_0}^{v} z(\rho) + \lambda z(\rho+v-1) = 0, & \rho \in \{0,\ldots,b+m_0\}, \\
\Delta^i z(v-n_0) = 0, & i \in \{0,\ldots,n_0-2\}, \\
\Delta_{v-n_0}^{\mu} z(b+m_0+v-\mu) = 0 &
\end{cases} \qquad (3.20)$$

and let z be a non-trivial solution of Prb. (3.20). If $z(\rho+v-1) \neq 0$ for $\rho \in \{0,\ldots,b+m_0\}$, then the eigenvalues λ of Prb. (3.20) satisfy the inequality

$$\begin{aligned}
|\lambda| &\geq \frac{\Gamma(v-\mu)}{M_0(b+m_0)} \times \left[\frac{(b+m_0+v)^{\underline{b+m_0+1}}}{(b+m_0+v-\mu)^{\underline{b+m_0+1}}} - 1\right]^{-1} \\
&= \frac{\Gamma(v-\mu)}{M_0(b+m_0)} \times \left[\frac{\Gamma(b+m_0+v+1)\times\Gamma(v-\mu)}{\Gamma(b+m_0+v-\mu+1)\times\Gamma(v)} - 1\right]^{-1},
\end{aligned}$$

where the constant M_0 is defined in (3.13).

3.1. *Concluding Remarks*

We conclude this paper by clarifying some highlights. Lyapunov-type inequalities have so many applications as we explained in the beginning of the paper. For example, with the help of these inequalities, an idea can be obtained about the existence or non-existence of the real zeros of Mittag-Leffler functions well-known in the study of these inequalities. A lower bound can be found for eigenvalues of eigenvalue problems generated by the corresponding boundary value problems. Furthermore, by using them, stability criteria for some certain systems and equations can be acquired. We refer the reader to the monograph by Agarwal et al. [1] which includes much more information on these issues.

In the light of these information, the inequalities obtained in this paper are valuable since they can be applied on many fields. These inequalities presents a lower bound for eigenvalues of eigenvalue problem which is obtained from the higher-order discrete fractional boundary value problem as seen in the Example 1. As an application of our study, bounds for eigenvalues of special cases of our problem can be considered. Also, so-called inequalities can be used to obtain an interval in which the Mittag-Leffler function in question has no real zeros. Moreover, the inequalities can help to establish conditions that guarantee the stability of some related systems and equations.

Such applications and more are left to the researchers. It will be of interest to study on these application. On the other hand, considering that the idea of studying the Hartman-type inequalities for higher-order discrete boundary value problems is new, we think that these inequalities are promising for applications and we hope that the area of discrete fractional equations will be developed with their use.

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REFERENCES

[1] R. P. Agarwal, M. Bohner, and A. Özbekler, *Lyapunov inequalities and applications*. Springer, Cham, [2021] ©2021. doi: 10.1007/978-3-030-69029-8.

[2] G. Borg, “On a Liapounoff criterion of stability,” *Amer. J. Math.*, vol. 71, pp. 67–70, 1949, doi: 10.2307/2372093.

[3] S.-S. Cheng, “Lyapunov inequalities for differential and difference equations,” 1991, no. 23, pp. 25–41 (1992), third International Seminar on Ordinary Differential Equations (Poznań, 1990).

[4] P. L. De Nápoli and J. P. Pinasco, “Estimates for eigenvalues of quasilinear elliptic systems,” *J. Differential Equations*, vol. 227, no. 1, pp. 102–115, 2006, doi: 10.1016/j.jde.2006.01.004.

[5] R. A. C. Ferreira, "Some discrete fractional Lyapunov-type inequalities," *Fract. Differ. Calc.*, vol. 5, no. 1, pp. 87–92, 2015, doi: 10.7153/fdc-05-08.
[6] R. A. C. Ferreira, "Addendum to "Some discrete fractional Lyapunov-type inequalities" [Fract. Differ. Calc. **5** (2015), no. 1, 87-92] [MR3384641]," *Fract. Differ. Calc.*, vol. 8, no. 2, pp. 357–359, 2018, doi: 10.7153/fdc-2018-08-22.
[7] C. Goodrich and A. C. Peterson, *Discrete fractional calculus*. Springer, Cham, 2015. doi: 10.1007/978-3-319-25562-0.
[8] G. S. Guseinov and B. Kaymakçalan, "Lyapunov inequalities for discrete linear Hamiltonian systems," 2003, vol. 45, no. 6-9, pp. 1399–1416, advances in difference equations, IV, doi: 10.1016/S0898-1221(03)00095-6.
[9] G. S. Guseinov and A. Zafer, "Stability criteria for linear periodic impulsive Hamiltonian systems," *J. Math. Anal. Appl.*, vol. 335, no. 2, pp. 1195–1206, 2007, doi: 10.1016/j.jmaa.2007.01.095.
[10] P. Hartman, *Ordinary differential equations*. John Wiley & Sons, Inc., New York-London-Sydney, 1964.
[11] M. Holm, *The theory of discrete fractional calculus: Development and application*. ProQuest LLC, Ann Arbor, MI, 2011, thesis (Ph.D.)–The University of Nebraska - Lincoln.
[12] M. Jleli, J. J. Nieto, and B. Samet, "Lyapunov-type inequalities for a higher order fractional differential equation with fractional integral boundary conditions," *Electron. J. Qual. Theory Differ. Equ.*, pp. Paper No. 16, 17, 2017, doi: 10.14232/ejqtde.2017.1.16.
[13] A. Liapounoff, *Problème Général de la Stabilité du Mouvement*, ser. Annals of Mathematics Studies, no. 17. Princeton University Press, Princeton, N. J.; Oxford University Press, London, 1947.
[14] B. López, J. Rocha, and K. Sadarangani, "Lyapunov type inequality for a nonlinear fractional hybrid boundary value problem," *Z. Anal. Anwend.*, vol. 38, no. 1, pp. 97–106, 2019, doi: 10.4171/ZAA/1629.
[15] Q. Ma, C. Ma, and J. Wang, "A Lyapunov-type inequality for a fractional differential equation with Hadamard derivative," *J. Math. Inequal.*, vol. 11, no. 1, pp. 135–141, 2017, doi: 10.7153/jmi-11-13.
[16] Z. Nehari, "Some eigenvalue estimates," *J. Analyse Math.*, vol. 7, pp. 79–88, 1959, doi: 10.1007/BF02787681.
[17] W. T. Reid, "A matrix equation related to a non-oscillation criterion and Liapunov stability," *Quart. Appl. Math.*, vol. 23, pp. 83–87, 1965, doi: 10.1090/qam/176997.
[18] B. Singh, "Forced oscillations in general ordinary differential equations," *Tamkang J. Math.*, vol. 6, pp. 5–11, 1975.
[19] V. Slyn'ko and C. Tunç, "Stability of abstract linear switched impulsive differential equations," *Automatica J. IFAC*, vol. 107, pp. 433–441, 2019, doi: 10.1016/j.automatica.2019.06.001.
[20] C. Tunç and O. Tunç, "On the boundedness and integration of non-oscillatory solutions of certain linear differential equations of second order," *Journal of Advanced Research*, vol. 7, no. 1, pp. 165–168, 2016, doi: https://doi.org/10.1016/j.jare.2015.04.005.
[21] A. Wintner, "On the non-existence of conjugate points," *Amer. J. Math.*, vol. 73, pp. 368–380, 1951, doi: 10.2307/2372182.

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