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ON A REDUCTION OF A NONLINEAR AUTONOMOUS BOUNDARY-VALUE PROBLEM TO A CRITICAL CASE OF THE FIRST ORDER

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Abstract. We found constructive conditions of solvability and the scheme for constructing solutions of the nonlinear autonomous boundary-value problem in the case of multiple solutions of the equation for generating constants. The convergent iterative scheme was constructed for finding approximations to solutions of the nonlinear autonomous boundary-value problem for the system of ordinary differential equations in the case of multiple solutions of the equation for generating constants. We found the approximations to solutions of the periodic boundary-value problem for autonomous equation of Duffing type as an example of applying the constructed iterative scheme. We applied residuals in the original equation for controlling the accuracy of the found approximations to solutions of the periodic boundary-value problem for the autonomous equation of Duffing type.

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1. INTRODUCTION

Traditionally, the study of periodic and Noether boundary-value problems in critical cases was associated with the assumption that the differential equation, as well as the boundary condition, are known and fixed [1, 3, 17]. As a rule, the study of periodic problems in the case of parametric resonance was limited to the study of stability issues [18, 20]. Studies of linear autonomous boundary-value problems lead to the study of boundary-value problems in the case of parametric resonance, since the replacement of the independent variable [1, 3, 17] used in the critical case defines a non-autonomous boundary-value problem with an additional unknown. The aims of this article is to use the results, obtained in the study of nonlinear boundary-value problems in the case of parametric resonance, to solve linear autonomous boundary-value problems. We investigate the problem of finding the solutions [1, 3, 17]

$$z(t,\varepsilon):\ z(\cdot,\varepsilon)\in\mathbb{C}^1[a,b(\varepsilon)],\ \ z(t,\cdot)\in\mathbb{C}[0,\varepsilon_0],\ \ b(\varepsilon)\in\mathbb{C}[0,\varepsilon_0]$$

of autonomous system of ordinary differential equations

$$dz(t,\varepsilon)/dt = Az(t,\varepsilon) + f + \varepsilon Z(z(t,\varepsilon),\varepsilon), \tag{1.1}$$

satisfying the boundary condition

$$\ell z(\cdot,\varepsilon) = \alpha + \varepsilon J(z(\cdot,\varepsilon),\varepsilon), \quad \alpha \in \mathbb{R}^m. \tag{1.2}$$

We will seek the solutions of boundary-value problem (1.1), (1.2) in a small neighborhood of a solution

$$z_0(t) \in \mathbb{C}^1[a,b_0], \quad b_0 = b(0)$$

of the generating Noether $(m \neq n)$ boundary-value problem

$$dz_0/dt = Az_0 + f, \quad f \in \mathbb{R}^n, \quad \ell z_0(\cdot) = \alpha. \tag{1.3}$$

Here, A is an $(n\times n)$ constant matrix, $Z(z,\varepsilon)$ is a nonlinear function which is twice continuously differentiable with respect to the unknown $z(t,\varepsilon)$ and continuously differentiable with respect to a small parameter ε in a small neighborhood of a solution of the generating problem and on the segment $[0,\varepsilon_0]$; $\ell z(\cdot,\varepsilon)$ and $J(z(\cdot,\varepsilon),\varepsilon)$ are, respectively, a linear and nonlinear vector functionals $\ell z(\cdot,\varepsilon), J(z(\cdot,\varepsilon),\varepsilon) : \mathbb{C}[a,b(\varepsilon)] \to \mathbb{R}^m$. Moreover, the second functional is twice continuously differentiable with respect to the unknown $z(t,\varepsilon)$ and continuously differentiable with respect to the small parameter ε in a small neighborhood of a solution of the generating problem and on the segment $[0,\varepsilon_0]$. The problem posed continues the study of a nonlinear autonomous boundary-value problem for a system of ordinary differential equations, including in the case of parametric resonance [1, 3, 6, 17, 18, 20]. In the critical case, $(P_{Q^*} \neq 0)$ provided

$$P_{Q_d^*}\{\alpha - \ell K[f](\cdot)\} = 0 \tag{1.4}$$

the generating problem (1.3) possesses a family of solutions [3]

$$z_0(t,c_r) = X_r(t)c_r + G[f;\alpha](t), \quad X_r(t) = X(t)P_{Q_r}, \quad c_r \in \mathbb{R}^r.$$

Here, $Q := \ell X(\cdot)$ is an $(m\times n)$ matrix, rank $Q := n_1$, $n - n_1 = r$, P_{Q^*} – $(m\times m)$ is an matrix-orthoprojector

$$P_{Q^*} : \mathbb{R}^m \to \mathbb{N}(Q^*),$$

$X(t)$ is the normal $(X(a) = I_n)$ fundamental matrix of the homogeneous part of differential system (1.3); P_{Q_r} is the $(n\times r)$ matrix formed by r linearly independent columns of the $(n\times n)$ matrix-orthoprojector $P_Q : \mathbb{R}^n \to \mathbb{N}(Q)$;

$$G[f;\alpha](t) = X(t)Q^+\{\alpha - \ell K[f](\cdot)\} + K[f](t)$$

is the generalized Green operator of the boundary-value problem (1.3), Q^+ is the Moore–Penrose pseudoinverse matrix [3],

$$K[f](t) = X(t)\int_a^t X^{-1}(s) f ds$$

is the Green operator of the Cauchy problem for differential system (1.3), I_n is the identity $(n\times n)$ matrix; $P_{Q_d^*}$ is the matrix formed by d linearly independent rows of the matrix-orthoprojector P_{Q^*}.

Suppose that the equation for generating constants for a nonlinear autonomous boundary-value problem for a system of ordinary differential equations (1.1), (1.2) has multiple roots [1,3,17]. In this case, the study for an autonomous boundary-value problem for the system of ordinary differential equations (1.1), (1.2) with respect to scheme [1, 3] is not possible. Along with the autonomous boundary-value problem for the system of ordinary differential equations (1.1), (1.2) we consider the problem of finding solutions [1, 3, 17]

$$z(t,\varepsilon):\ z(\cdot,\varepsilon)\in\mathbb{C}^1[a,b(\varepsilon)],\ \ z(t,\cdot)\in\mathbb{C}[0,\varepsilon_0],\ \ b(\varepsilon),\ h(\varepsilon)\in\mathbb{C}[0,\varepsilon_0]$$

of an autonomous system of ordinary differential equations in the case of parametric resonance [6, 18]

$$dz(t,\varepsilon)/dt = Az(t,\varepsilon)+f+\varepsilon\,\mathcal{Z}(z(t,\varepsilon),h(\varepsilon),\varepsilon), \tag{1.5}$$

satisfying the boundary condition

$$\ell z(\cdot,\varepsilon)=\alpha+\varepsilon J(z(\cdot,\varepsilon),h(\varepsilon),\varepsilon),\quad \alpha\in\mathbb{R}^m. \tag{1.6}$$

We will seek the solutions of boundary-value problem (1.5), (1.6) in a small neighborhood of a solution

$$z_0(t)\in\mathbb{C}^1[a,b_0],\quad b_0=b(0),\quad h_0=h(0)\in\mathbb{R}^q$$

of the generating Noether $(m\neq n)$ boundary-value problem

$$dz_0(t)/dt=Az_0(t)+f,\quad f\in\mathbb{R}^n,\quad \ell z_0(\cdot)=\alpha. \tag{1.7}$$

Here, $B(z(t,\varepsilon))$ is an nonlinear $(n\times q)$ matrix which is twice continuously differentiable with respect to the unknown $z(t,\varepsilon)$ in a small neighborhood of the generating solution

$$\mathcal{Z}(z(t,\varepsilon),h(\varepsilon),\varepsilon):=Z(z(t,\varepsilon),\varepsilon)+B(z(t,\varepsilon))\,h(\varepsilon),$$

is a nonlinear function which is twice continuously differentiable with respect to the unknowns $z(t,\varepsilon)$ and $h(\varepsilon)$ in a small neighborhood of the generating solution and continuously differentiable with respect to the small parameter ε on the segment $[0,\varepsilon_0]$; $\ell z(\cdot,\varepsilon)$ and $J(z(\cdot,\varepsilon),h(\varepsilon),\varepsilon)$ are, respectively, a linear and nonlinear vector functionals $\ell z(\cdot,\varepsilon)$, $J(z(\cdot,\varepsilon),h(\varepsilon),\varepsilon):\mathbb{C}[a,b(\varepsilon)]\to\mathbb{R}^m$. Moreover, the second functional is twice continuously differentiable with respect to the unknown $z(t,\varepsilon)$ in a small neighborhood of a solution of the generating problem (1.7) and continuously differentiable with respect to the small parameter ε in a small neighborhood of a solution of the generating problem and on the segment $[0,\varepsilon_0]$. Note that the solutions of the generating problem (1.7) and problem (1.3), as well as the problems themselves, coincide. In the critical case, the problem (1.5), (1.6) differs significantly from similar non-autonomous boundary-value problems; unlike the latter, the right end $b(\varepsilon)$

in the interval $[a, b(\varepsilon)]$, on which we are looking for a solution to the problem (1.5), (1.6), unknown and to be determined in the process of constructing a solution.

The change of independent variables [1]

$$t = a + (\tau - a)(1 + \varepsilon\beta(\varepsilon)), \;\; b(\varepsilon) = b^* + \varepsilon(b^* - a)\beta(\varepsilon), \;\; \beta(\varepsilon) \in \mathbb{C}[0, \varepsilon_0], \;\; \beta(0) = \beta^*,$$

in problem (1.5), (1.6) we obtain the problem of finding a solution

$$z(\cdot, \varepsilon) \in \mathbb{C}^1[a, b_0], \;\; z(\tau, \cdot) \in \mathbb{C}[0, \varepsilon_0], \;\; \beta(\varepsilon), \; h(\varepsilon) \in \mathbb{C}[0, \varepsilon_0]$$

to a system of differential equations

$$\begin{aligned} dz(\tau, \varepsilon)/d\tau = {} & Az(\tau, \varepsilon) + f + \varepsilon\, Z(z(\tau, \varepsilon), h(\varepsilon), \varepsilon) \\ & + \varepsilon\beta(\varepsilon)[Az(\tau, \varepsilon) + \varepsilon\, Z(z(\tau, \varepsilon), h(\varepsilon), \varepsilon)], \end{aligned}$$

that satisfy the boundary condition

$$\ell z(\cdot, \varepsilon) = \alpha + \varepsilon J(z(\cdot, \varepsilon), h(\varepsilon), \varepsilon) + \varepsilon\beta(\varepsilon)[\alpha + \varepsilon J(z(\cdot, \varepsilon), h(\varepsilon), \varepsilon)].$$

Next, we change the variable

$$z(\tau, \varepsilon) = z_0(\tau, c_r) + x(\tau, \varepsilon), \quad h(\varepsilon) = h_0 + \mu(\varepsilon), \quad \beta(\varepsilon) = \beta_0 + \eta(\varepsilon),$$

and arrive at the problem of finding a solution

$$x(\cdot, \varepsilon) \in \mathbb{C}^1[a, b_0], \;\; x(\tau, \cdot) \in \mathbb{C}[0, \varepsilon_0], \;\; \mu(\varepsilon), \; \eta(\varepsilon) \in \mathbb{C}[0, \varepsilon_0]$$

to a system of differential equations

$$dx(\tau, \varepsilon)/d\tau = Ax(\tau, \varepsilon) + \varepsilon Y(z_0(\tau, c_r) + x(\tau, \varepsilon), h_0 + \mu(\varepsilon), \beta_0 + \eta(\varepsilon), \varepsilon), \tag{1.8}$$

that satisfy the boundary condition

$$\ell x(\cdot, \varepsilon) = \varepsilon H(z_0(\cdot, c_r) + x(\cdot, \varepsilon), h_0 + \mu(\varepsilon), \beta_0 + \eta(\varepsilon), \varepsilon). \tag{1.9}$$

Here,

$$\begin{aligned} & Y(z_0(\tau, c_r) + x(\tau, \varepsilon), h_0 + \mu(\varepsilon), \beta_0 + \eta(\varepsilon), \varepsilon) \\ & := (1 + \varepsilon\beta(\varepsilon)) \times Z(z_0(\tau, c_r) + x(\tau, \varepsilon), h_0 + \mu(\varepsilon), \varepsilon) + \beta(\varepsilon) A\,(z_0(\tau, c_r) + x(\tau, \varepsilon)), \\ & H(z_0(\cdot, c_r) + x(\cdot, \varepsilon), h_0 + \mu(\varepsilon), \beta_0 + \eta(\varepsilon), \varepsilon) \\ & := (1 + \varepsilon\beta(\varepsilon)) \times J(z_0(\cdot, c_r) + x(\cdot, \varepsilon), h_0 + \mu(\varepsilon), \varepsilon) + \alpha\beta(\varepsilon) : \; \mathbb{C}[a, b_0] \to \mathbb{R}^m. \end{aligned}$$

In the critical case, $(P_{Q^*} \neq 0)$ provided that (1.4) the boundary-value problem (1.8), (1.9) solvable under the condition

$$\begin{aligned} & P_{Q_d^*}\{H(z_0(\cdot, c_r) + x(\cdot, \varepsilon), h_0 + \mu(\varepsilon), \eta(\varepsilon), \varepsilon) \\ & \quad - \ell K[Y(z_0(\tau, c_r) + x(\tau, \varepsilon), h_0 + \mu(\varepsilon), \beta_0 + \eta(\varepsilon), \varepsilon)](\cdot)\} = 0. \end{aligned} \tag{1.10}$$

We denote vectors

$$\check{c}_0 := \begin{bmatrix} c_r \\ \beta_0 \\ h_0 \end{bmatrix} \in \mathbb{R}^{r+q+1}, \;\; \check{c}_0^* := \begin{bmatrix} c_r^* \\ \beta_0^* \\ h_0^* \end{bmatrix} \in \mathbb{R}^{r+q+1}, \;\; \check{c}(\varepsilon) := \begin{bmatrix} c_r(\varepsilon) \\ \mu(\varepsilon) \\ \eta(\varepsilon) \end{bmatrix} \in \mathbb{R}^{r+q+1}.$$

Since the nonlinear function $Y(z,\beta(\varepsilon),h(\varepsilon),\tau,\varepsilon)$ and the nonlinear vector functional $H(z(\cdot,\varepsilon),h(\varepsilon),\beta(\varepsilon),\varepsilon)$ are continuous with respect to z, β and h in a small neighborhood of the generating problem (1.7), in a small neighborhood of the initial value h_0 of the function $h(\varepsilon)$ and the initial value β_0 of the function $\beta(\varepsilon)$, we arrive at the following equation

$$\mathcal{F}(\check{c}_0) := P_{Q_d^*}\Big\{\alpha\beta_0 + J(z_0(\cdot,c_r),h_0,0) \\ - \ell K\big[\beta_0 A z_0(s,c_r) + \mathcal{Z}(z_0(s,c_r),h_0,s,0)\big](\cdot)\Big\} = 0.$$

Thus, the necessary conditions of the existence of a solution to the nonlinear autonomous boundary-value problem (1.5), (1.6) in the case of parametric resonance is determined by the following lemma. This lemma is a generalization of the corresponding statements from [1, 3] on the case of parametric resonance and from [6] on the case of an autonomous boundary-value problem (1.5), (1.6).

Lemma 1. *Suppose that the boundary-value problem (1.5), (1.6) corresponds to the critical case* $(P_{Q^*} \neq 0)$ *and the solvability condition of generating Noether* $(m \neq n)$ *problem (1.7) is satisfied condition (1.4). Also, suppose that in a small neighborhood of the generating solution*

$$z_0(t,c_r^*) \in \mathbb{C}^1[a,b_0^*], \quad b_0^* = b(0), \quad h_0^* = h(0) \in \mathbb{R}^q$$

the weakly nonlinear boundary-value problem (1.5), (1.6) has a solution

$$z(t,\varepsilon):\ z(\cdot,\varepsilon) \in \mathbb{C}^1[a,b(\varepsilon)],\ \ z(t,\cdot) \in \mathbb{C}[0,\varepsilon_0],\ \ b(\varepsilon),\ h(\varepsilon) \in \mathbb{C}[0,\varepsilon_0].$$

Moreover, in a sufficiently small neighborhood of the vector h_0^* *there exists an eigenfunction* $h(\varepsilon) \in \mathbb{C}[0,\varepsilon_0]$. *Then the equality*

$$\mathcal{F}(\check{c}_0^*) = 0. \tag{1.11}$$

is satisfied.

By analogy with weakly nonlinear boundary-value problems in the critical case [1, 3] and periodic boundary-value problems [20], we say that equation (1.11) is an equation for generating constants of the boundary-value problem (1.5), (1.6) in the case of parametric resonance. Assume that equation (1.11) has real root. We fix one of the solutions $\check{c}_0 \in \mathbb{R}^{r+q+1}$ of equation (1.11) and arrive at the problem of finding the solutions

$$x(\cdot,\varepsilon) \in \mathbb{C}^1[a,b_0],\ \ x(\tau,\cdot) \in \mathbb{C}[0,\varepsilon_0],\ \ \mu(\varepsilon),\ \eta(\varepsilon) \in \mathbb{C}[0,\varepsilon_0]$$

of the problem (1.8), (1.9) in the neighborhood of the generating solution

$$z_0(t,c_r^*) = X_r(t)\,c_r^* + G[f;\alpha](t), \quad c_r^* \in \mathbb{R}^r,$$

and finding functions

$$h(\varepsilon) := h_0^* + \mu(\varepsilon),\ \ \beta(\varepsilon) = \beta_0^* + \eta(\varepsilon), \quad \mu(\varepsilon),\ \eta(\varepsilon) \in C[0,\varepsilon_0]$$

in the neighborhood of points h_0^* and β_0^*.

2. On a reduction of an autonomous Noether boundary-value problem to a critical case of the first order

In the critical case ($P_{Q^*} \neq 0$) under condition (1.4) to construct an operator system that is used to solve the boundary-value problem (1.8), (1.9), the linearization of the solvability condition was previously applied (1.10). Wherein, in the case of simplicity of the roots [3, 12] by the equations of the generating constants (1.11)

$$P_{C_0^*(\check{c}_0^*)} P_{Q_d^*} = 0, \quad C_0\left(\check{c}_0^*\right) = \mathcal{F}'_{\check{c}(\varepsilon)}\left(\check{c}(\varepsilon)\right)\Big|_{\ \check{c}(\varepsilon) = \check{c}_0^*}$$

solution were constructed using the method of simple iterations [3, 6]. Note that the solutions of the boundary-value problem (1.5), (1.6) in the case of parametric resonance and the problem (1.1), (1.2), in the case of their existence, as well as the problems themselves, coincide for $h(\varepsilon) = 0$. To find a solution

$$x(\tau,\varepsilon) = X_r(\tau)c_r(\varepsilon) + x^{(1)}(\tau,\varepsilon), \qquad h(\varepsilon) = h_0^* + \mu(\varepsilon),$$

$$x^{(1)}(\tau,\varepsilon) = \varepsilon \cdot G\Big[Y(z_0(s,c_r^*) + x(s,\varepsilon), h_0^* + \mu(\varepsilon), \beta_0^* + \eta(\varepsilon), \varepsilon);$$

$$H(z_0(\cdot,c_r^*) + x(\cdot,\varepsilon), h_0^* + \mu(\varepsilon), \beta_0^* + \eta(\varepsilon), \varepsilon)\Big](\tau)$$

of the boundary value-problem (1.8), (1.9) the Newton–Kantorovich method can be used [8, 13].

Lemma 2. *Suppose that the following conditions are fulfilled for the equation*

$$\mathcal{F}(\check{c}(\varepsilon)) = 0. \tag{2.1}$$

1. *The nonlinear vector function $\mathcal{F}(\check{c}(\varepsilon)) : \mathbb{R}^{r+q+1} \to \mathbb{R}^d$ is twice continuously differentiable in a neighborhood of zero and has the root $\check{c}(\varepsilon)$.*
2. *In a neighborhood of zero, the following inequalities hold:*

$$\left|\left|J_j^+\right|\right| \leq \sigma_1(j), \quad \left|\left|d^2\mathcal{F}(\xi_j; \check{c}(\varepsilon) - \check{c}_j(\varepsilon))\right|\right| \leq \sigma_2(j) \cdot ||\check{c}(\varepsilon) - \check{c}_j(\varepsilon)||.$$

3. *There is a constant*

$$\theta := \sup_{j\in N}\left\{\frac{\sigma_1(j)\sigma_2(j)}{2}\right\}.$$

Then, under the conditions

$$P_{J_j^*} = 0, \quad J_j := \mathcal{F}'(\check{c}_j(\varepsilon)) \in \mathbb{R}^{d\times(r+q+1)}, \quad \theta \cdot |\check{c}(\varepsilon)| < 1 \tag{2.2}$$

or finding a solution $\check{c}(\varepsilon)$ of the equation (2.1) an iterative scheme can be used

$$\check{c}_{j+1}(\varepsilon) = \check{c}_j(\varepsilon) - J_j^+ \mathcal{F}(\check{c}_j(\varepsilon)), \quad j = 0,\ 1,\ 2,\ \ldots\ . \tag{2.3}$$

In this case, the rate of convergence of the sequence $\{\check{c}_j(\varepsilon)\}$ *to the solution* $\check{c}(\varepsilon)$ *of the equation (2.1) will be quadratic. Here,* $P_{J_j^*}: \mathbb{R}^d \to \mathbb{N}(J_j^*)$ *is an matrix-orthoprojector* J_k^*.

Note that the condition (2.2) is equivalent to the requirement of completeness of the rank of the matrix J_k and is possible only in the case $d \le p+q+1$.

Theorem 1. *Suppose that for the generating problem (1.7) corresponds to the critical case* $P_{Q^*} \neq 0$ *and the condition of its solvability is satisfied (1.4). We also assume that the conditions of Lemma 2 are satisfied for the equation (2.1). Then, for each root* $c_r^* \in \mathbb{R}^r$, $h_0^* \in \mathbb{R}^q$ *of the equation for the generating constants (1.11) in the neighborhood generating solution* $z_0(\varepsilon, c_r^*)$, *and in the neighborhood of the points* h_0^* *and* β_0^* *to find at least one solution*

$$z(\cdot,\varepsilon) \in \mathbb{C}^1[a, b(\varepsilon)], \quad z(\tau,\cdot) \in \mathbb{C}[0,\varepsilon_0], \quad h(\varepsilon),\ \beta(\varepsilon) \in \mathbb{C}[0,\varepsilon_0]$$

of the problem (1.5), (1.6) in the case of parametric resonance, an iterative scheme can be used

$$\begin{gathered}
z_{k+1}(\tau,\varepsilon) = z_0(\tau, c_r^*) + x_{k+1}(\tau,\varepsilon),\ \beta_{k+1}(\varepsilon) = \beta_0^* + \eta_{k+1}(\varepsilon),\ h_{k+1}(\varepsilon) = h_0^* + \mu_{k+1}(\varepsilon), \\
x_{k+1}(\tau,\varepsilon) = X_r(\tau) c_{r_{k+1}}(\varepsilon) + x_{k+1}^{(1)}(\tau,\varepsilon), \quad \mathcal{F}(\check{c}_{k+1}(\varepsilon)) = 0, \quad k = 0,\ 1,\ 2,\ \ldots, \qquad (2.4) \\
x_{k+1}^{(1)}(\tau,\varepsilon) = \varepsilon \cdot G\Big[Y(z_0(s,c_r^*) + x_k(s,\varepsilon), h_0^* + \mu_k(\varepsilon), \beta_0^* + \eta_k(\varepsilon), \varepsilon); \\
H(z_0(\cdot, c_r^*) + x_k(\cdot,\varepsilon), h_0^* + \mu_k(\varepsilon), \beta_0^* + \eta_k(\varepsilon), \varepsilon)\Big](\tau).
\end{gathered}$$

Returning to the nonlinear autonomous boundary-value problem for the system of ordinary differential equations (1.1), (1.2), suppose that the equation for generating constants for an autonomous Noether boundary-value problem for a system of ordinary differential equations (1.1), (1.2) has multiples roots [1,3,17], while equations for generating constants (1.11) for the boundary-value problem (1.5), (1.6) in the case of parametric resonance has simple roots:

$$P_{C_0^*(\check{c}_0^*)} P_{Q_d^*} = 0,$$

among which $\check{c}_0^* \in \mathbb{R}^{r+q+1}$, $h_0^* = 0$. Suppose also that at each step of the iterative scheme (2.4) among the roots of the equation $\mathcal{F}(\check{c}_k(\varepsilon)) = 0$ is a root for which $\mu_k(\varepsilon) = 0$. In this case, the iteration scheme (2.4) can be used to find solutions an autonomous boundary-value problem (1.1), (1.2). Therefore, solutions to the boundary-value problem (1.5), (1.6) in the case of parametric resonance and the problem (1.1), (1.2), if they exist, coincide for $h(\varepsilon) = 0$.

Corollary 1. *Assume that, to find at least one solution*

$$z(\cdot,\varepsilon) \in \mathbb{C}^1[a, b(\varepsilon)], \quad z(\tau,\cdot) \in \mathbb{C}[0,\varepsilon_0], \quad h(\varepsilon),\ \beta(\varepsilon) \in \mathbb{C}[0,\varepsilon_0]$$

of the problem (1.5), (1.6) in the case of parametric resonance, an iterative scheme can be used (2.4). Assume also that, among the roots of the equation for generating constants (1.11) for the boundary-value problem (1.5), (1.6) in the case of parametric resonance is the simple root

$$\check{c}_0^* \in \mathbb{R}^{r+q+1}, \quad h_0^* = 0.$$

Suppose also that, at each step of the iterative scheme (2.4) among the roots of the equation

$$\mathcal{F}(\check{c}_k(\varepsilon)) = 0, \quad k = 0,\ 1,\ 2,\ \ldots$$

there is a root for which $\mu_k(\varepsilon) = 0$. The iterative scheme (2.4) can be used to know the solutions autonomous boundary-value problem (1.1), (1.2) in this case.

If the equation for generating constants for an autonomous boundary-value problem for a system of ordinary differential equations (1.1), (1.2) has multiple roots [1, 3, 17], for this equation for generating constants (1.11) for the boundary-value problem (1.5), (1.6) in the case of parametric resonance has simple roots, among which $\check{c}_0^* \in \mathbb{R}^{r+q+1}$, $h_0^* = 0$. The conditions of the proved corollary are also satisfied, similarly [1, 3, 9, 15], then we will say that an autonomous Noether boundary-value problem (1.1), (1.2) is reduced to a critical case of the first order.

3. Autonomous periodic problem for the Duffing equation

The conditions of the proven corollary are satisfied in the case of an autonomous periodic problem for the Duffing equation [17]

$$y'' + y = \varepsilon y^3. \tag{3.1}$$

The generating periodic problem for the Duffing equation (3.1) is solvable and, with an appropriate fixation of the origin of the independent variable, has a general solution

$$z_0(t, c_r) = X_r(t)\, c_r, \quad X_r(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}, \quad c_r \in \mathbb{R}^1.$$

Equation for generating amplitudes in the case of an autonomous periodic problem for the Duffing equation (3.1)

$$\frac{\pi c_r}{4} \begin{pmatrix} 3\,c_r^2 - 8\beta_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

obviously has only multiple roots:

$$B_0 = \frac{\pi c_r^*}{2} \begin{pmatrix} 3c_r^* & -4 \\ 0 & 0 \end{pmatrix}, \quad \beta^* = \frac{3\,c_r^*}{8}.$$

Along with the autonomous periodic problem for the equation (3.1) consider the problem of finding periodic solutions to the equation Duffing-type with parametric perturbation

$$y'' + y = \varepsilon y^3 + \varepsilon h(\varepsilon)\,(y' + y'^3). \tag{3.2}$$

Note that, in contrast to the article [6] the Duffing-type equation (3.2) with parametric perturbation is autonomous. The equation (3.2) is reduced to the form (1.5) for

$$z(t,\varepsilon)=\begin{bmatrix} z^{(a)}(t,\varepsilon)\\ z^{(b)}(t,\varepsilon)\end{bmatrix},\quad A=\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix},$$

$$Z(z(t,\varepsilon),h(\varepsilon),t,\varepsilon)=\begin{bmatrix} 0\\ h(\varepsilon)z^{(b)}(t,\varepsilon)+h(\varepsilon)\Big(z^{(b)}(t,\varepsilon)\Big)^3+\Big(z^{(a)}(t,\varepsilon)\Big)^3\end{bmatrix}.$$

The generating periodic problem for a Duffing-type equation (3.2) with parametric perturbation is also solvable with a corresponding fixation of the origin of the independent variable and has a general solution [17]

$$z_0(t,c_r)=X_r(t)\,c_r,\quad X_r(t)=\begin{bmatrix} \cos t\\ -\sin t\end{bmatrix},\quad c_r\in\mathbb{R}^1.$$

The equation for generating constants (1.11), in the case of a periodic problem for an equation of Duffing-type (3.2) with parametric perturbation

$$\pi c_r(3c_r^2-8\beta_0)=0,\quad \pi c_r h_0(3c_r^2+4)=0,$$

has a simple root [1, 3, 6, 12]

$$c_r^*=\frac{1}{10},\quad \beta_0^*=\frac{3}{800},\quad h_0^*=0,\quad C_0\Big(\check{c}_0^*\Big)=\frac{\pi}{4000}\begin{bmatrix} 0 & -800 & 0\\ 403 & 0 & 403\end{bmatrix}.$$

At the first step of the iterative scheme (2.4) due to the completeness of the rank of the matrix J_0 the condition (2.2) is satisfied. Assuming that $\varepsilon := 0,1$. The first approximation to the solution of the periodic problem for the equation (3.2)

$$y_1(\tau,\varepsilon)=c_{r_1}(\varepsilon)\cos\tau+y_1^{(1)}(\tau,\varepsilon),\quad h_1(\varepsilon)=h_0^*+\mu_1(\varepsilon)$$

define functions

$$y_1^{(1)}(\tau,\varepsilon)=\frac{\varepsilon}{32\,000}\Big(\cos\tau-\cos 3\tau\Big),$$

as well as

$$c_{r_1}(\varepsilon)\approx-\frac{8413}{25\,079\,771\,422},\quad \beta_1(\varepsilon)\approx\frac{7\,140\,587}{1\,903\,038\,646},\quad h_1(\varepsilon)=\mu_1(\varepsilon)=0.$$

At the second step of the iterative scheme (2.4) due to the completeness of the rank of the matrix J_1 the condition (2.2) is also satisfied, wherein

$$y_2(\tau,\varepsilon)=c_{r_2}(\varepsilon)\cos\tau+y_2^{(1)}(\tau,\varepsilon),\quad h_2(\varepsilon)=h_0^*+\mu_2(\varepsilon);$$

here

$$y_2^{(1)}(\tau,\varepsilon)\approx\frac{155\,839\cos\tau}{49\,833\,136\,807}-\frac{1\,12138\cos 3\tau}{35\,857\,607\,281}+\frac{67\cos 5\tau}{685\,548\,603\,382}$$
$$-\frac{\cos 7\tau}{654\,850\,200\,560\,485}-\frac{\cos 9\tau}{104\,778\,954\,869\,870\,365\,088},$$

as well as

$$c_{r_2}(\varepsilon) \approx -\frac{34\,312}{102\,284\,783\,173}, \quad \beta_2(\varepsilon) \approx \frac{31\,802\,653}{8\,475\,728\,456}, \quad h_2(\varepsilon) = \mu_2(\varepsilon) = 0.$$

At the third step of the iterative scheme (2.4) due to the completeness of the rank of the matrix J_2 the condition (2.2) is also satisfied, wherein

$$y_3(\tau,\varepsilon) = c_{r_3}(\varepsilon)\cos\tau + y_3^{(1)}(\tau,\varepsilon), \quad h_3(\varepsilon) = h_0^* + \mu_3(\varepsilon);$$

here

$$y_3^{(1)}(\tau,\varepsilon) \approx \frac{149\,943\cos\tau}{47\,947\,755\,026} - \frac{106\,169\cos 3\tau}{33\,948\,940\,663} + \frac{447\cos 5\tau}{4\,570\,493\,090\,387} + \frac{\cos 7\tau}{327\,064\,274\,993\,900} + \frac{\cos 9\tau}{12\,308\,773\,007\,388\,528\,938},$$

as well as

$$c_{r_3}(\varepsilon) \approx -\frac{39\,176}{116\,784\,468\,121}, \quad \beta_3(\varepsilon) \approx \frac{1\,318\,533}{351\,402\,371}, \quad h_3(\varepsilon) = \mu_3(\varepsilon) = 0.$$

The found zero-order and first three approximations to the periodic solution of the Duffing-type equation (3.2) with parametric perturbation and the function $h(\varepsilon)$ are characterized by the discrepancies

$$\Delta_k(\varepsilon) = \Big\| y_k''(t,\varepsilon) + y_k(t,\varepsilon) - \varepsilon y_k^3(t,\varepsilon) - \varepsilon h_k(\varepsilon) y_k'(t,\varepsilon) - \varepsilon h_k(\varepsilon) y_k'^3(t,\varepsilon) \Big\|_{\mathbb{C}[0;2\pi(1+\varepsilon\beta_k(\varepsilon))]}, \quad k = 0,\ 1,\ 2,\ 3.$$

In particular, for $\varepsilon = 0,1$ we get

$$\Delta_0(0,1) \approx 0,0000\,25, \qquad \Delta_1(0,1) \approx 1,99\,328 \times 10^{-8},$$
$$\Delta_2(0,1) \approx 1,72\,064 \times 10^{-12}, \qquad \Delta_3(0,1) \approx 1,03\,067 \times 10^{-16}.$$

The found zero-order and first three approximations to the periodic solution of the Duffing equation (3.1) characterize the discrepancies

$$\delta_k(\varepsilon) = \Big\| y_k''(t,\varepsilon) + y_k(t,\varepsilon) - \varepsilon y_k^3(t,\varepsilon) \Big\|_{\mathbb{C}[0;2\pi(1+\varepsilon\beta_k(\varepsilon))]}, \quad k = 0,\ 1,\ 2,\ 3.$$

In particular, for $\varepsilon = 0,1$ we get

$$\delta_0(0,1) \approx 0,000\,025\,061, \qquad \delta_1(0,1) \approx 1,99\,328 \times 10^{-8},$$
$$\delta_2(0,1) \approx 1,72\,065 \times 10^{-12}, \qquad \delta_3(0,1) \approx 1,249 \times 10^{-16}.$$

We now compare the obtained zero-order and first three approximations to the periodic solution of the Duffing-type equation (3.1) with the corresponding zero-order

$$y_{0p}(\tau, c_r^*) = \frac{\cos\tau}{10}, \quad \beta_0^* = \frac{3}{800}, \quad h_0^* = 0,$$

and first three approximations obtained by the Poincare method used to find the periodic solution of the same Duffing-type equation (3.1)

$$y_{1p}(\tau,\varepsilon) = y_0(\tau,c_r^*) + \varepsilon\, w_1(\tau), \quad \beta_{1p}(\varepsilon) = \frac{3}{800} + \frac{57\,\varepsilon}{2\;560\;000},$$

$$y_{2p}(\tau,\varepsilon) = y_0(\tau,c_r^*) + \varepsilon\, w_1(\tau) + \varepsilon^2 w_2(\tau),\ \beta_{2p}(\varepsilon) = \frac{3}{800} + \frac{57\,\varepsilon}{2\;560\;000} + \frac{63\,\varepsilon^2}{409\;600\;000},$$

$$y_{3p}(\tau,\varepsilon) = y_0(\tau,c_r^*) + \varepsilon\, w_1(\tau) + \varepsilon^2 w_2(\tau) + \varepsilon^3 w_3(\tau),$$

$$\beta_{3p}(\varepsilon) = \frac{3}{800} + \frac{57\,\varepsilon}{2\;560\;000} + \frac{63\,\varepsilon^2}{409\;600\;000} + \frac{6\;069\,\varepsilon^3}{5\;242\;880\;000\;000};$$

here

$$w_1(\tau) = \frac{1}{32\;000}\Big(\cos\tau - \cos 3\tau\Big),$$

$$w_2(\tau) = \frac{1}{102\;400\;000}\Big(23\cos\tau - 24\cos 3\tau + \cos 5\tau\Big),$$

$$w_3(\tau) = \frac{1}{327\;680\;000\;000}\Big(547\cos\tau - 594\cos 3\tau + 48\cos 5\tau - \cos 7\tau\Big).$$

Note that the zero-order and first three approximations obtained by the Poincare method for the periodic solution of the Duffing-type equation (3.1) are characterized by the discrepancies

$$\delta_{kp}(\varepsilon) = \Big|\Big| y''_{kp}(t,\varepsilon) + y_{kp}(t,\varepsilon) - \varepsilon\, y^3_{kp}(t,\varepsilon)\Big|\Big|_{\mathbb{C}[0;2\pi(1+\varepsilon\beta_{kp}(\varepsilon))]}, \quad k = 0,\ 1,\ 2,\ 3.$$

In particular, for $\varepsilon = 0,1$ we get

$$\delta_{0p}(0,1) \approx 0{,}000\;025\;061, \qquad \delta_{1p}(0,1) \approx 2{,}01\;363 \times 10^{-8},$$

$$\delta_{2p}(0,1) \approx 1{,}63\;890 \times 10^{-11}, \qquad \delta_{3p}(0,1) \approx 1{,}45\;439 \times 10^{-14}.$$

Thus, the zero-order and first three approximations obtained according to the iterative scheme (2.4) aimed at finding the periodic solution of the Duffing-type equation (3.1) are much more exact than the first three approximations given by the Poincare method for the periodic solution of the same Duffing-type equation.

By analogy [2,5,7], the proposed scheme of investigation of the nonlinear autonomous boundary-value problems in the case of parametric resonance can be generalized to the matrix boundary-value problems.

The proposed scheme for studying nonlinear autonomous boundary-value problems in the case of parametric resonance can be used in electronics [18], geodesy [16], plasma theory [19], nonlinear optics, mechanics [4] and machine-tool industry [14], along with finding solutions, it is necessary to calculate the eigenfunction of the corresponding differential equation, and similarly [10, 11] can be transferred to nonlinear Noether boundary-value problems unsolved with respect to the derivative.

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