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MULTIPLE SOLUTIONS FOR A FRACTIONAL P&Q-LAPLACIAN SYSTEM INVOLVING HARDY-SOBOLEV EXPONENTS

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Abstract. In this paper, we prove the existence of infinitely many solutions for a fractional p&q-Laplacian system involving Hardy-Sobolev exponents and obtain new conclusion under different conditions. The methods used here are based on variational methods and Ljusternik-Schnirelmann theory.

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1. INTRODUCTION

In this paper, we study the following fractional p&q-Laplacian system involving Hardy-Sobolev exponents:

$$\begin{cases} (-\triangle)_p^s u + (-\triangle)_q^s u = \lambda |u|^{r-2}u + \frac{2\alpha}{\alpha+\beta}\frac{|u|^{\alpha-2}u|v|^{\beta}}{|x|^{\theta}} & \text{in } \Omega, \\ (-\triangle)_p^s v + (-\triangle)_q^s v = \mu |v|^{r-2}v + \frac{2\beta}{\alpha+\beta}\frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^{\theta}} & \text{in } \Omega, \\ u = v = 0 & \text{on } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain containing the origin, $p \in (1,\infty)$, $s \in (0,1)$, $1 < r < q < p, 0 \leq \theta < sp < N$, $\alpha+\beta = p^*_{s,\theta}$ and $\lambda, \mu > 0$ are two parameters, $p^*_{s,\theta} = \frac{(N-\theta)p}{N-ps}$ is the fractional Hardy-Sobolev exponent, the fractional p-Laplacian operator $(-\triangle)_p^s$ is the nonlocal operator defined on smooth functions by

$$(-\triangle)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0^+} \int_{B^c_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy, \quad x \in \mathbb{R}^N.$$

This definition is consistent, up to normalization depending on N and s. We would like to point out that, in the last decades, problems involving fractional p-Laplacian

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are widely studied. We refer the readers to [2, 5, 6, 10, 13, 18–24] and references therein.

Then system (1.1) reduces to the critical fractional p-Laplacian equation with Hardy-Sobolev exponents

$$\begin{cases} (-\triangle)_p^s u = \lambda |u|^{r-2}u + \frac{|u|^{p^*_{s,\theta}-2}u}{|x|^\theta} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Ning, Wang and Zhang proved the existence, multiplicity and bifurcation results for the above problem in [15]. When $\theta = 0$, in [11], Khiddi proved that problem (1.2) has infinitely many solutions with negative energy by using Ljusternik-Schnirelmann theory. When $r = p$, Ghoussoub and Yuan obtained multiple solutions in [8]. In [16], Perera and Zou proved that this problem has a nontrivial solution where $s = 1$, $r = p$, $\lambda > \lambda_1$ is not an eigenvalue and λ_1 is the first eigenvalue of the eigenvalue problem.

However, only a few articles have studied the $p\&q$-Laplacian system, the results for the case can be seen in [4, 9, 12, 14], for example,

$$\begin{cases} (-\triangle)_p^s u + (-\triangle)_q^s u = \lambda |u|^{r-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta & \text{in } \Omega, \\ u = v = 0 & \text{on } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.3}$$

Obviously, system (1.3) is equivalent to $\theta = 0$ of system (1.1), Chen and Gui proved the existence of infinitely many solutions of problem (1.3) in [4]. Moreover, in [3], Chen and Deng proved that problem (1.3) has at least two positive solutions when $p = q$.

Yet even fewer authors study systems involving Hardy-potential and critical nonlinearities. Motivated by the above works, this paper discusses the fractional $p\&q$-Laplacian system with Hardy-Sobolev exponents, by some techniques to establish new estimates to overcome difficulties. We prove the existence of infinitely many solutions by using Ljusternik-Schnirelmann theory. This result extend some results in the literature for the fractional $p\&q$-Laplacian problem.

Before stating our main result, we introduce some notations. Let

$$[u]_{s,p} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}}$$

be the Galiardo seminorm of a measurable function $u \colon \mathbb{R}^N \to \mathbb{R}$, and let

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) \colon [u]_{s,p} < \infty\}$$

be the fractional Sobolev space endowed with the norm

$$\|u\|_{s,p} = (|u|_p^p + [u]_{s,p}^p)^{\frac{1}{p}},$$

where $|\cdot|_p$ is the norm in $L^p(\mathbb{R}^N)$. We define

$$X_p^s(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) \colon u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\}$$

equivalently renormed by setting $\|\cdot\|_{s,p} = [\cdot]_{s,p}$, which is a uniformly convex Banach space. We work in the closed linear subspaces $W_p = X_p^s(\Omega) \times X_p^s(\Omega)$ and $W_q = X_q^s(\Omega) \times X_q^s(\Omega)$, which are reflexive Banach spaces endowed with the norms

$$\|(u,v)\|_p = (\|u\|_{s,p}^p + \|v\|_{s,p}^p)^{\frac{1}{p}} \text{ and } \|(u,v)\|_q = (\|u\|_{s,q}^q + \|v\|_{s,q}^q)^{\frac{1}{q}}. \tag{1.4}$$

Set $E = W_p \cap W_q$ endowed the norm $\|(u,v)\| = \|(u,v)\|_p + \|(u,v)\|_q$. Defining

$$\mathcal{A}_{p,s}(u,\phi) = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}} dxdy. \tag{1.5}$$

We say that $(u,v) \in E$ is a weak solutions of problem (1.1), if $\forall(\phi,\psi) \in E$, the following holds

$$\begin{aligned}\mathcal{A}_{p,s}(u,\phi) + \mathcal{A}_{q,s}(u,\phi) + \mathcal{A}_{p,s}(v,\psi) + \mathcal{A}_{q,s}(v,\psi) = \lambda \int_\Omega |u|^{r-2}u\phi dx \\ + \mu \int_\Omega |v|^{r-2} v\psi dx + \frac{2\alpha}{p_{s,\theta}^*} \int_\Omega \frac{|u|^{\alpha-2}u|v|^\beta \phi}{|x|^\theta} dx + \frac{2\beta}{p_{s,\theta}^*} \int_\Omega \frac{|u|^\alpha |v|^{\beta-2} v\psi}{|x|^\theta} dx.\end{aligned}$$

The corresponding energy functional of system (1.1) is defined dy

$$\begin{aligned}J_{\lambda,\mu}(u,v) &= \frac{1}{p}\|(u,v)\|_p^p + \frac{1}{q}\|(u,v)\|_q^q \\ &\quad - \frac{1}{r}\left(\lambda \int_\Omega |u|^r dx + \mu \int_\Omega |v|^r dx\right) - \frac{2}{p_{s,\theta}^*} \int_\Omega \frac{|u|^\alpha |v|^\beta}{|x|^\theta} dx, \end{aligned}\tag{1.6}$$

for $(u,v) \in E$, it is easy to know that $J_{\lambda,\mu}$ is even, $J_{\lambda,\mu} \in C^1(E,\mathbb{R})$ and

$$\begin{aligned}\langle J'_{\lambda,\mu}(u,v),(u,v)\rangle &= \|(u,v)\|_p^p + \|(u,v)\|_q^q \\ &\quad - \left(\lambda \int_\Omega |u|^r dx + \mu \int_\Omega |v|^r dx\right) - 2\int_\Omega \frac{|u|^\alpha |v|^\beta}{|x|^\theta} dx. \end{aligned}\tag{1.7}$$

Our result can be stated as follows.

Theorem 1. *There exists $\Lambda_* > 0$ such that for each $0 < \lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}} < \Lambda_*$, the system (1.1) has infinitely many solutions with negative energy.*

Remark 1. The system in [4] is a special case of system (1.1). In [12], when $\theta = 0$, $s = 1$, Li and Yang proved that problem (1.3) has at least cat(Ω)+1 distinct positive solutions by Lusternik-Schnirelmann category under condition (A):

$$(A): N > p, \quad 1 < r < q < \frac{N(p-1)}{N-1} < p < p^* = \frac{Np}{N-p}.$$

The paper is organized as follows. In Section 2, we show that the $(PS)_c$ condition holds for the related energy functional in certain critical levels. In Section 3, we prove Theorem 1.

2. THE $(PS)_c$ CONDITION FOR $J_{\lambda,\mu}$

In this section, we show that the Palais-Smale condition, $(PS)_c$ holds for the related energy functional in certain critical levels.

Definition 1. Let $c \in \mathbb{R}$, E is a Banach space and $J_{\lambda,\mu} \in C^1(E,\mathbb{R})$. We say that $\{(u_k,v_k)\}$ is a $(PS)_c$ sequence for $J_{\lambda,\mu}$ in E if $J_{\lambda,\mu}(u_k,v_k) = c + o(1)$ and $J'_{\lambda,\mu}(u_k,v_k) = o(1)$ strongly in E^* (the dual space of the Sobolev space E) as $k \to \infty$. We say that $J_{\lambda,\mu}$ satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence $\{(u_k,v_k)\}$ for $J_{\lambda,\mu}$ in E has a convergent subsequence.

Let

$$S_{s,\theta} = \inf_{u\in X_p^s(\Omega)\setminus\{0\}} \frac{\|u\|_{s,p}^p}{(\int_\Omega \frac{|u|^{p^*_{s,\theta}}}{|x|^\theta})^{\frac{p}{p^*_{s,\theta}}}},$$

which is positive by the fractional Hardy-Sobolev constant of $X_p^s \hookrightarrow L^{p^*_{s,\theta}}(\mathbb{R}^N)$ and independent of Ω. In order to simplify calculation, set

$$V_\theta(\Omega) = \int_\Omega |x|^{\frac{r\theta}{p^*_{s,\theta}-r}} dx.$$

We need the following Lemmas.

Lemma 1. *If $\{(u_k,v_k)\} \subset E$ is a $(PS)_c$ sequence for $J_{\lambda,\mu}$, then $\{(u_k,v_k)\}$ is bounded in E.*

Proof. Let $\{(u_k,v_k)\} \subset E$ is a $(PS)_c$ sequence for $J_{\lambda,\mu}$ satisfying

$$J_{\lambda,\mu}(u_k,v_k) = c + o(1) \text{ and } J'_{\lambda,\mu}(u_k,v_k) = o(1) \text{ in } E^*.$$

From (1.6) and (1.7), we obtain

$$\begin{aligned} J_{\lambda,\mu}(u_k,v_k) - \frac{1}{p^*_{s,\theta}}\langle J'_{\lambda,\mu}(u_k,v_k),(u_k,v_k)\rangle &= (\frac{1}{p} - \frac{1}{p^*_{s,\theta}})\|(u_k,v_k)\|_p^p \\ &+ (\frac{1}{q} - \frac{1}{p^*_{s,\theta}})\|(u_k,v_k)\|_q^q \\ &- (\frac{1}{r} - \frac{1}{p^*_{s,\theta}})\int_\Omega (\lambda|u_k|^r + \mu|v_k|^r)dx. \quad (2.1) \end{aligned}$$

From the definition of $S_{s,\theta}$, Hölder and Sobolev inequalities, we conclude that

$$\begin{aligned} \int_\Omega (\lambda|u_k|^r + \mu|v_k|^r)dx &\le \lambda V_\theta(\Omega)^{\frac{p^*_{s,\theta}-r}{p^*_{s,\theta}}} \left(\int_\Omega \frac{|u_k|^{p^*_{s,\theta}}}{|x|^\theta}dx\right)^{\frac{r}{p^*_{s,\theta}}} \\ &+ \mu V_\theta(\Omega)^{\frac{p^*_{s,\theta}-r}{p^*_{s,\theta}}} \left(\int_\Omega \frac{|v_k|^{p^*_{s,\theta}}}{|x|^\theta}dx\right)^{\frac{r}{p^*_{s,\theta}}} \end{aligned}$$

$$\begin{aligned}
&\leq V_\theta(\Omega)^{\frac{p^*_{s,\theta}-r}{p^*_{s,\theta}}} S_{s,\theta}^{-\frac{r}{p}} (\lambda\|u_k\|_p^r + \mu\|v_k\|_p^r) \\
&\leq V_\theta(\Omega)^{\frac{p^*_{s,\theta}-r}{p^*_{s,\theta}}} S_{s,\theta}^{-\frac{r}{p}} (\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}})^{\frac{p-r}{p}} \|(u_k,v_k)\|_p^r \\
&\leq V_\theta(\Omega)^{\frac{p^*_{s,\theta}-r}{p^*_{s,\theta}}} S_{s,\theta}^{-\frac{r}{p}} (\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}})^{\frac{p-r}{p}} \|(u_k,v_k)\|^r. \qquad (2.2)
\end{aligned}$$

From (2.1) and (2.2), we deduce that

$$C(1+\|(u_k,v_k)\|+\|(u_k,v_k)\|^r) \geq (\frac{1}{p}-\frac{1}{p^*_{s,\theta}})\|(u_k,v_k)\|_p^p + (\frac{1}{q}-\frac{1}{p^*_{s,\theta}})\|(u_k,v_k)\|_q^q. \qquad (2.3)$$

We suppose the contrary, we may assume that $\|(u_k,v_k)\| \to \infty$ as $k \to \infty$, we need to consider the following three cases:

Case 1: If $\|(u_k,v_k)\|_p \to \infty$ and $\|(u_k,v_k)\|_q \to \infty$. For k large enough, we obtain $\|(u_k,v_k)\|_p \geq 1$ and $\|(u_k,v_k)\|_p^p \geq \|(u_k,v_k)\|_q^q$. Indeed, using the inequality $(a+b)^q \leq C_q(a^q+b^q)$ and (2.3), we deduce that

$$\begin{aligned}
C(1+\|(u_k,v_k)\|+\|(u_k,v_k)\|^r) &\geq \min\{\frac{1}{p}-\frac{1}{p^*_{s,\theta}}, \frac{1}{q}-\frac{1}{p^*_{s,\theta}}\}(\|(u_k,v_k)\|_p^q + \|(u_k,v_k)\|_q^q) \\
&\geq (\frac{1}{p}-\frac{1}{p^*_{s,\theta}})C_q^{-1}(\|(u_k,v_k)\|_p + \|(u_k,v_k)\|_q)^q \\
&= (\frac{1}{p}-\frac{1}{p^*_{s,\theta}})C_q^{-1}\|(u_k,v_k)\|^q,
\end{aligned}$$

both sides are divided by $\|(u_k,v_k)\|^r$, we have

$$C(\frac{1}{\|(u_k,v_k)\|^r} + \frac{1}{\|(u_k,v_k)\|^{r-1}} + 1) \geq (\frac{1}{p}-\frac{1}{p^*_{s,\theta}})C_q^{-1}\|(u_k,v_k)\|^{q-r},$$

we get $C \geq \infty$, as $k \to \infty$, this is impossible.

Case 2: If $\|(u_k,v_k)\|_p$ is bounded and $\|(u_k,v_k)\|_q \to \infty$. By (2.1) and (2.2), we obtain

$$C(1+\|(u_k,v_k)\|_p+\|(u_k,v_k)\|_q+\|(u_k,v_k)\|_p^r) \geq (\frac{1}{q}-\frac{1}{p^*_{s,\theta}})\|(u_k,v_k)\|_q^q,$$

both sides are divided by $\|(u_k,v_k)\|_q^q$, we conclude that

$$C(\frac{1}{\|(u_k,v_k)\|_q^q} + \frac{\|(u_k,v_k)\|_p}{\|(u_k,v_k)\|_q^q} + \frac{1}{\|(u_k,v_k)\|_q^{q-1}} + \frac{\|(u_k,v_k)\|_p^r}{\|(u_k,v_k)\|_q^q}) \geq \frac{1}{q}-\frac{1}{p^*_{s,\theta}},$$

we get $0 \geq \frac{1}{q}-\frac{1}{p^*_{s,\theta}} > 0$, as $k \to \infty$, this is a contradiction.

Case 3: If $\|(u_k,v_k)\|_p \to \infty$ and $\|(u_k,v_k)\|_q$ is bounded. Similar to Case 2.

From Cases 1-3, we can conclude that $\{(u_k,v_k)\}$ is bounded in E. □

Lemma 2. *If $\{(u_k,v_k)\}$ is a $(PS)_c$ sequence for $J_{\lambda,\mu}$ with $(u_k,v_k)\rightharpoonup(u,v)$ in E, then $J'_{\lambda,\mu}(u,v)=0$, and there exists a positive constant C_0 such that*

$$J_{\lambda,\mu}(u,v)\geq -C_0(\lambda^{\frac{p}{p-r}}+\mu^{\frac{p}{p-r}}).$$

Proof. If $\{(u_k,v_k)\}$ is a $(PS)_c$ sequence for $J_{\lambda,\mu}$ with $(u_k,v_k)\rightharpoonup(u,v)$ in E, then

$$J'_{\lambda,\mu}(u_k,v_k)=o(1) \text{ strongly in } E^* \text{ as } k\to\infty.$$

Let $(\phi,\psi)\in E$, we have

$$\begin{aligned}\langle J'_{\lambda,\mu}(u_k,v_k)-J'_{\lambda,\mu}(u,v),(\phi,\psi)\rangle &= \mathcal{A}_{p,s}(u_k,\phi)-\mathcal{A}_{p,s}(u,\phi)+\mathcal{A}_{q,s}(u_k,\phi)-\mathcal{A}_{q,s}(u,\phi)\\ &\quad+\mathcal{A}_{p,s}(v_k,\psi)-\mathcal{A}_{p,s}(v,\psi)+\mathcal{A}_{q,s}(v_k,\psi)-\mathcal{A}_{q,s}(v,\psi)\\ &\quad-\lambda\int_\Omega(|u_k|^{r-2}u_k-|u|^{r-2}u)\phi dx\\ &\quad-\mu\int_\Omega(|v_k|^{r-2}v_k-|v|^{r-2}v)\psi dx\\ &\quad-\frac{2\alpha}{p^*_{s,\theta}}\int_\Omega\left(\frac{|u_k|^{\alpha-2}u_k|v_k|^\beta}{|x|^\theta}-\frac{|u|^{\alpha-2}u|v|^\beta}{|x|^\theta}\right)\phi dx\\ &\quad-\frac{2\beta}{p^*_{s,\theta}}\int_\Omega\left(\frac{|u_k|^{\alpha}|v_k|^{\beta-2}v_k}{|x|^\theta}-\frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^\theta}\right)\psi dx,\end{aligned}$$

where $\mathcal{A}_{p,s}$ is defined in (1.5). Since $\{(u_k,v_k)\}$ is bounded in E, up to subsequence, this implies the following:

$$\begin{aligned}&u_k\rightharpoonup u \text{ in } X^s_p(\Omega),\\ &u_k\to u \text{ a.e. in } \Omega,\\ &u_k\to u \text{ in } L^r(\Omega),\ 1\leq r<p^*_{s,\theta}\end{aligned}\tag{2.4}$$

and

$$\frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))}{|x-y|^{\frac{N+sp}{p'}}} \text{ is bounded in } L^{p'}(\mathbb{R}^N),$$

therefore,

$$\frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))}{|x-y|^{\frac{N+sp}{p'}}}\rightharpoonup\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{\frac{N+sp}{p'}}} \text{ in } L^{p'}(\mathbb{R}^N),$$

where $\frac{1}{p}+\frac{1}{p'}=1$, and $\frac{\phi(x)-\phi(y)}{|x-y|^{\frac{N+sp}{p}}}\in L^p(\mathbb{R}^N)$, so

$$\mathcal{A}_{p,s}(u_k,\phi)\to\mathcal{A}_{p,s}(u,\phi), \text{ as } k\to\infty.\tag{2.5}$$

Similarly, we obtain

$$\mathcal{A}_{q,s}(u_k,\phi)\to\mathcal{A}_{q,s}(u,\phi), \text{ as } k\to\infty.\tag{2.6}$$

Similar to (2.4), we have

$$\begin{aligned} &v_k \rightharpoonup v \text{ in } X_p^s(\Omega), \\ &v_k \to v \text{ a.e. in } \Omega, \\ &v_k \to v \text{ in } L^r(\Omega),\ 1 \le r < p_{s,\theta}^* \end{aligned} \tag{2.7}$$

and

$$\frac{|v_k(x)-v_k(y)|^{p-2}(v_k(x)-v_k(y))}{|x-y|^{\frac{N+sp}{p'}}} \text{ is bounded in } L^{p'}(\mathbb{R}^N),$$

it follows that

$$\frac{|v_k(x)-v_k(y)|^{p-2}(v_k(x)-v_k(y))}{|x-y|^{\frac{N+sp}{p'}}} \rightharpoonup \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{\frac{N+sp}{p'}}} \text{ in } L^{p'}(\mathbb{R}^N)$$

and

$$\frac{\psi(x)-\psi(y)}{|x-y|^{\frac{N+sp}{p}}} \in L^p(\mathbb{R}^N),$$

so

$$\mathcal{A}_{p,s}(v_k,\psi) \to \mathcal{A}_{p,s}(v,\psi), \text{ as } k \to \infty. \tag{2.8}$$

Similarly, we obtain

$$\mathcal{A}_{q,s}(v_k,\psi) \to \mathcal{A}_{q,s}(v,\psi), \text{ as } k \to \infty. \tag{2.9}$$

Moreover, by (2.4) and (2.7), we conclude that

$$|u_k|^{r-2}u_k \rightharpoonup |u|^{r-2}u,\ |v_k|^{r-2}v_k \rightharpoonup |v|^{r-2}v \quad \text{in } L^{r'}(\Omega),$$

$$\frac{|u_k|^{\alpha-2}u_k|v_k|^{\beta}}{|x|^{\theta}} \rightharpoonup \frac{|u|^{\alpha-2}u|v|^{\beta}}{|x|^{\theta}},\ \frac{|u_k|^{\alpha}|v_k|^{\beta-2}v_k}{|x|^{\theta}} \rightharpoonup \frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^{\theta}} \quad \text{in } L^{\frac{p_{s,\theta}^*}{p_{s,\theta}^*-1}}(\Omega),$$

thus,

$$\begin{aligned} &\int_\Omega (|u_k|^{r-2}u_k|u|^{r-2}u)\phi dx \to 0, \\ &\int_\Omega (|v_k|^{r-2}v_k|v|^{r-2}v)\psi dx \to 0, \text{ as } k \to \infty, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &\int_\Omega \left(\frac{|u_k|^{\alpha-2}u_k|v_k|^{\beta}}{|x|^{\theta}} - \frac{|u|^{\alpha-2}u|v|^{\beta}}{|x|^{\theta}} \right) \phi dx \to 0, \\ &\int_\Omega \left(\frac{|u_k|^{\alpha}|v_k|^{\beta-2}v_k}{|x|^{\theta}} - \frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^{\theta}} \right) \psi dx \to 0, \text{ as } k \to \infty. \end{aligned} \tag{2.11}$$

From (2.5)-(2.6) and (2.8)-(2.11), we deduce that

$$\langle J'_{\lambda,\mu}(u_k,v_k) - J'_{\lambda,\mu}(u,v),(\phi,\psi)\rangle \to 0 \text{ for all } (\phi,\psi) \in E,$$

furthermore, we obtain $\langle J'_{\lambda,\mu}(u,v),(u,v)\rangle = 0$, it implies that

$$2\int_\Omega \frac{|u|^\alpha|v|^\beta}{|x|^\theta}dx = \|(u,v)\|_p^p + \|(u,v)\|_q^q - \int_\Omega (\lambda|u|^r + \mu|v|^r)dx,$$

therefore,

$$\begin{aligned} J_{\lambda,\mu}(u,v) &= (\frac{1}{p} - \frac{1}{p^*_{s,\theta}})\|(u,v)\|_p^p + (\frac{1}{q} - \frac{1}{p^*_{s,\theta}})\|(u,v)\|_q^q \\ &\quad - (\frac{1}{r} - \frac{1}{p^*_{s,\theta}})\int_\Omega (\lambda|u|^r + \mu|v|^r)dx \\ &\geq \frac{ps-\theta}{(N-\theta)p}\|(u,v)\|_p^p - (\frac{1}{r} - \frac{1}{p^*_{s,\theta}})\int_\Omega (\lambda|u|^r + \mu|v|^r)dx. \qquad (2.12) \end{aligned}$$

By Sobolev embedding, Hölder and Young inequalities, we deduce that

$$\begin{aligned} \lambda\int_\Omega |u|^r dx &\leq V_\theta(\Omega)^{\frac{p^*_{s,\theta}-r}{p^*_{s,\theta}}} S_{s,\theta}^{-\frac{r}{p}} \lambda \|u\|_p^r \\ &\leq \frac{ps-\theta}{(N-\theta)p}\left(\frac{1}{r} - \frac{1}{p^*_{s,\theta}}\right)^{-1} \|u\|_p^p \\ &\quad + \frac{p-r}{p}\left[\frac{p}{r}\frac{ps-\theta}{(N-\theta)p}\left(\frac{1}{r} - \frac{1}{p^*_{s,\theta}}\right)^{-1}\right]^{-\frac{r}{p-r}} \cdot V_\theta(\Omega)^{\frac{p(p^*_{s,\theta}-r)}{p^*_{s,\theta}(p-r)}} S_{s,\theta}^{-\frac{r}{p-r}} \lambda^{\frac{p}{p-r}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mu\int_\Omega |v|^r dx &\leq \frac{ps-\theta}{(N-\theta)p}\left(\frac{1}{r} - \frac{1}{p^*_{s,\theta}}\right)^{-1} \|v\|_p^p \\ &\quad + \frac{p-r}{p}\left[\frac{p}{r}\frac{ps-\theta}{(N-\theta)p}\left(\frac{1}{r} - \frac{1}{p^*_{s,\theta}}\right)^{-1}\right]^{-\frac{r}{p-r}} \cdot V_\theta(\Omega)^{\frac{p(p^*_{s,\theta}-r)}{p^*_{s,\theta}(p-r)}} S_{s,\theta}^{-\frac{r}{p-r}} \mu^{\frac{p}{p-r}}, \end{aligned}$$

so,

$$\begin{aligned} \int_\Omega (\lambda|u|^r + \mu|v|^r)dx &\leq \frac{ps-\theta}{(N-\theta)p}\left(\frac{1}{r} - \frac{1}{p^*_{s,\theta}}\right)^{-1} \|(u,v)\|_p^p \\ &\quad + \frac{p-r}{p}\left[\frac{p}{r}\frac{ps-\theta}{(N-\theta)p}\left(\frac{1}{r} - \frac{1}{p^*_{s,\theta}}\right)^{-1}\right]^{-\frac{r}{p-r}} \\ &\quad \cdot V_\theta(\Omega)^{\frac{p(p^*_{s,\theta}-r)}{p^*_{s,\theta}(p-r)}} S_{s,\theta}^{-\frac{r}{p-r}} (\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}}) \end{aligned}$$

$$= \frac{ps-\theta}{(N-\theta)p}\left(\frac{1}{r}-\frac{1}{p^*_{s,\theta}}\right)^{-1}\|(u,v)\|_p^p+\widetilde{C}(\lambda^{\frac{p}{p-r}}+\mu^{\frac{p}{p-r}}),$$

where $\widetilde{C}=\frac{p-r}{p}\left[\frac{p}{r}\frac{ps-\theta}{(N-\theta)p}\left(\frac{1}{r}-\frac{1}{p^*_{s,\theta}}\right)^{-1}\right]^{-\frac{r}{p-r}}V_\theta(\Omega)^{\frac{p(p^*_{s,\theta}-r)}{p^*_{s,\theta}(p-r)}}S_{s,\theta}^{-\frac{r}{p-r}}$. By (2.12), we conclude that

$$J_{\lambda,\mu}(u,v)\geq-\left(\frac{1}{r}-\frac{1}{p^*_{s,\theta}}\right)\widetilde{C}(\lambda^{\frac{p}{p-r}}+\mu^{\frac{p}{p-r}}),$$

it follows that, let $C_0=\left(\frac{1}{r}-\frac{1}{p^*_{s,\theta}}\right)\widetilde{C}$, we can get

$$J_{\lambda,\mu}(u,v)\geq-C_0(\lambda^{\frac{p}{p-r}}+\mu^{\frac{p}{p-r}}).$$

□

Lemma 3. *$J_{\lambda,\mu}$ satisfies the $(PS)_c$ condition with*

$$c<\frac{2(ps-\theta)}{(N-\theta)p}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N-\theta}{ps-\theta}}-C_0(\lambda^{\frac{p}{p-r}}+\mu^{\frac{p}{p-r}}),$$

where $S_{\alpha,\beta}=\inf_{(u,v)\in E\setminus\{(0,0)\}}\frac{\|(u,v)\|_p^p}{\left(\int_\Omega\frac{|u|^\alpha|v|^\beta}{|x|^\theta}dx\right)^{\frac{p}{p^*_{s,\theta}}}}$.

Proof. Let $\{(u_k,v_k)\}$ be a $(PS)_c$ sequence for $J_{\lambda,\mu}$ in E, these hold that

$$\begin{aligned}c+o(1)=&\frac{1}{p}\|(u_k,v_k)\|_p^p+\frac{1}{q}\|(u_k,v_k)\|_q^q\\&-\frac{1}{r}\int_\Omega(\lambda|u_k|^r+|v_k|^r)\,dx-\frac{2}{p^*_{s,\theta}}\int_\Omega\frac{|u_k|^\alpha|v_k|^\beta}{|x|^\theta}dx\end{aligned}$$

and

$$\begin{aligned}o(1)\|(u_k,v_k)\|=&\|(u_k,v_k)\|_p^p+\|(u_k,v_k)\|_q^q\\&-\int_\Omega(\lambda|u_k|^r+|v_k|^r)dx-2\int_\Omega\frac{|u_k|^\alpha|v_k|^\beta}{|x|^\theta}dx.\end{aligned}\tag{2.13}$$

By Lemma 1, $\{(u_k,v_k)\}$ is bounded in E, then up to a subsequence, we have $(u_k,v_k)\rightharpoonup(u,v)$ in E and we deduce from Lemma 2 that $J'_{\lambda,\mu}(u,v)=0$, we will show that $(u_k,v_k)\to(u,v)$ in E. Since $u_k\to u$, $v_k\to v$ strongly in $L^r(\Omega)$, so

$$\int_\Omega(\lambda|u_k|^r+|v_k|^r)\,dx\to\int_\Omega(\lambda|u|^r+\mu|v|^r)dx,\ \text{as } k\to\infty.$$

Applying Brezis-Lieb Lemma [25, Lemma 1.32], it follows that

$$\begin{aligned}\|(u_k,v_k)\|_p^p&=\|(u_k-u,v_k-v)\|_p^p+\|(u,v)\|_p^p+o(1),\\\|(u_k,v_k)\|_q^q&=\|(u_k-u,v_k-v)\|_q^q+\|(u,v)\|_q^q+o(1)\end{aligned}\tag{2.14}$$

and

$$\int_\Omega \frac{|u_k|^\alpha |v_k|^\beta}{|x|^\theta} dx = \int_\Omega \frac{|u_k - u|^\alpha |v_k - v|^\beta}{|x|^\theta} dx + \int_\Omega \frac{|u|^\alpha |v|^\beta}{|x|^\theta} dx + o(1). \tag{2.15}$$

From (2.14)-(2.15), we conclude that

$$\begin{aligned} c - J_{\lambda,\mu}(u,v) + o(1) = &\frac{1}{p}\|(u_k - u, v_k - v)\|_p^p + \frac{1}{q}\|(u_k - u, v_k - v)\|_p^p \\ &- \frac{2}{p^*_{s,\theta}} \int_\Omega \frac{|u_k - u|^\alpha |v_k - v|^\beta}{|x|^\theta} dx. \end{aligned} \tag{2.16}$$

By (2.13), we obtain

$$\|(u_k - u, v_k - v)\|_p^p + \|(u_k - u, v_k - v)\|_q^q = 2\int_\Omega \frac{|u_k - u|^\alpha |v_k - v|^\beta}{|x|^\theta} dx + o(1). \tag{2.17}$$

Therefore, we may assume that

$$\begin{aligned} &\|(u_k - u, v_k - v)\|_p^p \to d, \\ &\|(u_k - u, v_k - v)\|_q^q \to l, \\ &2\int_\Omega \frac{|u_k - u|^\alpha |v_k - v|^\beta}{|x|^\theta} dx \to m, \text{ as } k \to \infty. \end{aligned}$$

Moreover, $d, l \geq 0$ and $d + l = m$. If $d = 0$, then $(u_k, v_k) \to (u, v)$, the proof is done. Suppose that $d > 0$, by the definition of $S_{\alpha,\beta}$, we have $d \leq m \leq 2S_{\alpha,\beta}^{-\frac{p^*_{s,\theta}}{p}} d^{\frac{p^*_{s,\theta}}{p}}$, or,

$$d \geq 2\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N-\theta}{ps-\theta}}.$$

From (2.16), we deduce that

$$\begin{aligned} c &= \frac{d}{p} + \frac{l}{q} - \frac{m}{p^*_{s,\theta}} + J_{\lambda,\mu}(u,v) \\ &= \left(\frac{1}{p} - \frac{1}{p^*_{s,\theta}}\right)d + \left(\frac{1}{q} - \frac{1}{p^*_{s,\theta}}\right)l + J_{\lambda,\mu}(u,v) \\ &\geq \frac{2(ps-\theta)}{(N-\theta)p}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N-\theta}{ps-\theta}} - C_0\left(\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}}\right), \end{aligned}$$

this is a contradiction with the definition of c. □

3. THE PROOF OF OUR MAIN THEOREM

Given the function $J_{\lambda,\mu}$ defined by (1.6) and (2.2), we obtain that

$$J_{\lambda,\mu}(u,v) \geq \frac{1}{p}\|(u,v)\|_p^p - \frac{2}{p^*_{s,\theta}} \int_\Omega \frac{|u|^\alpha |v|^\beta}{|x|^\theta} dx - \frac{1}{r}\int_\Omega (\lambda |u|^r dx + \mu |v|^r) dx$$

$$\geq \frac{1}{p}\|(u,v)\|_p^p - \frac{2}{p_{s,\theta}^*} S_{\alpha,\beta}^{-\frac{p_{s,\theta}^*}{p}} \|(u,v)\|_p^{p_{s,\theta}^*}$$
$$- \frac{1}{r} C_{\theta,\Omega}(\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}})^{\frac{p-r}{p}} \|(u,v)\|_p^r$$
$$= c_1\|(u,v)\|_p^p - c_2\|(u,v)\|_p^{p_{s,\theta}^*} - c_3(\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}})^{\frac{p-r}{p}} \|(u,v)\|_p^r,$$

where $c_1 = \frac{1}{p}$, $c_2 = \frac{2}{p_{s,\theta}^*} S_{\alpha,\beta}^{-\frac{p_{s,\theta}^*}{p}}$, $c_3 = \frac{1}{r} C_{\theta,\Omega}$ with $C_{\theta,\Omega} = V_\theta(\Omega)^{\frac{p_{s,\theta}^* - r}{p_{s,\theta}^*}} S_{s,\theta}^{-\frac{r}{p}}$. If we define for $t \geq 0$, $h(t) := c_1 t^p - c_2 t^{p_{s,\theta}^*} - c_3(\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}})^{\frac{p-r}{p}} t^r$, then

$$J_{\lambda,\mu}(u,v) \geq h(\|(u,v)\|_p). \tag{3.1}$$

It is easy to see that there exists Λ_* such that for any $0 < \lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}} < \Lambda_*$, we have:

(1) h has exactly two distinct positive zero points denoted by R_0 and R_1;
(2) h attains nonnegative maximum at R and it verifies $R_0 < R < R_1$;
(3) $\frac{2(ps-\theta)}{(N-\theta)p}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N-\theta}{ps-\theta}} - C_0(\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}}) \geq 0$, where C_0 is given in Lemma 2.

From the structure of $h(t)$, we see that there are constants $0 < R_0 < R_1$, such that $h(R_0) = h(R_1) = 0$ and

$$\begin{aligned} h(t) &\leq 0 \quad \text{if} \quad t \leq R_0, \\ h(t) &> 0 \quad \text{if} \quad R_0 < t < R_1, \\ h(t) &< 0 \quad \text{if} \quad t > R_1. \end{aligned}$$

We now introduce the following truncation of the functional $J_{\lambda,\mu}$. Taking the nonincreasing function $\tau\colon \mathbb{R}^+ \to [0,1]$ and $C^\infty(\mathbb{R}^+)$ such that $\tau(t) = 1$ if $t \leq R_0$; $\tau(t) = 0$ if $t \geq R_1$. Let $\varphi(u,v) = \tau(\|(u,v)\|_p)$, consider the truncated functional

$$\begin{aligned} I_{\lambda,\mu}(u,v) = \frac{1}{p}\|(u,v)\|_p^p + \frac{1}{q}\|(u,v)\|_q^q - \frac{1}{r}\int_\Omega (\lambda|u|^r dx + \mu|v|^r) dx \\ - \frac{2}{p_{s,\theta}^*}\varphi(u,v)\int_\Omega \frac{|u|^\alpha|v|^\beta}{|x|^\theta} dx. \end{aligned} \tag{3.2}$$

Similarly to (3.1), we have

$$I_{\lambda,\mu}(u,v) \geq \widetilde{h}(\|(u,v)\|_p), \tag{3.3}$$

where $\widetilde{h}(t) := c_1 t^p - c_2 t^{p_{s,\theta}^*}\tau(t) - c_3(\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}})^{\frac{p-r}{p}} t^r$.

By the definition of $h(t)$ and $0 \leq \tau(t) \leq 1$ for $t \geq 0$, we obtain $\widetilde{h}(t) \geq h(t)$ if $t \geq 0$. It follows from $\tau(t) = 1$ if $0 \leq t \leq R_0$ that $\widetilde{h}(t) = h(t)$ if $0 \leq t \leq R_0$. From $h(R_0) = h(R_1) = 0$ and $h(t) > 0$ if $R_0 < t < R_1$, we deduce that $\widetilde{h}(t) \geq 0$ if $R_0 < t \leq R_1$.

Moreover, $\widetilde{h}(t) = t^r(c_1 t^{p-r} - c_3(\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}})^{\frac{p-r}{p}})$ is strictly increasing if $t > R_1$, then $\widetilde{h}(t) > 0$ if $t > R_1$. Consequently

$$\widetilde{h}(t) \geq 0 \ \text{ for } \ t \geq R_0. \tag{3.4}$$

Lemma 4. *We have the following results:*

(1) $I_{\lambda,\mu} \in C^1(E,\mathbb{R})$.
(2) *If* $I_{\lambda,\mu}(u,v) \leq 0$, *then* $\|(u,v)\|_p \leq R_0$. *Moreover,* $I_{\lambda,\mu}(\widetilde{u},\widetilde{v}) = J_{\lambda,\mu}(\widetilde{u},\widetilde{v})$ *for all* $(\widetilde{u},\widetilde{v})$ *in a small enough neighborhood of* (u,v).
(3) *There exists* $\Lambda_* > 0$ *such that if* $0 < \lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}} < \Lambda_*$, *then* $I_{\lambda,\mu}$ *satisfies a local* $(PS)_c$ *condition for* $c < 0$.

Proof. Since $\varphi \in C^\infty$ and $\varphi(u,v) = 1$ for (u,v) near (0,0), $I_{\lambda,\mu} \in C^1(E,\mathbb{R})$ and assertion (1) holds.

By taking $I_{\lambda,\mu}(u,v) \leq 0$, we can deduce from (3.3) that

$$\widetilde{h}(\|(u,v)\|_p) \leq 0$$

and by (3.4), we have

$$\|(u,v)\|_p \leq R_0$$

implying (2).

For the proof of (3), let $\{(u_k,v_k)\} \subset E$ be a $(PS)_c$ sequence of $I_{\lambda,\mu}$ with $c < 0$. Then we may assume that $I_{\lambda,\mu}(u_k,v_k) < 0$, $I'_{\lambda,\mu}(u_k,v_k) \to 0$. By (2), there exists $\Lambda_* > 0$ such that $0 < \lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}} < \Lambda_*$, $\|(u_k,v_k)\|_p \leq R_0$, so $I_{\lambda,\mu}(u_k,v_k) = J_{\lambda,\mu}(u_k,v_k)$ and $I'_{\lambda,\mu}(u_k,v_k) = J'_{\lambda,\mu}(u_k,v_k)$. By Lemma 3, $J_{\lambda,\mu}$ satisfies the $(PS)_c$ condition for $c < 0$, thus $I_{\lambda,\mu}$ satisfies the $(PS)_c$ condition for $c < 0$, this completes the proof. □

It is possible to prove the existence of level sets of $I_{\lambda,\mu}$ with arbitrarily large genus. Now, we use the idea in [7] to construct negative critical value of $I_{\lambda,\mu}$ via genus, more precisely:

Lemma 5. $\forall\, k \in \mathbb{N}$, $\exists\, \varepsilon = \varepsilon(k) > 0$ *such that*

$$\gamma(\{(u,v) \in E\colon I_{\lambda,\mu} \leq -\varepsilon(k)\}) \geq k.$$

Proof. Let $k \in \mathbb{N}$. We consider E_k be a k-dimensional subspaces of E. Let $(u,v) \in E_k$ with norm $\|(u,v)\|_p = 1$. For $0 < \rho < R_0$, we have

$$\begin{aligned} J_{\lambda,\mu}(\rho u,\rho v) &= I_{\lambda,\mu}(\rho u,\rho v) \\ &= \frac{1}{p}\rho^p + \frac{\rho^q}{q}\|(u,v)\|_q^q - \frac{\rho^r}{r}\int_\Omega (\lambda|u|^r + \mu|v|^r)dx \\ &\quad - \frac{2\rho^{p^*_{s,\theta}}}{p^*_{s,\theta}}\varphi(u,v)\int_\Omega \frac{|u|^\alpha|v|^\beta}{|x|^\theta}dx. \end{aligned}$$

Since E_k is a space of finite dimension, all the norms in E_k are equivalent. If we define

$$\alpha_k := \sup\{\|(u,v)\|_q^q : (u,v) \in E_k, \|(u,v)\|_p = 1\} < \infty$$

and

$$\beta_k := \inf\{\int_\Omega |u|^r dx : (u,v) \in E_k, \|(u,v)\|_p = 1\} > 0.$$

Then we have

$$I_{\lambda,\mu}(\rho u, \rho v) \leq \frac{1}{p}\rho^p + \frac{\rho^q}{q}\alpha_k - \beta_k \min\{\lambda,\mu\}\frac{\rho^r}{r}.$$

Then, there exist $\varepsilon(k) > 0$ and $0 < \rho < R_0$ such that $I_{\lambda,\mu}(\rho u, \rho v) \leq -\varepsilon(k)$ for $(u,v) \in E_k$ with $\|(u,v)\|_p = 1$. Let $S_\rho = \{(u,v) \in E : \|(u,v)\|_p = \rho\}$, so

$$S_\rho \cap E_k \subset \{(u,v) \in E : I_{\lambda,\mu}(u,v) \leq -\varepsilon(k)\},$$

therefore, by the property of genus in [17] and the fact $\gamma(S_\rho \cap E_k) = k$, it implies that

$$\gamma(\{(u,v) \in E : I_{\lambda,\mu}(u,v) \leq -\varepsilon(k)\}) \geq \gamma(S_\rho \cap E_k) = k.$$

□

Now, we prove our main result.

Proof of Theorem 1. For $k \in \mathbb{N}$, set

$$\Gamma_k = \{A \subset E \backslash \{(0,0)\} : A \text{ is closed}, A = -A, \gamma(A) \geq k\},$$

where $\gamma(A)$ is the genus of A. Let us set

$$c_k = \inf_{A \in \Gamma_k} \sup_{(u,v) \in A} I_{\lambda,\mu}(u,v),$$

and

$$K_c = \{(u,v) \in E : I_{\lambda,\mu}(u,v) = c, I'_{\lambda,\mu}(u,v) = 0\}.$$

Suppose $0 < \lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}} < \Lambda_*$, where Λ_* is the constant given by Lemma 4. In fact, if we denote $I_{\lambda,\mu}^{-\varepsilon} = \{(u,v) \in E : I_{\lambda,\mu}(u,v) \leq -\varepsilon\}$, by Lemma 5, there exists $\varepsilon(k) > 0$ such that $\gamma(I_{\lambda,\mu}^{-\varepsilon(k)}) \geq k$ for $k \in \mathbb{N}$. Because $I_{\lambda,\mu}$ is continuous and even, $I_{\lambda,\mu}^{-\varepsilon(k)} \in \Gamma_k$, then $c_k \leq -\varepsilon(k) < 0$ for all $k \in \mathbb{N}$. But $I_{\lambda,\mu}$ is bounded from below, hence $c_k > -\infty$ for all $k \in \mathbb{N}$.

Let us assume that $c = c_k = c_{k+1} = \cdots = c_{k+l}$. Note that $c < 0$, therefore, $I_{\lambda,\mu}$ satisfies the $(PS)_c$ condition, and it is easy to see that K_c is a compact set.

If $\gamma(K_c) \leq l$, then there is a closed and symmetric set U with $K_c \subset U$ and $\gamma(U) \leq l$ by the continuity property of genus [17]. By [1, Lemma 1.3], there is an odd homeomorphism $\eta : E \to E$ such that $\eta(I_{\lambda,\mu}^{c+\delta} - U) \subset I_{\lambda,\mu}^{c-\delta}$ for some $\delta > 0$. By definition,

$$c = c_k = c_{k+l} = \inf_{A \in \Gamma_{k+l}} \sup_{(u,v) \in A} I_{\lambda,\mu}(u,v),$$

there exists $A \in \Gamma_{k+l}$ such that $\sup_{(u,v)\in A} I_{\lambda,\mu}(u,v) < c+\delta$, i.e. $A \subset I_{\lambda,\mu}^{c+\delta}$ and $\eta(A-U) \subset \eta(I_{\lambda,\mu}^{c+\delta} - U) \subset I_{\lambda,\mu}^{c-\delta}$, that is

$$\sup_{(u,v)\in\eta(A-U)} I_{\lambda,\mu}(u,v) \leq c-\delta. \tag{3.5}$$

But $\gamma(\overline{A-U}) \geq \gamma(A) - \gamma(U) \geq k$, and $\gamma(\eta(\overline{A-U})) = \gamma(\overline{A-U}) \geq k$, then $\eta(\overline{A-U}) \in \Gamma_k$. This is a contradiction. In fact, $\eta(\overline{A-U}) \in \Gamma_k$ implies that

$$\sup_{(u,v)\in\eta(\overline{A-U})} I_{\lambda,\mu}(u,v) \geq c_k = c,$$

which contradicts to (3.5). So we have proved that $\gamma(K_c) \geq l+1$.

We are now ready to show that $I_{\lambda,\mu}$ has infinitely many critical points. Note c_k is nondecreasing and strictly negative. We distinguish two cases:

Case 1: Suppose that there are $1 < k_1 < \cdots < k_i < \cdots$, satisfying

$$c_{k_1} < \cdots < c_{k_i} < \cdots,$$

then $\gamma(K_c) \geq 1$, and we see that $\{c_{k_i}\}$ is a sequence of distinct critical values of $I_{\lambda,\mu}$.

Case 2: We assume in this case that, for some positive integer k_0, there is $l \geq 1$ such that $c = c_{k_0} = c_{k_0+1} = \cdots = c_{k_0+l}$, then $\gamma(K_{c_{k_0}}) \geq l+1$, which shows that $K_{c_{k_0}}$ contains infinitely many distinct elements.

Since $J_{\lambda,\mu}(u,v) = I_{\lambda,\mu}(u,v)$ if $I_{\lambda,\mu}(u,v) < 0$, we see that there are infinitely many critical points of $J_{\lambda,\mu}(u,v)$, that is to say, there are negative energy solutions to problem (1.1). This completes the proof of Theorem 1. □

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