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HADAMARD HOMOMORPHISMS AND HADAMARD DERIVATIONS ON BANACH ALGEBRAS

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Abstract. In this paper, we prove the stability of Hadamard homomorphisms and Hadamard derivations in Banach algebras. This is applied to investigate Hadamard isomorphisms between Banach algebras.

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1. INTRODUCTION

In 1941, D. H. Hyers [3] gave the first partial solution to Ulam's question for the case of approximate additive mappings under the assumption that G and G' are Banach spaces. In 1978, Th. M. Rassias [14] generalized the theorem of Hyers [3] by considering the stability problem with unbounded Cauchy differences. In 1991, Z. Gajda [1], following the same approach as that by Th. M. Rassias [14] gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [1] as well as by Th. M¿ Rassias and P. Šemrl [15], that one cannot prove a Rassias-type theorem when $p = 1$. P. Găvruta [2] obtained the generalized result of the Rassias theorem which allows the Cauchy difference to be controlled by a general unbounded function.

The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4,5,7]. The method provided by D. H. Hyers [3] which produces the additive function will be called a direct method. This method is the most important and useful tool to study the stability of different functional equations.

During the last two decades, a number of articles and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings, for example, Cauchy-Jensen mappings, k-additive mappings, invariant means, multiplicative mappings, bounded

nth differences, convex functions, generalized orthogonality mappings, Euler-Lagrange functional equations, differential equations, and Navier-Stokes equations (see [9, 16–20]).

The theory of stability is a significant branch of the qualitative theory of differential equations and dynamical systems. During the recent decades many interesting results have been studied on some types differential equations (see [10, 11, 21]).

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results, containing ternary homomorphisms and ternary derivations, concerning this problem (see [8, 13]).

Let $(A, ||.||)$ be a normed linear space over real or complex number field. For $X = (x_1, x_2) \in A^2$, we define

$$||(x_1, x_2)||_\infty := \max\{||x_1||, ||x_2||\}.$$

Let A and B be complex vector spaces. A mapping $f : A \times A \longrightarrow B \times B$ is 2-additive if

$$f(x_1 + x_2, y_1 + y_2) = \sum_{i,j=1}^{2} f(x_i, y_j)$$

for all $x_1, y_1, x_2, y_2 \in A$, and a mapping $f : A \times A \to B \times B$ is called a $\mathbb{C}$-2-linear mapping if f is $\mathbb{C}$-linear for each variable.

Lemma 1. *[12] Let A and B be real or complex normed linear spaces. Then the mapping $f : A \times A \longrightarrow B \times B$ is 2-additive if and only if*

$$f(x_1 + x_2, y_1 - y_2) + f(x_1 - x_2, y_1 + y_2) = 2f(x_1, y_1) - 2f(x_2, y_2) \tag{1.1}$$

for all $x_1, x_2, y_1, y_2 \in A$.

2. Stability of Hadamard homomorpisms and Hadamard isomorphisms in complex Banach algebras

Let A and B be complex Banach algebras. Throughout this section, $A \times A$ (resp. $B \times B$) will be complex Banach algebra with norm $|| \cdot ||_\infty$ and unital (e, e) (resp. (e', e')).

Definition 1. Let A be a complex Banach algebra. For $X = (x_1, y_1), Y = (x_2, y_2) \in A^2$, the *inner Hadamard product* (entry-wise product) of X and Y is defined by

$$X \bullet_H Y = (x_1, y_1) \bullet_H (x_2, y_2) := (x_1 x_2, y_1 y_2).$$

Definition 2. Let A and B be two complex Banach algebras. A $\mathbb{C}$-2-linear mapping $\varphi : A \times A \to B \times B$ is called a *Hadamard homomorphism* if it satisfies

$$\varphi(X \bullet_H Y) = \varphi(X) \bullet_H \varphi(Y)$$

for all $X = (x_1, y_1), Y = (x_2, y_2) \in A^2$.

Example 1. Let $\varphi:\mathbb{R}^2\to\mathbb{R}^2$ be defined by $\varphi(x,y)=(xy,xy)$. Then φ is a Hadamard homomorphism.

Example 2. Let M_{mn} be the set of all $m\times n$ real matrices, and $\varphi:\mathbf{M_{mn}}\to\mathbf{M_{mn}}$ be defined by $\varphi(X,Y)=(X._HY,0)$ for $X=[x_{ij}],Y=[y_{ij}]\in M_{mn}$, where the Hadamard product (entry-wise product) of X and Y is defined by

$$X._HY=Z=[z_{ij}],\quad \text{where}\quad z_{ij}=x_{ij}y_{ij}\quad \text{for all}\quad 1\le i\le m, 1\le j\le n.$$

Then φ is a Hadamard homomorphism.

Lemma 2. *[12] Let A and B be complex Banach algebras and $f:A\times A\to B\times B$ be a 2-additive mapping such that $f(\mu x,\nu y)=\mu\nu f(x,y)$ for all $\mu,\nu\in\mathbb{T}^1:=\{\xi\in\mathbb{C}:|\xi|=1\}$ and all $x,y\in A$. Then f is 2-linear over $\mathbb{C}$.*

Proof. Let $\lambda,\beta\in\mathbb{C}$ be nonzero numbers and M,N be two integers such that $M>4|\lambda|$ and $N>4|\beta|$. Then $|\frac{\lambda}{M}|<\frac{1}{4}<1-\frac{2}{3}$ and $|\frac{\beta}{N}|<\frac{1}{4}<1-\frac{2}{3}$. By [6] Theorem 1, there exist $\mu_1,\mu_2,\mu_3,\nu_1,\nu_2,\nu_3\in\mathbb{T}^1$ such that $3\frac{\lambda}{M}=\mu_1+\mu_2+\mu_3$ and $3\frac{\beta}{N}=\nu_1+\nu_2+\nu_3$. Since f is 2-additive, $f(x,y)=f(3\cdot\frac{1}{3}x,3\cdot\frac{1}{3}y)=9f(\frac{1}{3}x,\frac{1}{3}y)$ for all $x,y\in A$. Thus $f(\frac{1}{3}x,\frac{1}{3}y)=\frac{1}{9}f(x,y)$. So

$$\begin{aligned}f(\lambda x,\beta y)&=f\left(\frac{M}{3}\cdot 3\frac{\lambda}{M}x,\frac{N}{3}\cdot 3\frac{\beta}{N}y\right)=MNf\left(\frac{1}{3}\cdot 3\frac{\lambda}{M}x,\frac{1}{3}\cdot 3\frac{\beta}{N}y\right)\\&=\frac{MN}{9}f\left(3\frac{\lambda}{M}x,3\frac{\beta}{N}y\right)=\frac{MN}{9}f(\mu_1x+\mu_2x+\mu_3x,\nu_1y+\nu_2y+\nu_3y)\\&=\frac{MN}{9}(\mu_1+\mu_2+\mu_3)(\nu_1+\nu_2+\nu_3)f(x,y)=\frac{MN}{9}\cdot 3\frac{\lambda}{M}\cdot 3\frac{\beta}{N}f(x,y)\\&=\lambda\beta f(x,y)\end{aligned}$$

for all $x,y\in A$. Therefore the mapping $f:A\times A\to B\times B$ is 2-linear over $\mathbb{C}$. □

Lemma 3. *Let A and B be complex Banach algebras and $f:A\times A\to B\times B$ be a mapping such that*

$$f(\mu(x_1+x_2),\nu(y_1-y_2))+f(\mu(x_1-x_2),\nu(y_1+y_2))=2\mu\nu f(x_1,y_1)-2\mu\nu f(x_2,y_2)$$

for all $\mu,\nu\in\mathbb{T}^1$ and all $x,y\in A$, then f is 2-linear over $\mathbb{C}$.

Proof. By the use of Lemma 1 with $\mu=\nu=1$ and Lemma 2 with $x_2=y_2=0$, the mapping f is 2-linear over $\mathbb{C}$. □

Theorem 1. *Let A and B be complex Banach algebras. Let $p+q<2, r+s<2$ and η be positive real numbers. Suppose that $f:A\times A\to B\times B$ is a mapping satisfying $f(0,0)=0$ and the inequality*

$$\begin{aligned}&||f(\lambda(x_1+x_2),\mu(y_1-y_2))+f(\lambda(x_1-x_2),\mu(y_1+y_2))-2\lambda\mu f(x_1,y_1)\\&\quad+2\lambda\mu f(x_2,y_2)||_\infty\le\eta\{||x_1||^p||x_2||^q+||y_1||^r||y_2||^s\}\end{aligned}\tag{2.1}$$

for all $\lambda,\mu \in \mathbb{T}^1$ *and* $x_1,y_1,x_2,y_2 \in A$. *If the mapping* $f : A\times A \to B\times B$ *satisfies*

$$||f(X._H Y) - f(X)._H f(Y)||_\infty \le \eta ||x_1||^p ||x_2||^q ||y_1||^r ||y_2||^s \tag{2.2}$$

for all $X = (x_1,y_1), Y = (x_2,y_2) \in A^2$, *then there exists a unique Hadamard homomorphism* $\varphi : A\times A \to B\times B$ *such that*

$$||\varphi(x,y) - f(x,y)||_\infty \le 3\eta \left(\frac{||x||^{p+q}}{4-2^{p+q}} + \frac{||y||^{r+s}}{4-2^{r+s}} \right) \tag{2.3}$$

for all $x,y \in A$.

Proof. Letting $\lambda = \mu = 1$, $x_1 = x_2$ and $y_2 = -y_1$ in (2.1), we have

$$||f(2x_1,2y_1) - 2f(x_1,y_1) + 2f(x_1,-y_1)||_\infty \le \eta(||x_1||^{p+q} + ||y_1||^{r+s}). \tag{2.4}$$

for all $x_1,y_1 \in A$. Setting $\lambda = \mu = 1$, $x_1 = y_1 = 0$ in (2.1), we get

$$f(x_2,-y_2) + f(-x_2,y_2) + 2f(x_2,y_2) = 0$$

for all $x_2,y_2 \in A$. Replacing x_2 by x_1 and y_2 by y_1 in the above inequality, we obtain

$$f(x_1,-y_1) + f(-x_1,y_1) + 2f(x_1,y_1) = 0 \tag{2.5}$$

for all $x_1,y_1 \in A$. Putting $\lambda = \mu = 1$, $x_2 = -x_1$ and $y_1 = y_2$ in (2.1), we obtain

$$||f(2x_1,2y_1) - 2f(x_1,y_1) + 2f(-x_1,y_1)||_\infty \le \eta(||x_1||^{p+q} + ||y_1||^{r+s}) \tag{2.6}$$

for all $x_1,y_1 \in A$. By (2.4) and (2.5), we get

$$||f(2x_1,2y_1) - 4f(x_1,y_1) + f(x_1,-y_1) - f(-x_1,y_1)||_\infty \le 2\eta(||x_1||^{p+q} + ||y_1||^{r+s}),$$

for all $x_1,y_1 \in A$. By (2.4) and (2.6), we get

$$||f(x_1,-y_1) - f(-x_1,y_1)||_\infty \le \eta(||x_1||^{p+q} + ||y_1||^{r+s})$$

for all $x_1,y_1 \in A$. By the above two inequalities, we have

$$||f(2x_1,2y_1) - 4f(x_1,y_1)||_\infty \le 3\eta(||x_1||^{p+q} + ||y_1||^{r+s}) \tag{2.7}$$

for all $x_1,y_1 \in A$. Replacing x_1 by $2^n x_1$ and y_1 by $2^n y_1$ and dividing 4^{n+1} in (2.7), we obtain that

$$\left|\left| \frac{1}{4^{n+1}} f(2^{n+1}x_1, 2^{n+1}y_1) - \frac{1}{4^n} f(2^n x_1, 2^n y_1) \right|\right|_\infty \le \frac{3\eta}{4^{n+1}} \left(2^{n(p+q)} ||x_1||^{p+q} + 2^{n(r+s)} ||y_1||^{r+s} \right)$$

for all $x_1,y_1 \in A$ and all $n = 0,1,2,\cdots$. For given integers $k,m (0 \le k \le m)$, we get

$$\left|\left| \frac{1}{4^m} f(2^m x_1, 2^m y_1) - \frac{1}{4^k} f(2^k x_1, 2^k y_1) \right|\right|_\infty \le \sum_{n=k}^{m-1} \frac{3\eta}{4^{n+1}} \left(2^{n(p+q)} ||x_1||^{p+q} + 2^{n(r+s)} ||y_1||^{r+s} \right) \tag{2.8}$$

for all $x_1, y_1 \in A$. By the use of (2.8), the sequence $\left\{\frac{1}{4^n} f(2^n x_1, 2^n y_1)\right\}$ is a Cauchy sequence for all $x_1, y_1 \in A$. Since $B \times B$ is complete, the sequence $\left\{\frac{1}{4^n} f(2^n x_1, 2^n y_1)\right\}$ converges for all $x_1, y_1 \in A$. Define $\varphi : A \times A \longrightarrow B \times B$ by

$$\varphi(x_1, y_1) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x_1, 2^n y_1)$$

for all $x_1, y_1 \in A$, we have

$$\begin{aligned}
&||\varphi(x_1+x_2, y_1-y_2) + \varphi(x_1-x_2, y_1+y_2) - 2\varphi(x_1,y_1) + 2\varphi(x_2,y_2)||_\infty \\
&\quad = \lim_{n\to\infty} \Big|\Big| \frac{1}{4^n} f(2^n(x_1+x_2), 2^n(y_1+y_2)) + \frac{1}{4^n} f(2^n(x_1-x_2), 2^n(y_1+y_2)) \\
&\qquad - \frac{2}{4^n} f(2^n x_1, 2^n y_1) + \frac{2}{4^n} f(2^n x_2, 2^n y_2) \Big|\Big|_\infty \\
&\quad \le \lim_{n\to\infty} \frac{\eta}{4^n} [2^{n(p+q)} ||x_1||^p ||x_2||^q + 2^{n(r+s)} ||y_1||^r ||y_2||^s] \\
&\quad \le \lim_{n\to\infty} \frac{2^{\zeta n}}{4^n} \eta [||x_1||^p ||x_2||^q + ||y_1||^r ||y_2||^s]
\end{aligned}$$

where $\zeta = \max\{p+q, r+s\}$, $x_1, x_2, y_1, y_2 \in A$ and $n = 0, 1, 2, \ldots$. As $n \to \infty$, by Lemma 3, the mapping $\varphi : A \times A \to B \times B$ is 2-linear. Setting $k = 0$ and taking $m \to \infty$ in (2.8), one can obtain the inequality (2.3). Let $X = (x_1, y_1), Y = (x_2, y_2)$. From (2.2) we have

$$\begin{aligned}
&||\varphi(X \bullet_H Y) - \varphi(X) \bullet_H \varphi(Y)||_\infty \\
&\quad = \lim_{n\to\infty} \frac{1}{16^n} ||f(2^n x_1 . 2^n x_2, 2^n y_1 . 2^n y_2) - f(2^n x_1, 2^n y_1) \bullet_H f(2^n x_2, 2^n y_2)|| \\
&\quad \le \lim_{n\to\infty} \frac{2^{n(p+q+r+s)}}{16^n} ||x_1||^p ||x_2||^q ||y_1||^r ||y_2||^s = 0,
\end{aligned}$$

for all $x_1, y_1, x_2, y_2 \in A$, since $p+q+r+s < 4$. Thus

$$\varphi(X \bullet_H Y) = \varphi(X) \bullet_H \varphi(Y).$$

Now, let $\psi : A \times A \to B \times B$ be another 2-additive mapping satisfying (2.3). Then we have

$$\begin{aligned}
||\varphi(x,y) - \psi(x,y)||_\infty &= \frac{1}{4^n} ||\varphi(2^n x, 2^n y) - \psi(2^n x, 2^n y)||_\infty \\
&\le \frac{1}{4^n} \big(||\varphi(2^n x, 2^n y) - f(2^n x, 2^n y)||_\infty \\
&\qquad + ||f(2^n x, 2^n y) - \psi(2^n x, 2^n y)||_\infty \big) \\
&\le \frac{3\eta}{4^n} \left(\frac{2^{n(p+q)} ||x||^{p+q}}{4 - 2^{p+q}} + \frac{2^{n(r+s)} ||y||^{r+s}}{4 - 2^{r+s}} \right)
\end{aligned}$$

$$\leq \frac{3\eta \times 2^{n\zeta}}{4^n(4-2^\zeta)}(||x||^{p+q}+||y||^{r+s}),$$

where $\zeta = \max\{p+q, r+s\}$. This tends to zero as $n \to \infty$ for all $x, y \in A$. So we can conclude that $\varphi(x,y) = \psi(x,y)$ for all $x, y \in A$. This proves the uniqueness of φ. □

Theorem 2. *Let A and B be complex Banach algebras. Let $p+q<2, r+s<2$ and η be positive real numbers. Let $f : A\times A \to B\times B$ be a bijective mapping satisfying (2.1) such that*

$$f(X._H Y) = f(X)._H f(Y) \tag{2.9}$$

for all $X,Y \in A^2$. If $f(\alpha x, \beta y)$ is continuous in $\alpha, \beta \in \mathbb{C}$ for each fixed $X = (x,y) \in A^2$ and $\frac{1}{4^n} f(2^n e, 2^n e) = (e', e')$, then the mapping $f : A\times A \to B\times B$ is a Hadamard isomorphism.

Proof. Since $f(X._H Y) = f(X)._H f(Y)$ for all $X,Y \in A^2$, the mapping $f : A\times A \to B\times B$ satisfies (2.1). By Theorem 1, there exists a Hadamard homomorphism $\varphi : A^2 \longrightarrow B^2$ satisfying (2.3). The mapping $\varphi : A^2 \longrightarrow B^2$ is defined by

$$\varphi(x,y) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y)$$

for all $x, y \in A$.

It follows from (2.9) that

$$\begin{aligned}
\varphi(x,y) = \varphi((e,e)._H(x,y)) &= \lim_{n\to\infty} \frac{1}{4^n} f(2^n ex, 2^n ey) \\
&= \lim_{n\to\infty} \frac{1}{4^n} f((2^n e, 2^n e)._H(x,y)) = \lim_{n\to\infty} \frac{1}{4^n}(f(2^n e, 2^n e)._H f(x,y)) \\
&= \left(\lim_{n\to\infty} \frac{1}{4^n} f(2^n e, 2^n e)\right)._H f(x,y) = (e', e')._H f(x,y) = f(x,y)
\end{aligned}$$

for all $x, y \in A$. So the bijective mapping $f : A\times A \to B\times B$ is a Hadamard isomorphism. □

3. Stability of Hadamard derivations on complex Banach algebras

In this section, we prove the stability of Hadamard derivations on complex Banach algebras associated with the 2-additive functional equation.

Definition 3. Let A be a complex Banach algebra. A $\mathbb{C}$-2-linear mapping $D : A\times A \longrightarrow A\times A$ is a *Hadamard derivation* if D satisfies

$$D(X._H Y) = D(X)._H Y^2 + X^2._H D(Y)$$

for all $X,Y \in A^2$.

Theorem 3. *Let A be a complex Banach algebra. Let $p+q<2, r+s<2$ and η be positive real numbers. Let $f : A\times A \to A\times A$ be a mapping satisfying (2.1) such that*

$$||f(X._H Y) - f(X)._H Y^2 - X^2._H f(Y)||_\infty \le \eta ||x_1||^p ||x_2||^q ||y_1||^r ||y_2||^s \tag{3.1}$$

for all $x_1, y_1, x_2, y_2 \in A$. Then there exists a unique Hadamard derivation $\varphi : A\times A \to A\times A$ such that

$$||D(x,y) - f(x,y)|| \le 3\eta \left(\frac{||x||^{p+q}}{4-2^{p+q}} + \frac{||y||^{r+s}}{4-2^{r+s}} \right). \tag{3.2}$$

Proof. By (2.8), the sequence $\left\{\frac{1}{4^n} f(2^n x_1, 2^n y_1)\right\}$ is a Cauchy sequence for all $x_1, y_1 \in A$. Since A is complete, the sequence $\left\{\frac{1}{4^n} f(2^n x_1, 2^n y_1)\right\}$ converges. So one can define the mapping $D : A\times A \longrightarrow A\times A$ by

$$D(x,y) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x_1, 2^n y_1)$$

for all $x_1, y_1 \in A$. By the same reasoning as in the proof of Theorem 1, we get the fact that D is 2-bilinear.

It follows from (3.1) that

$$\begin{aligned}
&||D(X._H Y) - D(X)._H Y^2 - X^2._H D(Y)||_\infty \\
&\quad = \lim_{n\to\infty} \Big|\Big| \frac{1}{4^n} f(2^n x_1 x_2, 2^n y_1 y_2) - \frac{1}{4^n} f(2^n x_1, 2^n y_1)._H (x_2, y_2)^2 \\
&\qquad - \frac{1}{4^n}(x_1, y_1)^2._H f(2^n x_2, 2^n y_2)\Big|\Big|_\infty \\
&\quad = \lim_{n\to\infty} \frac{1}{2^{4n}} \Big|\Big| f(2^n x_1.2^n x_2, 2^n y_1.2^n y_2) - f(2^n x_1, 2^n y_1)._H (2^n x_2, 2^n y_2)^2 \\
&\qquad - (2^n x_1, 2^n y_1)^2._H f(2^n x_2, 2^n y_2)\Big|\Big|_\infty \\
&\quad \le \lim_{n\to\infty} \frac{2^{n(p+q+r+s)}}{2^{4n}} ||x_1||^p ||x_2||^q ||y_1||^r ||y_2||^s = 0
\end{aligned}$$

for all $x_1, y_1, x_2, y_2 \in A$. So

$$D(X._H Y) = D(X)._H Y^2 + X^2._H D(Y)$$

for all $x_1, y_1, x_2, y_2 \in A$. Now, let Ψ be another mapping satisfying (3.2). Then we have

$$\begin{aligned}
||D(x,y) - \psi(x,y)||_\infty &= \lim_{n\to\infty} \frac{1}{4^n} ||D(2^n x, 2^n y) - \psi(2^n x, 2^n y)||_\infty \\
&\le \lim_{n\to\infty} \frac{1}{4^n} \left(||D(2^n x, 2^n y) - f(2^n x, 2^n y)||_\infty + ||f(2^n x, 2^n y) - \psi(2^n x, 2^n y)||_\infty \right) \\
&\le \lim_{n\to\infty} \frac{3\eta}{4^n} \left(\frac{2^{n(p+q)} ||x||^{p+q}}{4-2^{p+q}} + \frac{2^{n(r+s)} ||y||^{r+s}}{4-2^{r+s}} \right)
\end{aligned}$$

$$\leq \lim_{n\to\infty} \frac{3\eta \times 2^{n\zeta}}{4^n(4-2^\zeta)}(||x||^{p+q}+||y||^{r+s}).$$

By the same argument as in the proof of Theorem 1, the mapping $D : A\times A \longrightarrow A\times A$ is a unique Hadamard derivation satisfying (3.2). □

4. Conclusion

In this work, we have proved the Hyers-Ulam stability of Hadamard homomorphisms and Hadamard derivations in Banach algebras associated to the bi-additive functional equation (1.1). This has been applied to investigate Hadamard isomorphisms between Banach algebras.

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