MULTIPLE SOLUTIONS FOR A FRACTIONAL P&Q-LAPLACIAN SYSTEM INVOLVING HARDY-SOBOLEV EXPOSANTS

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Abstract. In this paper, we prove the existence of infinitely many solutions for a fractional p&q-Laplacian system involving Hardy-Sobolev exponents and obtain new conclusion under different conditions. The methods used here are based on variational methods and Ljusternik-Schnirelmann theory.

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1. INTRODUCTION

In this paper, we study the following fractional p&q-Laplacian system involving Hardy-Sobolev exponents:

\[
\begin{align*}
\begin{cases}
(-\Delta)_p^s u + (-\Delta)_q^s u &= \lambda |u|^{r-2} u + \frac{2\alpha}{\alpha + \beta} \frac{|u|^{\alpha-2} u}{|x|^\theta} \
(-\Delta)_p^s v + (-\Delta)_q^s v &= \mu |v|^{r-2} v + \frac{2\beta}{\alpha + \beta} \frac{|u|^{\alpha} |v|^{\beta-2} v}{|x|^\theta}
\end{cases}
\end{align*}
\]

in \(\Omega\),

\[u = v = 0\]
on \(\mathbb{R}^N \setminus \Omega\),

(1.1)

where \(\Omega \subseteq \mathbb{R}^N\) is a bounded domain containing the origin, \(p \in (1, \infty), s \in (0, 1), 1 < r < q < p, 0 \leq \theta < sp < N, \alpha + \beta = p^*_s,\) and \(\lambda, \mu > 0\) are two parameters, \(p^*_s = \frac{(N-\theta)p}{N-ps}\) is the fractional Hardy-Sobolev exponent, the fractional p-Laplacian operator \((-\Delta)_p^s\) is the nonlocal operator defined on smooth functions by

\[
(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{B(x) \setminus \bar{B}(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+sp}} dy, \quad x \in \mathbb{R}^N.
\]

This definition is consistent, up to normalization depending on \(N\) and \(s\). We would like to point out that, in the last decades, problems involving fractional p-Laplacian
are widely studied. We refer the readers to \[2, 5, 6, 10, 13, 18–24\] and references therein.

Then system (1.1) reduces to the critical fractional \( p \)-Laplacian equation with Hardy-Sobolev exponents

\[
\begin{aligned}
(-\triangle)_p u &= \lambda |u|^{r-2} u + \frac{|u|_s^{p_s-2} u}{|x|^s} \\
u &= 0
\end{aligned}
\tag{1.2}
\]

in \( \Omega \),
on \( \partial \Omega \).

Ning, Wang and Zhang proved the existence, multiplicity and bifurcation results for the above problem in \[15\]. When \( \theta = 0 \), in \[11\], Khiddi proved that problem (1.2) has infinitely many solutions with negative energy by using Ljusternik-Schnirelmann theory. When \( r = p \), Ghoussoub and Yuan obtained multiple solutions in \[8\]. In \[16\], Perera and Zou proved that this problem has a nontrivial solution where \( s = 1 \), \( r = p \), \( \lambda > \lambda_1 \) is not an eigenvalue and \( \lambda_1 \) is the first eigenvalue of the eigenvalue problem.

However, only a few articles have studied the \( p&q \)-Laplacian system, the results for the case can be seen in \[4, 9, 12, 14\], for example,

\[
\begin{aligned}
(-\triangle)_p u + (-\triangle)_q u &= \lambda |u|^{r-2} u + \frac{2\alpha}{\alpha + \beta} |u|^{\alpha-2} u |v|^{\beta} \\
u = v &= 0
\end{aligned}
\tag{1.3}
\]

in \( \Omega \), on \( \mathbb{R}^N \setminus \Omega \).

Obviously, system (1.3) is equivalent to \( \theta = 0 \) of system (1.1), Chen and Gui proved the existence of infinitely many solutions of problem (1.3) in \[4\]. Moreover, in \[3\], Chen and Deng proved that problem (1.3) has at least two positive solutions when \( p = q \).

Yet even fewer authors study systems involving Hardy-potential and critical non-linearities. Motivated by the above works, this paper discusses the fractional \( p&q \)-Laplacian system with Hardy-Sobolev exponents, by some techniques to establish new estimates to overcome difficulties. We prove the existence of infinitely many solutions by using Ljusternik-Schnirelmann theory. This result extend some results in the literature for the fractional \( p&q \)-Laplacian problem.

Before stating our main result, we introduce some notations. Let

\[
[u]_{s,p} = \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}}
\]

be the Gagliardo seminorm of a measurable function \( u : \mathbb{R}^N \to \mathbb{R} \), and let

\[
W^{s,p}(\mathbb{R}^N) = \{ u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty \}
\]

be the fractional Sobolev space endowed with the norm

\[
\|u\|_{s,p} = (\|u\|_p^p + [u]_{s,p}^p)^{\frac{1}{p}}
\]

where \( \| \cdot \|_p \) is the norm in \( L^p(\mathbb{R}^N) \). We define

\[
X_{s,p}^r(\Omega) = \{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}
\]
equivalently renormed by setting $\| \cdot \|_{s, p} = [\cdot]_{s, p}$, which is a uniformly convex Banach space. We work in the closed linear subspaces $W_p = X_p^s(\Omega) \times X_q^s(\Omega)$ and $W_q = X_q^s(\Omega) \times X_p^s(\Omega)$, which are reflexive Banach spaces endowed with the norms

$$
\|(u, v)\|_p = (\|u\|_{s, p}^p + \|v\|_{s, p}^p)^{\frac{1}{p}}\text{ and } \|(u, v)\|_q = (\|u\|_{s, q}^q + \|v\|_{s, q}^q)^{\frac{1}{q}}.
$$

(1.4)

Set $E = W_p \cap W_q$ endowed the norm $\|(u, v)\|_E = \|(u, v)\|_p + \|(u, v)\|_q$. Defining

$$
\mathcal{A}_{p, q}(u, \phi) = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+sp}} \, dx dy.
$$

(1.5)

We say that $(u, v) \in E$ is a weak solutions of problem (1.1), if $\forall (\phi, \psi) \in E$, the following holds

$$
\mathcal{A}_{p, q}(u, \phi) + \mathcal{A}_{q, s}(u, \phi) + \mathcal{A}_{r, s}(v, \psi) + \mathcal{A}_{s, q}(v, \psi) = \lambda \int_\Omega |u|^{-2} u \phi \, dx + 
+ \mu \int_\Omega |v|^{-2} v \psi \, dx + \frac{2\alpha}{p_{s, \theta}} \int_\Omega \frac{|u|^{\alpha-2} u |v|^{\beta} \phi}{|x|^\theta} \, dx + \frac{2\beta}{q_{s, \theta}} \int_\Omega \frac{|u|^{\alpha} |v|^{\beta-2} v \psi}{|x|^\theta} \, dx.
$$

The corresponding energy functional of system (1.1) is defined by

$$
J_{p, q}(u, v) = \frac{1}{p} \|(u, v)\|_p^p + \frac{1}{q} \|(u, v)\|_q^q - \frac{1}{r} \left( \lambda \int_\Omega |u|^r \, dx + \mu \int_\Omega |v|^r \, dx \right) - \frac{2}{p_{s, \theta}} \int_\Omega \frac{|u|^{\alpha} |v|^{\beta}}{|x|^\theta} \, dx,
$$

(1.6)

for $(u, v) \in E$, it is easy to know that $J_{p, q}$ is even, $J_{p, q} \in C^1(E, \mathbb{R})$ and

$$
\langle J_{p, q}'(u, v), (u, v) \rangle = \|(u, v)\|_p^p + \|(u, v)\|_q^q - \left( \lambda \int_\Omega |u|^r \, dx + \mu \int_\Omega |v|^r \, dx \right) - 2 \int_\Omega \frac{|u|^{\alpha} |v|^{\beta}}{|x|^\theta} \, dx.
$$

(1.7)

Our result can be stated as follows.

**Theorem 1.** There exists $\Lambda_0 > 0$ such that for each $0 < \lambda^{\frac{1}{2p}} + \mu^{\frac{1}{2q}} < \Lambda_0$, the system (1.1) has infinitely many solutions with negative energy.

**Remark 1.** The system in [4] is a special case of system (1.1). In [12], when $\theta = 0$, $s = 1$, Li and Yang proved that problem (1.3) has at least $\text{cat}(\Omega)+1$ distinct positive solutions by Lusternik-Schnirelmann category under condition (A):

$$
(A) : N > p, \quad 1 < r < q < \frac{N(p-1)}{N-1} < p < p^* = \frac{Np}{N-p}.
$$

The paper is organized as follows. In Section 2, we show that the $(PS)_c$ condition holds for the related energy functional in certain critical levels. In Section 3, we prove Theorem 1.
2. The \((PS)_c\) Condition for \(J_{\lambda,\mu}\)

In this section, we show that the Palais-Smale condition, \((PS)_c\) holds for the related energy functional in certain critical levels.

**Definition 1.** Let \(c \in \mathbb{R}\), \(E\) is a Banach space and \(J_{\lambda,\mu} \in C^1(E,\mathbb{R})\). We say that \(\{(u_k,v_k)\}\) is a \((PS)_c\) sequence for \(J_{\lambda,\mu}\) in \(E\) if \(J_{\lambda,\mu}(u_k,v_k) = c + o(1)\) and \(J'_{\lambda,\mu}(u_k,v_k) = o(1)\) strongly in \(E^*\) (the dual space of the Sobolev space \(E\)) as \(k \to \infty\). We say that \(J_{\lambda,\mu}\) satisfies the \((PS)_c\) condition if any \((PS)_c\) sequence \(\{(u_k,v_k)\}\) for \(J_{\lambda,\mu}\) in \(E\) has a convergent subsequence.

Let

\[
S_{\lambda,\theta} = \inf_{u \in X_{\mu}^r(\Omega) \setminus \{0\}} \frac{\|u\|^{p}_{\frac{p}{s},\theta}}{\int_{\Omega} \frac{|\nabla u|^p}{|x|^s} dx},
\]

which is positive by the fractional Hardy-Sobolev constant of \(X_{\mu}^r \hookrightarrow L^{p_{\mu,\theta}}(\mathbb{R}^N)\) and independent of \(\Omega\). In order to simplify calculation, set

\[
V_\theta(\Omega) = \int_\Omega |x|^\frac{\theta}{s} dx.
\]

We need the following Lemmas.

**Lemma 1.** If \(\{(u_k,v_k)\} \subset E\) is a \((PS)_c\) sequence for \(J_{\lambda,\mu}\), then \(\{(u_k,v_k)\}\) is bounded in \(E\).

**Proof.** Let \(\{(u_k,v_k)\} \subset E\) is a \((PS)_c\) sequence for \(J_{\lambda,\mu}\) satisfying

\[
J_{\lambda,\mu}(u_k,v_k) = c + o(1) \text{ and } J'_{\lambda,\mu}(u_k,v_k) = o(1) \text{ in } E^*.
\]

From (1.6) and (1.7), we obtain

\[
J_{\lambda,\mu}(u_k,v_k) - \frac{1}{p_{s,\theta}^*} \langle J'_{\lambda,\mu}(u_k,v_k), (u_k,v_k) \rangle = (\frac{1}{p} - \frac{1}{p_{s,\theta}^*}) \| (u_k,v_k) \|_p^p + \frac{1}{q} \| (u_k,v_k) \|_q^q - (\frac{1}{r} - \frac{1}{p_{s,\theta}^*}) \int_\Omega (\lambda |u_k|^r + \mu |v_k|^r) dx. \tag{2.1}
\]

From the definition of \(S_{r,\theta}\), Hölder and Sobolev inequalities, we conclude that

\[
\int_\Omega (\lambda |u_k|^r + \mu |v_k|^r) dx \leq \lambda V_\theta(\Omega) \frac{p_{s,\theta}^*}{r_{s,\theta}} \left( \int_\Omega \frac{|u_k|^{p_{s,\theta}^*}}{|x|^s} dx \right)^{\frac{r}{r_{s,\theta}}} + \mu V_\theta(\Omega) \frac{p_{s,\theta}^*}{r_{s,\theta}} \left( \int_\Omega \frac{|v_k|^{p_{s,\theta}^*}}{|x|^s} dx \right)^{\frac{r}{r_{s,\theta}}}.
\]
If Case 3: \( \geq \) we get 0

From (2.1) and (2.2), we deduce that
\[
\begin{align*}
\|u_k\|_p + \|v_k\|_p & \leq V_\theta(\Omega) \frac{r}{r-\gamma} \frac{\bar{r}}{\bar{r}-\overline{\gamma}} S^\theta_{\bar{r}, \overline{\gamma}} (\lambda^\theta p^\theta + \mu^\theta q^\theta) \| (u_k, v_k) \|_p^r, \\
\|u_k\|_q + \|v_k\|_q & \leq V_\theta(\Omega) \frac{r}{r-\gamma} \frac{\bar{r}}{\bar{r}-\overline{\gamma}} S^\theta_{\bar{r}, \overline{\gamma}} (\lambda^\theta p_\theta^\theta + \mu^\theta q_\theta^\theta) \| (u_k, v_k) \|_q^r.
\end{align*}
\]  
\( (2.2) \)

From (2.1) and (2.2), we deduce that
\[
C(1 + \| (u_k, v_k) \|_p + \| (u_k, v_k) \|_q^r) \geq \frac{1}{1} \left( 1 - \frac{1}{p^\theta} \right) \| (u_k, v_k) \|_p^q + \frac{\| (u_k, v_k) \|_q^r}{q^\theta}.
\]
\( (2.3) \)

We suppose the contrary, we may assume that \( \| (u_k, v_k) \|_p \to \infty \) as \( k \to \infty \), we need to consider the following three cases:

**Case 1:** If \( \| (u_k, v_k) \|_p \to \infty \) and \( \| (u_k, v_k) \|_q \to \infty \). For \( k \) large enough, we obtain \( \| (u_k, v_k) \|_p \geq 1 \) and \( \| (u_k, v_k) \|_q^r \geq \| (u_k, v_k) \|_q^q \). Indeed, using the inequality \( (a+b)^q \leq C_q (a^q + b^q) \) and (2.3), we deduce that
\[
C(1 + \| (u_k, v_k) \|_p + \| (u_k, v_k) \|_q^r) \geq \min \left\{ \frac{1}{p} - \frac{1}{q}, \frac{1}{q^\theta} \right\} \| (u_k, v_k) \|_p^q + \| (u_k, v_k) \|_q^r
\]
\[
\geq \left( \frac{1}{p} - \frac{1}{q^\theta} \right) C_q^{-1} \| (u_k, v_k) \|_p + \| (u_k, v_k) \|_q^q,
\]
both sides are divided by \( \| (u_k, v_k) \|_q^q \), we have
\[
C \left( \frac{1}{\| (u_k, v_k) \|_p^q + \| (u_k, v_k) \|_q^q} \right) + \frac{1}{\| (u_k, v_k) \|_q^q - \| (u_k, v_k) \|_q^q} \geq \left( \frac{1}{p} - \frac{1}{q^\theta} \right) C_q^{-1} \| (u_k, v_k) \|_q^q,
\]
we get \( C \geq \infty \), as \( k \to \infty \), this is impossible.

**Case 2:** If \( \| (u_k, v_k) \|_p \) is bounded and \( \| (u_k, v_k) \|_q \to \infty \). By (2.1) and (2.2), we obtain
\[
C(1 + \| (u_k, v_k) \|_p + \| (u_k, v_k) \|_q^r) \geq \left( \frac{1}{q} - \frac{1}{p^\theta} \right) \| (u_k, v_k) \|_q^q,
\]
both sides are divided by \( \| (u_k, v_k) \|_q^q \), we conclude that
\[
C \left( \frac{1}{\| (u_k, v_k) \|_p^q} + \frac{1}{\| (u_k, v_k) \|_q^r} \right) + \frac{1}{\| (u_k, v_k) \|_q^q - \| (u_k, v_k) \|_q^q} \geq \left( \frac{1}{q} - \frac{1}{p^\theta} \right) C_q^{-1} \| (u_k, v_k) \|_q^q,
\]
we get \( 0 \geq \frac{1}{q} - \frac{1}{p^\theta} > 0 \), as \( k \to \infty \), this is a contradiction.

**Case 3:** If \( \| (u_k, v_k) \|_p \to \infty \) and \( \| (u_k, v_k) \|_q \) is bounded. Similar to Case 2.

From Cases 1-3, we can conclude that \( \{ (u_k, v_k) \} \) is bounded in \( E \).
Lemma 2. If \( \{ (u_k, v_k) \} \) is a \( (PS)_c \) sequence for \( J_{\lambda, \mu} \) with \( (u_k, v_k) \rightharpoonup (u, v) \) in \( E \), then \( J'_{\lambda, \mu}(u, v) = 0 \), and there exists a positive constant \( C_0 \) such that

\[
J_{\lambda, \mu}(u, v) \geq -C_0(\lambda^{r-\frac{1}{r}} + \mu^{r-\frac{1}{r}}).
\]

Proof. If \( \{ (u_k, v_k) \} \) is a \( (PS)_c \) sequence for \( J_{\lambda, \mu} \) with \( (u_k, v_k) \rightharpoonup (u, v) \) in \( E \), then

\[
J'_{\lambda, \mu}(u_k, v_k) = o(1) \text{ strongly in } E' \text{ as } k \to \infty.
\]

Let \( (\phi, \psi) \in E \), we have

\[
\langle J'_{\lambda, \mu}(u_k, v_k) - J'_{\lambda, \mu}(u, v), (\phi, \psi) \rangle = \mathcal{A}_{p,s}(u_k, \phi) - \mathcal{A}_{p,s}(u, \phi) + \mathcal{A}_{q,s}(u_k, \phi) - \mathcal{A}_{q,s}(u, \phi)
\]

\[
\quad + \mathcal{A}_{p,s}(v_k, \psi) - \mathcal{A}_{p,s}(v, \psi) + \mathcal{A}_{q,s}(v_k, \psi) - \mathcal{A}_{q,s}(v, \psi)
\]

\[- \lambda \int_{\Omega} (|u_k|^{r-2} u_k - |u|^{r-2} u) \phi dx
\]

\[- \mu \int_{\Omega} (|v_k|^{r-2} v_k - |v|^{r-2} v) \psi dx
\]

\[- \frac{2\alpha}{p^*_s, \theta} \int_{\Omega} \left( \frac{|u_k|^{\alpha-2} u_k |v_k|^\beta}{|x|^{\theta}} - \frac{|u|^{\alpha-2} u |v|^\beta}{|x|^{\theta}} \right) \phi dx
\]

\[- \frac{2\beta}{p^*_s, \theta} \int_{\Omega} \left( \frac{|u_k|^{\alpha} |v_k|^\beta-2 v_k}{|x|^{\theta}} - \frac{|u|^{\alpha} |v|^\beta-2 v}{|x|^{\theta}} \right) \psi dx,
\]

where \( \mathcal{A}_{p,s} \) is defined in (1.5). Since \( \{ (u_k, v_k) \} \) is bounded in \( E \), up to subsequence, this implies the following:

\[
u_k \to u \text{ in } X_p^s(\Omega),
\]

\[
u_k \to u \text{ a.e. in } \Omega,
\]

\[
u_k \to u \text{ in } L^r(\Omega), \quad 1 \leq r < p^*_s, \theta
\]

and

\[
|u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y)) \quad \text{is bounded in } L^p(\mathbb{R}^N),
\]

therefore,

\[
|u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y)) \to |u(x) - u(y)|^{p-2}(u(x) - u(y)) \quad \text{in } L^p(\mathbb{R}^N),
\]

where \( \frac{1}{p} + \frac{1}{p'} = 1 \), and \( |\phi(y)| \frac{1}{|x-y|^{\frac{\theta}{p}}} \in L^p(\mathbb{R}^N) \), so

\[
\mathcal{A}_{p,s}(u_k, \phi) \to \mathcal{A}_{p,s}(u, \phi), \quad \text{as } k \to \infty.
\]

Similarly, we obtain

\[
\mathcal{A}_{q,s}(u_k, \phi) \to \mathcal{A}_{q,s}(u, \phi), \quad \text{as } k \to \infty.
\]
Similar to (2.4), we have

\[ v_k \rightarrow v \text{ in } X^p_{\rho}(\Omega), \]
\[ v_k \rightarrow v \text{ a.e. in } \Omega, \]
\[ v_k \rightarrow v \text{ in } L^r(\Omega), \quad 1 < r < p_{r_\theta}^* \]  \hfill (2.7)

and

\[ \frac{|v_k(x) - v_k(y)|^{p-2}(v_k(x) - v_k(y))}{|x-y|^{\frac{N+sp}{p}}} \text{ is bounded in } L^p(\mathbb{R}^N), \]

it follows that

\[ \frac{|v_k(x) - v_k(y)|^{p-2}(v_k(x) - v_k(y))}{|x-y|^{\frac{N+sp}{p}}} \rightarrow \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x-y|^{\frac{N+sp}{p}}} \text{ in } L^p(\mathbb{R}^N) \]

and

\[ \psi(x) - \psi(y) \rightarrow \psi(x) - \psi(y) \quad \text{in} \quad L^p(\mathbb{R}^N), \]

so

\[ \mathcal{A}_{p,s}(v_k, \psi) \rightarrow \mathcal{A}_{p,s}(v, \psi), \quad \text{as } k \rightarrow \infty. \]  \hfill (2.8)

Similarly, we obtain

\[ \mathcal{A}_{q,s}(v_k, \psi) \rightarrow \mathcal{A}_{q,s}(v, \psi), \quad \text{as } k \rightarrow \infty. \]  \hfill (2.9)

Moreover, by (2.4) and (2.7), we conclude that

\[ |u_k|^{r-2}u_k \rightarrow |u|^{r-2}u, \quad |v_k|^{r-2}v_k \rightarrow |v|^{r-2}v \quad \text{in } L^r(\Omega), \]
\[ \frac{|u_k|^{\alpha-2}u_k|v_k|^2}{|x|^6} \rightarrow \frac{|u|^{\alpha-2}u|v|^2}{|x|^6}, \quad \frac{|u_k|^{\alpha}|v_k|^{\beta-2}v_k}{|x|^6} \rightarrow \frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^6} \quad \text{in } L^{p_{r_\theta}^*}(\Omega), \]

thus,

\[ \int_{\Omega} (|u_k|^{r-2}u_k|u|^{r-2}u) \phi dx \rightarrow 0, \]
\[ \int_{\Omega} (|v_k|^{r-2}v_k|v|^{r-2}v) \psi dx \rightarrow 0, \quad \text{as } k \rightarrow \infty, \]  \hfill (2.10)

and

\[ \int_{\Omega} \left( \frac{|u_k|^{\alpha-2}u_k|v_k|^2}{|x|^6} - \frac{|u|^{\alpha-2}u|v|^2}{|x|^6} \right) \phi dx \rightarrow 0, \]
\[ \int_{\Omega} \left( \frac{|u_k|^{\alpha}|v_k|^{\beta-2}v_k}{|x|^6} - \frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^6} \right) \psi dx \rightarrow 0, \quad \text{as } k \rightarrow \infty. \]  \hfill (2.11)

From (2.5)-(2.6) and (2.8)-(2.11), we deduce that

\[ \langle J'_{\lambda,s}(u_k, v_k) - J'_{\lambda,s}(u, v), (\phi, \psi) \rangle \rightarrow 0 \text{ for all } (\phi, \psi) \in E, \]
furthermore, we obtain \( J'_{\lambda,\mu}(u,v), (u,v) = 0 \), it implies that
\[
2 \int_{\Omega} \frac{|u|^{\alpha}|v|^p}{|x|^q} \, dx = \| (u,v) \|_p^p + \| (u,v) \|_q^q - \int_{\Omega} (\lambda|u|^r + \mu|v|^r) \, dx,
\]
therefore,
\[
J_{\lambda,\mu}(u,v) = \left( \frac{1}{p} - \frac{1}{p^*_{\lambda,\mu}} \right) \| (u,v) \|_p^p + \left( \frac{1}{q} - \frac{1}{p^*_{\lambda,\mu}} \right) \| (u,v) \|_q^q - \left( \frac{1}{r} - \frac{1}{p^*_{\lambda,\mu}} \right) \int_{\Omega} (\lambda|u|^r + \mu|v|^r) \, dx
\]
\[
\geq \frac{ps - \theta}{(N - \theta)p} \| (u,v) \|_p^p - \left( \frac{1}{r} - \frac{1}{p^*_{\lambda,\mu}} \right) \int_{\Omega} (\lambda|u|^r + \mu|v|^r) \, dx. \tag{2.12}
\]
By Sobolev embedding, Hölder and Young inequalities, we deduce that
\[
\lambda \int_{\Omega} |u|^r \, dx \leq V_\theta(\Omega) \frac{p^*_{\lambda,\mu} - r}{p^*_{\lambda,\mu}} S_{\lambda,\mu}^{-\frac{r}{p^*_{\lambda,\mu}}} \lambda \| u \|_p^p
\]
\[
\leq \frac{ps - \theta}{(N - \theta)p} \left( \frac{1}{r} - \frac{1}{p^*_{\lambda,\mu}} \right) \| u \|_p^p
\]
\[
\quad + \frac{r - p}{p} \left[ \frac{p}{r} \frac{ps - \theta}{(N - \theta)p} \left( \frac{1}{r} - \frac{1}{p^*_{\lambda,\mu}} \right) \right]^{-1} \cdot V_\theta(\Omega) \frac{p^*_{\lambda,\mu} - r}{p^*_{\lambda,\mu}} S_{\lambda,\mu}^{-\frac{r}{p^*_{\lambda,\mu}}} \lambda^{-\frac{r}{p^*_{\lambda,\mu}}}.
\]
Similarly, we have
\[
\mu \int_{\Omega} |v|^r \, dx \leq \frac{ps - \theta}{(N - \theta)p} \left( \frac{1}{r} - \frac{1}{p^*_{\lambda,\mu}} \right) \| v \|_p^p
\]
\[
\leq \frac{ps - \theta}{(N - \theta)p} \left( \frac{1}{r} - \frac{1}{p^*_{\lambda,\mu}} \right) \| v \|_p^p
\]
\[
\quad + \frac{r - p}{p} \left[ \frac{p}{r} \frac{ps - \theta}{(N - \theta)p} \left( \frac{1}{r} - \frac{1}{p^*_{\lambda,\mu}} \right) \right]^{-1} \cdot V_\theta(\Omega) \frac{p^*_{\lambda,\mu} - r}{p^*_{\lambda,\mu}} S_{\lambda,\mu}^{-\frac{r}{p^*_{\lambda,\mu}}} \mu^{-\frac{r}{p^*_{\lambda,\mu}}},
\]
so,
\[
\int_{\Omega} (\lambda|u|^r + \mu|v|^r) \, dx \leq \frac{ps - \theta}{(N - \theta)p} \left( \frac{1}{r} - \frac{1}{p^*_{\lambda,\mu}} \right) \| (u,v) \|_p^p
\]
\[
\quad + \frac{r - p}{p} \left[ \frac{p}{r} \frac{ps - \theta}{(N - \theta)p} \left( \frac{1}{r} - \frac{1}{p^*_{\lambda,\mu}} \right) \right]^{-1} \cdot V_\theta(\Omega) \frac{p^*_{\lambda,\mu} - r}{p^*_{\lambda,\mu}} S_{\lambda,\mu}^{-\frac{r}{p^*_{\lambda,\mu}}} \lambda^{-\frac{r}{p^*_{\lambda,\mu}}}.
\]
By Lemma 1, and conclude that it follows that, let\( \text{Applying Brezis-Lieb Lemma [25, Lemma 1.32], it follows that} \)\( \text{where} \)

\[
\tilde{C} = \frac{p-r}{p} \left[ \frac{p}{r} \left( \frac{1}{r} - \frac{1}{p,s,\theta} \right)^{-1} \right]^{\frac{r(p,s,\theta-1)}{p-r}} V_0(\Omega) \left( \frac{p}{r} \right)^{p(s,\theta-1)} S_{s,\theta}^{-\frac{r}{r-1}}. \]

By (2.12), we conclude that

\[
J_{\lambda,\mu}(u,v) \geq -\left( \frac{1}{r} - \frac{1}{p,s,\theta} \right) \tilde{C}(\lambda)^{\frac{p}{r}} + \mu^{-\frac{p}{r}}. \]

it follows that, let \( C_0 = \left( \frac{1}{r} - \frac{1}{p,s,\theta} \right) \tilde{C} \), we can get

\[
J_{\lambda,\mu}(u,v) \geq -C_0(\lambda)^{\frac{p}{r}} + \mu^{-\frac{p}{r}}. \]

\[\square\]

**Lemma 3.** \( J_{\lambda,\mu} \) satisfies the (PS)\(_c\) condition with

\[
c < 2 \left( \frac{ps - \theta}{(N - \theta)p} \right) \left( \frac{S_{s,\theta}}{2} \right)^{\frac{N - \theta}{p}} - C_0(\lambda)^{\frac{p}{r}} + \mu^{-\frac{p}{r}}, \]

where \( S_{s,\theta} = \inf_{(u,v) \in E \setminus \{(0,0)\}} \frac{\| (u,v) \|^p_p}{\left( \int_\Omega |u|^s |v|^\theta \, dx \right)^{\frac{p}{p,s,\theta}}} \)

**Proof.** Let \( \{ (u_k, v_k) \} \) be a (PS)\(_c\) sequence for \( J_{\lambda,\mu} \) in \( E \), these hold that

\[
c + o(1) = \frac{1}{p} \| (u_k, v_k) \|^p_p + \frac{1}{q} \| (u_k, v_k) \|^q_q - \frac{1}{r} \int_\Omega (\lambda |u_k|^r + |v_k|^r) \, dx - \frac{2}{p,s,\theta} \int_\Omega \frac{|u_k|^s |v_k|^\theta}{|x|^\theta} \, dx \]

and

\[
o(1) \| (u_k, v_k) \| = \| (u_k, v_k) \|^p_p + \| (u_k, v_k) \|^q_q - \int_\Omega (\lambda |u_k|^r + |v_k|^r) \, dx - \int_\Omega \frac{|u_k|^s |v_k|^\theta}{|x|^\theta} \, dx. \tag{2.13} \]

By Lemma 1, \( \{ (u_k, v_k) \} \) is bounded in \( E \), then up to a subsequence, we have \( (u_k, v_k) \rightharpoonup (u, v) \) in \( E \) and we deduce from Lemma 2 that \( J'_{\lambda,\mu}(u, v) = 0 \), we will show that \( (u_k, v_k) \rightharpoonup (u, v) \) in \( E \). Since \( u_k \to u, v_k \to v \) strongly in \( L'(\Omega) \), so

\[
\int_\Omega (\lambda |u_k|^r + |v_k|^r) \, dx \to \int_\Omega (\lambda |u|^r + \mu |v|^r) \, dx, \text{ as } k \to \infty. \]

Applying Brezis-Lieb Lemma [25, Lemma 1.32], it follows that

\[
\| (u_k, v_k) \|^p_p = \| (u_k - u, v_k - v) \|^p_p + \| (u, v) \|^p_p + o(1), \]

\[
\| (u_k, v_k) \|^q_q = \| (u_k - u, v_k - v) \|^q_q + \| (u, v) \|^q_q + o(1). \tag{2.14} \]
and

\[
\int_{\Omega} \frac{|u_k|^\alpha |v_k|^\beta}{|x|^\theta} dx = \int_{\Omega} \frac{|u_k - u|^\alpha |v_k - v|^\beta}{|x|^\theta} dx + \int_{\Omega} \frac{|u|^\alpha |v|^\beta}{|x|^\theta} dx + o(1). \tag{2.15}
\]

From (2.14)-(2.15), we conclude that

\[
c - J_{k, \mu}(u, v) + o(1) = \frac{1}{p} \| (u_k - u, v_k - v) \|_p^p + \frac{1}{q} \| (u_k - u, v_k - v) \|_q^q
\]

\[
- \frac{2}{p^*_s, \theta} \int_{\Omega} \frac{|u_k - u|^\alpha |v_k - v|^\beta}{|x|^\theta} dx.
\]

By (2.13), we obtain

\[
\| (u_k - u, v_k - v) \|_p^p + \| (u_k - u, v_k - v) \|_q^q = 2 \int_{\Omega} \frac{|u_k - u|^\alpha |v_k - v|^\beta}{|x|^\theta} dx + o(1). \tag{2.17}
\]

Therefore, we may assume that

\[
\| (u_k - u, v_k - v) \|_p^p \to d,
\]

\[
\| (u_k - u, v_k - v) \|_q^q \to l,
\]

\[
2 \int_{\Omega} \frac{|u_k - u|^\alpha |v_k - v|^\beta}{|x|^\theta} dx \to m, \quad \text{as } k \to \infty.
\]

Moreover, \( d, l \geq 0 \) and \( d + l = m \). If \( d = 0 \), then \( (u_k, v_k) \to (u, v) \), the proof is done.

Suppose that \( d > 0 \), by the definition of \( S_{\alpha, \beta} \), we have \( d \leq m \leq 2S_{\alpha, \beta} \frac{p^*_s, \theta}{p^*_s, \theta} d \frac{p^*_s, \theta}{p^*_s, \theta} \), or,

\[
d \geq \frac{d}{2} \left( \frac{S_{\alpha, \beta}}{2} \right)^{\frac{p^*_s, \theta}{p^*_s, \theta}}.
\]

From (2.16), we deduce that

\[
c = \frac{d}{p} + \frac{l}{q} - \frac{m}{p^*_s, \theta} + J_{k, \mu}(u, v)
\]

\[
= \left( \frac{1}{p} \frac{1}{p^*_s, \theta} \right) d + \left( \frac{1}{q} - \frac{1}{p^*_s, \theta} \right) l + J_{k, \mu}(u, v)
\]

\[
\geq \frac{2}{p^*_s, \theta} \left( \frac{S_{\alpha, \beta}}{2} \right)^{\frac{p}{p^*_s, \theta}} - C_1 (\lambda^{\mu r} + \mu^{\mu r}),
\]

this is a contradiction with the definition of \( c \).

\[\square\]

3. The Proof of Our Main Theorem

Given the function \( J_{k, \mu} \) defined by (1.6) and (2.2), we obtain that

\[
J_{k, \mu}(u, v) \geq \frac{1}{p} \| (u, v) \|_p^p - \frac{2}{p^*_s, \theta} \int_{\Omega} \frac{|u|^\alpha |v|^\beta}{|x|^\theta} dx - \frac{1}{r} \int_{\Omega} (\lambda |u|^r dx + \mu |v|^r) dx
\]
From the structure of $h$, we have:

$$J_{h,\mu}(u,v) \geq h(||(u,v)||_p^p).$$

It is easy to see that there exists $\Lambda_*$ such that for any $0 < \lambda \frac{\mu}{p} + \mu \frac{\mu}{p} < \Lambda_*$, we have:

1. $h$ has exactly two distinct positive zero points denoted by $R_0$ and $R_1$;
2. $h$ attains nonnegative maximum at $R$ and it verifies $R_0 < R < R_1$;
3. $\frac{2(p-q-\theta)}{(N-\theta)p} \left( \frac{\mu_\beta}{p} \right)^{\frac{\mu_\theta}{p}} - C_0(\lambda \frac{\mu}{p} + \mu \frac{\mu}{p}) \geq 0$, where $C_0$ is given in Lemma 2.

From the structure of $h(t)$, we see that there are constants $0 < R_0 < R_1$, such that $h(R_0) = h(R_1) = 0$ and

$$h(t) \leq 0 \quad \text{if} \quad t \leq R_0,$$
$$h(t) > 0 \quad \text{if} \quad R_0 < t < R_1,$$
$$h(t) < 0 \quad \text{if} \quad t > R_1.$$

We now introduce the following truncation of the functional $J_{h,\mu}$. Taking the nonincreasing function $\tau: \mathbb{R}^+ \to [0,1]$ and $C^\infty(\mathbb{R}^+)$ such that $\tau(t) = 1$ if $t \leq R_0$; $\tau(t) = 0$ if $t \geq R_1$. Let $\varphi(u,v) = \tau(||(u,v)||_p)$, consider the truncated functional

$$I_{h,\mu}(u,v) = \frac{1}{p} ||(u,v)||_p^p + \frac{1}{q} ||(u,v)||_q^q - \frac{1}{r} \int_\Omega (\lambda |u|^q dx + \mu |v|^q) dx - \frac{2}{p_\alpha} \varphi(u,v) \int_\Omega \frac{|u|^a |v|^\beta}{|x|^\theta} dx. \quad (3.2)$$

Similarly to (3.1), we have

$$I_{h,\mu}(u,v) \geq \tilde{h}(||u,v||_p), \quad (3.3)$$

where $\tilde{h}(t) = c_1 t^p - c_2 t^{p_\beta} \tau(t) - c_3(\lambda \frac{\mu}{p} + \mu \frac{\mu}{p}) \frac{\mu}{p} t^r$.

By the definition of $h(t)$ and $0 \leq \tau(t) \leq 1$ for $t \geq 0$, we obtain $\tilde{h}(t) \geq h(t)$ if $t \geq 0$. It follows from $\tau(t) = 1$ if $0 \leq t \leq R_0$ that $\tilde{h}(t) = h(t)$ if $0 \leq t \leq R_0$. From $h(R_0) = h(R_1) = 0$ and $h(t) > 0$ if $R_0 < t < R_1$, we deduce that $\tilde{h}(t) \geq 0$ if $R_0 < t < R_1$.
Moreover, \( \tilde{h}(t) = t^r (c_1 t^{p-r} - c_3 (\lambda + \mu)^{e+1} t^{-r}) \) is strictly increasing if \( t > R_1 \), then \( \tilde{h}(t) > 0 \) if \( t > R_1 \). Consequently

\[
\tilde{h}(t) \geq 0 \quad \text{for} \quad t \geq R_0.
\]  

(3.4)

**Lemma 4.** We have the following results:

1. \( I_{\lambda, \mu} \in C^1(E, \mathbb{R}) \).
2. If \( I_{\lambda, \mu}(u, v) \leq 0 \), then \( \|(u, v)\| \leq R_0 \). Moreover, \( I_{\lambda, \mu}(\bar{u}, \bar{v}) = I_{\lambda, \mu}(u, v) \) for all \( (\bar{u}, \bar{v}) \) in a small enough neighborhood of \( (u, v) \).
3. There exists \( \Lambda_\epsilon > 0 \) such that if \( 0 < \lambda + \mu < \Lambda_\epsilon \), then \( I_{\lambda, \mu} \) satisfies a local \((PS)_\epsilon\) condition for \( \epsilon < 0 \).

**Proof.** Since \( \varphi \in C^\infty \) and \( \varphi(u, v) = 1 \) for \( (u, v) \) near \( (0,0) \), \( I_{\lambda, \mu} \in C^1(E, \mathbb{R}) \) and assertion (1) holds.

By taking \( I_{\lambda, \mu}(u, v) \leq 0 \), we can deduce from (3.3) that

\[
\tilde{h}(\|(u, v)\|) \leq 0
\]

and by (3.4), we have

\[
\|(u, v)\| \leq R_0
\]

implying (2).

For the proof of (3), let \( \{(u_k, v_k)\} \subset E \) be a \((PS)_\epsilon\) sequence of \( I_{\lambda, \mu} \) with \( \epsilon < 0 \). Then we may assume that \( I_{\lambda, \mu}(u_k, v_k) \leq 0, I_{\lambda, \mu}'(u_k, v_k) \to 0 \). By (2), there exists \( \Lambda_\epsilon > 0 \) such that \( 0 < \lambda + \mu < \Lambda_\epsilon \), \( \|(u_k, v_k)\| \leq R_0 \), so \( I_{\lambda, \mu}(u_k, v_k) = I_{\lambda, \mu}'(u_k, v_k) \) and \( I_{\lambda, \mu}'(u_k, v_k) = J_{\lambda, \mu}'(u_k, v_k) \). By Lemma 3, \( I_{\lambda, \mu} \) satisfies the \((PS)_\epsilon\) condition for \( \epsilon < 0 \), thus \( I_{\lambda, \mu} \) satisfies the \((PS)_\epsilon\) condition for \( \epsilon < 0 \), this completes the proof. \( \square \)

It is possible to prove the existence of level sets of \( I_{\lambda, \mu} \) with arbitrarily large genus. Now, we use the idea in [7] to construct negative critical value of \( I_{\lambda, \mu} \) via genus, more precisely:

**Lemma 5.** \( \forall k \in \mathbb{N}, \exists \epsilon = \epsilon(k) > 0 \) such that

\[
\gamma(\{(u, v) \in E : I_{\lambda, \mu} \leq -\epsilon(k)\}) \geq k.
\]

**Proof.** Let \( k \in \mathbb{N} \). We consider \( E_k \) be a \( k \)-dimensional subspaces of \( E \). Let \( (u, v) \in E_k \) with norm \( \|(u, v)\| \neq 1 \). For \( 0 < r < R_0 \), we have

\[
J_{\lambda, \mu}(pu, pv) = I_{\lambda, \mu}(pu, pv) = \frac{1}{p} p^\beta + \frac{\rho^q}{q} \|(u, v)\|^q - \frac{\rho^r}{r} \int_\Omega (|u|^r + |v|^r) dx
\]

\[ - 2 \rho^{\rho - \beta} \varphi(u, v) \int_\Omega |u|^\alpha |v|^\beta dx. \]
Since $E_k$ is a space of finite dimension, all the norms in $E_k$ are equivalent. If we define
$$
\alpha_k := \sup \{ \| (u, v) \|_p^2 : (u, v) \in E_k, \| (u, v) \|_p = 1 \} < \infty
$$
and
$$
\beta_k := \inf \int_{\Omega} |u|^p \, dx : (u, v) \in E_k, \| (u, v) \|_p = 1 > 0.
$$
Then we have
$$
I_{\lambda, \mu}(\rho u, \rho v) \leq \frac{1}{p} \rho^p + \frac{q}{q} \alpha_k - \beta_k \min(\lambda, \mu) \rho^r.
$$
Then, there exist $\varepsilon(k) > 0$ and $0 < \rho < R_0$ such that $I_{\lambda, \mu}(\rho u, \rho v) \leq -\varepsilon(k)$ for $(u, v) \in E_k$ with $\| (u, v) \|_p = 1$. Let $S_\rho = \{ (u, v) \in E : \| (u, v) \|_p = \rho \}$, so $S_\rho \cap E_k \subset \{ (u, v) \in E : I_{\lambda, \mu}(u, v) \leq -\varepsilon(k) \}$, therefore, by the property of genus in [17] and the fact $\gamma(S_\rho \cap E_k) = k$, it implies that
$$
\gamma(\{ (u, v) \in E : I_{\lambda, \mu}(u, v) \leq -\varepsilon(k) \}) \geq \gamma(S_\rho \cap E_k) = k.
$$

Now, we prove our main result.

**Proof of Theorem 1.** For $k \in \mathbb{N}$, set
$$
\Gamma_k = \{ A \subset E \setminus \{ (0, 0) \} : A \text{ is closed}, A = -A, \gamma(A) \geq k \},
$$
where $\gamma(A)$ is the genus of $A$. Let us set
$$
c_k = \inf_{A \in \Gamma_k} \sup_{(u, v) \in A} I_{\lambda, \mu}(u, v),
$$
and
$$
K_c = \{ (u, v) \in E : I_{\lambda, \mu}(u, v) = c, I'_{\lambda, \mu}(u, v) = 0 \}.
$$
Suppose $0 < \lambda \frac{p}{p-q} + \mu \frac{p}{p-r} < \Lambda_*$, where $\Lambda_*$ is the constant given by Lemma 4. In fact, if we denote $I_{\lambda, \mu}^\pm = \{ (u, v) \in E : I_{\lambda, \mu}(u, v) \leq \mp \varepsilon \}$, by Lemma 5, there exists $\varepsilon(k) > 0$ such that $\gamma(I_{\lambda, \mu}^{-\varepsilon(k)}) \geq k$ for $k \in \mathbb{N}$. Because $I_{\lambda, \mu}$ is continuous and even, $I_{\lambda, \mu}^{-\varepsilon(k)} \subset \Gamma_k$, then $c_k \leq -\varepsilon(k) < 0$ for all $k \in \mathbb{N}$. But $I_{\lambda, \mu}$ is bounded from below, hence $c_k > -\infty$ for all $k \in \mathbb{N}$.

Let us assume that $c = c_k = c_{k+1} = \cdots = c_{k+j}$. Note that $c < 0$, therefore, $I_{\lambda, \mu}$ satisfies the $(PS)_c$ condition, and it is easy to see that $K_c$ is a compact set.

If $\gamma(K_c) \leq l$, then there is a closed and symmetric set $U$ with $K_c \subset U$ and $\gamma(U) \leq l$ by the continuity property of genus [17]. By [1, Lemma 1.3], there is an odd homeomorphism $\eta : E \to E$ such that $\eta(K_{c+\delta} - U) \subset K_{c-\delta}$ for some $\delta > 0$. By definition,
$$
c = c = c_{k+1} = \inf_{A \in \Gamma_k} \sup_{(u, v) \in A} I_{\lambda, \mu}(u, v),
$$
there exists $A \in \Gamma_{k+1}$ such that $\sup_{(u,v) \in A} I_{h,\mu}(u,v) < c + \delta$, i.e. $A \subset \mathscr{I}^{c+\delta}_{h,\mu}$ and $\eta(A - U) \subset \mathscr{I}^{c+\delta}(U) \subset \mathscr{I}^{c-\delta}$, that is

$$\sup_{(u,v) \in \eta(A - U)} I_{h,\mu}(u,v) \leq c - \delta. \tag{3.5}$$

But $\gamma(A - U) \geq \gamma(A) - \gamma(U) \geq k$, and $\eta(\eta(A - U)) \gamma(A - U) \geq k$, then $\eta(A - U) \in \Gamma_k$. This is a contradiction. In fact, $\eta(\eta(A - U)) \in \Gamma_k$ implies that

$$\sup_{(u,v) \in \eta(A - U)} I_{h,\mu}(u,v) \geq c_k = c,$$

which contradicts to (3.5). So we have proved that $\gamma(K_c) \geq l + 1$.

We are now ready to show that $I_{h,\mu}$ has infinitely many critical points. Note $c_k$ is nondecreasing and strictly negative. We distinguish two cases:

**Case 1:** Suppose that there are $1 < k_1 < \cdots < k_i < \cdots$, satisfying

$$c_{k_1} < \cdots < c_{k_i} < \cdots,$$

then $\gamma(K_{c_i}) \geq 1$, and we see that $\{c_k\}$ is a sequence of distinct critical values of $I_{h,\mu}$.

**Case 2:** We assume in this case that, for some positive integer $k_0$, there is $l \geq 1$ such that $c = c_{k_0} = c_{k_0+1} = \cdots = c_{k_0+l}$, then $\gamma(K_{c_{k_0}}) \geq l + 1$, which shows that $K_{c_{k_0}}$ contains infinitely many distinct elements.

Since $J_{h,\mu}(u,v) = I_{h,\mu}(u,v)$ if $I_{h,\mu}(u,v) < 0$, we see that there are infinitely many critical points of $J_{h,\mu}(u,v)$, that is to say, there are negative energy solutions to problem (1.1). This completes the proof of Theorem 1. \hfill \Box

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