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BLOCK REPRESENTATIONS FOR THE g-DRAZIN INVERSE IN BANACH ALGEBRAS

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Abstract. We present new formulas for the g-Drazin inverse of the sum in a Banach algebra. The block representations for the g-Drazin inverse of a 2×2 block operator matrix are thereby established. These extend the known results obtained in [3, 7].

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1. Introduction

Let $\mathcal{A}$ be a complex Banach algebra with an identity 1. The commutant of $a\in\mathcal{A}$ is defined by $\mathrm{comm}(a)=\{x\in\mathcal{A}\mid xa=ax\}$. Let $\mathcal{A}^{\mathrm{qnil}}=\{a\in\mathcal{A}\mid 1-\mathrm{comm}(a)a\subseteq\mathcal{A}^{-1}\}$. As is well known, we have $a\in\mathcal{A}^{\mathrm{qnil}}\Leftrightarrow\lim\limits_{n\to\infty}\parallel a^n\parallel^{\frac{1}{n}}=0$. We say that $a\in\mathcal{A}$ has g-Drazin inverse (i.e. generalized Drazin inverse) if there exists $b\in\mathrm{comm}(a)$ such that $b=bab, a-a^2b\in\mathcal{A}^{\mathrm{qnil}}$. Such b is unique, if it exists, and it is denoted by a^d. When we have in the preceding $a-a^2b$ is nilpotent, then b is the Drazin inverse a^D of a. The Drazin and g-Drazin inverse are useful in matrix and operator theory. It has been applied in many fields such as ordinary differential equations, statistics and probability, Marcov chain, etc (see [2]).

The g-Drazin inverse of the sum of two elements in a Banach algebra has been studied by many authors, e.g. [3, 6, 8, 10, 11] and [12]. In [9, Theorem 2.1], Yang and Liu gave the representation of the Drazin inverse of two complex matrices P and Q such that $PQP=0, PQ^2=0$, which recovered the case $PQ=0$ studied by Hartwig et al. In [7], Ljubisavljevic and Cvetkovic-Ilic derived an expression of $(P+Q)^D$ under a weaker condition $PQP^2=0, PQPQ=0, PQ^2P=0$ and $PQ^3=0$. As the computational and applied requirements, the explicit representation of the g-Drazin inverse is expected under certain weaker conditions. This inspires us to derive the explicit formulas of $(a+b)^d$ in terms of a,b and their g-Drazin inverses in

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Banach algebras. In Section 2, we give the explicit representations for $(a+b)^d$ under the conditions $aba^2=0, abab=0, ab^2a=0$ and $ab^3=0$ in a Banach algebra $\mathcal{A}$. Furthermore, the representation of $(a+b)^d$ under the conditions $a^2ba=0, abab=0, ab^2a=0$ and $ab^3=0$ is presented. The known results are thereby extended to wider cases (see [7, Corollary 2.3] and [9, Theorem 2.1]).

Let $\mathcal{L}(X)$ denote the set of all bounded linear operators on a Banach space X. Let

$$M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad (*)$$

where $A\in\mathcal{L}(X), D\in\mathcal{L}(Y)$ have g-Drain inverses and X,Y are complex Banach spaces. Then M is a bounded linear operator on $X\oplus Y$. In Section 3, we apply our results and give the representations for the g-Drazin inverse of some operator matrix M whose Shur complement is zero, i.e. $D=CA^dB$. As Drazin and g-Drazin inverses coincide with each other for complex matrices, our results also extend [1, Theorem 4.1 and Theorem 4.2] with alternative formulas for the Drazin inverse of 2×2 block complex matrices.

2. ADDITIVE RESULTS

In [3, Theorem 2.2 and Theorem 2.4], the authors investigated the g-Drazin invertibility of $a+b$ under the preceding conditions. The purpose of this section is to further study such problems and establish the explicit formulas of the g-Drazin inverse of the sum. If $p=p^2\in\mathcal{A}$ is an idempotent, We can represent $a\in\mathcal{A}$ as $a=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_p$, where $a_{11}=pap, a_{12}=pa(1-p), a_{21}=(1-p)ap$ and $a_{22}=(1-p)a(1-p)$. We use a^{π} to stand for the spectral idempotent $1-aa^d$ of $a\in\mathcal{A}^d$. We begin with

Lemma 1. *Let*

$$x=\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}_p \text{ or } \begin{pmatrix} b & c \\ 0 & a \end{pmatrix}_p$$

Then

$$x^d=\begin{pmatrix} a^d & 0 \\ z & b^d \end{pmatrix}_p \text{ or } \begin{pmatrix} b^d & z \\ 0 & a^d \end{pmatrix}_p,$$

where

$$z=\sum_{i=0}^{\infty}(b^d)^{i+2}ca^ia^{\pi}+\sum_{i=0}^{\infty}b^ib^{\pi}c(a^d)^{i+2}-b^dca^d.$$

Proof. See [6, Theorem 2.1]. □

Lemma 2. *Let $a,b\in\mathcal{A}^d$. If $ab=0$, then $a+b\in\mathcal{A}^d$. In this case*

$$(a+b)^d=\sum_{i=0}^{\infty}b^ib^{\pi}(a^d)^{i+1}+\sum_{i=0}^{\infty}(b^d)^{i+1}a^ia^{\pi}.$$

Proof. See [4, Lemma 2.2]. □

We are now ready to extend [3, Theorem 2.2] and prove:

Theorem 1. *Let $a,b\in\mathcal{A}^d$. If $aba^2=0, abab=0, ab^2a=0$ and $ab^3=0$, then*

$$\begin{aligned}(a+b)^d &= a^d+(a^d)^3ba+(a^d)^2b+(a^d)^3b^2\\ &\quad +b(a^d)^2+b(a^d)^4ba+b(a^d)^3b+b(a^d)^4b^2+a(b^d)^9aba+(b^d)^8aba\\ &\quad +a(b^d)^9a^2b+a(b^d)^8ab+a(b^d)^9ab^2+(b^d)^8a^2b+(b^d)^7ab+(b^d)^8ab^2\\ &\quad +(ab+b^2)z(a+a^dba+aa^db+a^db^2)\\ &\quad +\big(a(b^d)^4+(b^d)^3\big)z(a^2ba+a^3b+a^2b^2),\end{aligned}$$

where

$$\begin{aligned}z &= -b^d(a^d)^3+b^{\pi}(a^d)^4+bb^{\pi}(a^d)^5+(b^d)^4a^{\pi}\\ &\quad -(b^d)^5a^d-(b^d)^4(a^d)^2+\sum_{n=1}^{\infty}u_n,\\ u_n &= b^{3n-1}b^{\pi}(a^d)^{3n+3}+b^{3n}b^{\pi}(a^d)^{3n+4}+b^{3n+1}b^{\pi}(a^d)^{3n+5}\\ &\quad +(b^d)^{3n+4}a^{3n}a^{\pi}+(b^d)^{3n+3}a^{3n-1}a^{\pi}+(b^d)^{3n+2}a^{3n-2}a^{\pi}.\end{aligned}$$

Proof. Set

$$M=\begin{pmatrix} a^3+a^2b+aba+ab^2 & a^3b+a^2b^2+abab+ab^3\\ a^2+ab+ba+b^2 & a^2b+ab^2+bab+b^3\end{pmatrix}.$$

Then $M=G+F$, where

$$G=\begin{pmatrix} a^2b+aba+ab^2 & a^3b+a^2b^2\\ 0 & a^2b+ab^2+bab\end{pmatrix},\quad F=\begin{pmatrix} a^3 & 0\\ a^2+ab+ba+b^2 & b^3\end{pmatrix}.$$

Clearly $F=A+B$, where

$$A=\begin{pmatrix} 0 & 0\\ ab & 0\end{pmatrix} \text{ and } B=\begin{pmatrix} a^3 & 0\\ a^2+ba+b^2 & b^3\end{pmatrix}.$$

Moreover we have $B=H+K$, where

$$H=\begin{pmatrix} a^3 & 0\\ a^2+ba & 0\end{pmatrix} \text{ and } K=\begin{pmatrix} 0 & 0\\ b^2 & b^3\end{pmatrix}.$$

In view of [3, Theorem 2.2], we have

$$(a+b)^d=(a+b,ab+b^2)M^d\begin{pmatrix} a\\ 1\end{pmatrix},$$

where $M^d=F^d+(F^d)^2G$ and

$$F^d=B^d+(B^d)^2A,$$

$$B^d=(I-KK^d)\big[\sum_{n=0}^{\infty}K^n(H^d)^n\big]H^d+K^d\big[\sum_{n=0}^{\infty}(K^d)^nH^n\big](I-HH^d),$$
$$H^d=\begin{pmatrix} (a^d)^3 & 0\\ (a^d)^4+b(a^d)^5 & 0\end{pmatrix},$$
$$K^d=\begin{pmatrix} 0 & 0\\ (b^d)^4 & (b^d)^3\end{pmatrix}.$$

We compute that

$$I-HH^d=\begin{pmatrix} a^{\pi} & 0\\ -(a+b)(a^d)^2 & 1\end{pmatrix};$$
$$(H-H^2H^d)^n=\begin{pmatrix} a^{3n}a^{\pi} & 0\\ (a+b)a^{3n-2}a^{\pi} & 0\end{pmatrix};$$
$$I-KK^d=\begin{pmatrix} 1 & 0\\ -b^d & b^{\pi}\end{pmatrix},$$
$$(K-K^2K^d)^n=\begin{pmatrix} 0 & 0\\ b^{3n-1}b^{\pi} & b^{3n}b^{\pi}\end{pmatrix}\quad n\in\mathbb{N}.$$

Moreover we have

$$(H^d)^{n+1}=\begin{pmatrix} (a^d)^{3n+3} & 0\\ (a^d)^{3n+4}+b(a^d)^{3n+5} & 0\end{pmatrix},\quad (K^d)^{n+1}=\begin{pmatrix} 0 & 0\\ (b^d)^{3n+4} & (b^d)^{3n+3}\end{pmatrix}.$$

Therefore we have

$$\begin{aligned}B^d&=(I-KK^d)H^d+\sum_{n=1}^{\infty}(K-K^2K^d)^n(H^d)^{n+1}\\&\quad+K^d(I-HH^d)+\sum_{n=1}^{\infty}(K^d)^{n+1}(H-H^2H^d)^n\\&=\begin{pmatrix} (a^d)^3 & 0\\ z & (b^d)^5\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}z&=-b^d(a^d)^3+b^{\pi}(a^d)^4+bb^{\pi}(a^d)^5+(b^d)^4a^{\pi}\\&\quad-(b^d)^5a^d-(b^d)^4(a^d)^2+\sum_{n=1}^{\infty}u_n,\\ u_n&=b^{3n-1}b^{\pi}(a^d)^{3n+3}+b^{3n}b^{\pi}(a^d)^{3n+4}+b^{3n+1}b^{\pi}(a^d)^{3n+5}\\&\quad+(b^d)^{3n+4}a^{3n}a^{\pi}+(b^d)^{3n+3}a^{3n-1}a^{\pi}+(b^d)^{3n+2}a^{3n-2}a^{\pi}.\end{aligned}$$

Then we have

$$F^d=B^d(I+B^dA)$$

$$= \begin{pmatrix} (a^d)^3 & 0 \\ z & (b^d)^5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (b^d)^5 ab & 1 \end{pmatrix} = \begin{pmatrix} (a^d)^3 & 0 \\ z+(b^d)^{10}ab & (b^d)^5 \end{pmatrix}.$$

Moreover we have

$$M^d = F^d(I+F^dG) = \begin{pmatrix} (a^d)^3 & 0 \\ z+(b^d)^{10}ab & (b^d)^5 \end{pmatrix}$$
$$\begin{pmatrix} 1+a^db+(a^d)^2ba+(a^d)^2b^2 & aa^db+a^db^2 \\ z(a^2b+aba+ab^2) & z(a^3b+a^2b^2)+(b^d)^5(a^2b+bab+ab^2) \end{pmatrix}.$$

Therefore $(a+b)^d = (a+b, ab+b^2)M^d \begin{pmatrix} a \\ 1 \end{pmatrix} = vw$, where

$$\begin{aligned} v &= (a+b, ab+b^2)\begin{pmatrix} (a^d)^3 & 0 \\ z+(b^d)^{10}ab & (b^d)^5 \end{pmatrix} \\ &= \big((a^d)^2+b(a^d)^3+(a+b)bz+(a+b)(b^d)^9ab, (a+b)(b^d)^4\big), \\ w &= \begin{pmatrix} 1+a^db+(a^d)^2ba+(a^d)^2b^2 & aa^db+a^db^2 \\ z(a^2b+aba+ab^2) & z(a^3b+a^2b^2)+(b^d)^5(a^2b+bab+ab^2) \end{pmatrix} \\ &\quad \begin{pmatrix} a \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} a+a^dba+aa^db+a^db^2 \\ z(a^2ba+a^3b+a^2b^2)+(b^d)^5(a^2b+bab+ab^2) \end{pmatrix}. \end{aligned}$$

By the direct computation, we complete the proof. □

Bu and Zhang considered the Drazin inverses of block complex matrices under the conditions $PQP^2 = 0, QPQP = 0$ and $Q^2 = 0$ (see [1, Remark 3.1]). As an immediate consequence of Theorem 1, we now derive

Corollary 1. *Let $a,b \in \mathcal{A}^d$. If $aba^2 = 0, abab = 0$ and $b^2 = 0$, then*

$$(a+b)^d = a^d + (a^d)^2b + (a^d)^3ba + b(a^d)^2 + b(a^d)^3b + b(a^d)^4ba.$$

Proof. Using the notations in Theorem 1, we have $u_n = 0, z = (a^d)^4 + b(a^d)^5$. In view of Theorem 1, $a+b$ has g-Drazin inverse and

$$\begin{aligned} (a+b)^d &= a^d + (a^d)^3ba + (a^d)^2b + b(a^d)^2 + b(a^d)^4ba + b(a^d)^3b \\ &\quad + abz(a+a^dba+aa^db) \\ &= a^d + (a^d)^2b + (a^d)^3ba + b(a^d)^2 + b(a^d)^3b + b(a^d)^4ba, \end{aligned}$$

as asserted. □

Using the previous corollary, we now extend [3, Theorem 2.4].

Theorem 2. *Let $a,b\in\mathcal{A}^d$. If $a^2ba=0, abab=0, ab^2a=0$ and $ab^3=0$, then*
$$\begin{aligned}(a+b)^d &= \left[(a+b)^2b\Gamma+(a+b)\Lambda\right]\left[\Gamma+\Delta(a+b)a\right]\\ &\quad+\left[(a+b)^2b\Delta+(a+b)\Xi\right]\left[\Lambda+\Xi(a+b)a\right],\end{aligned}$$
where
$$\begin{aligned}
\Gamma &= \alpha^d+z_2\beta\alpha+\alpha^d z_2\beta\alpha^2+z_1\beta^d\alpha^2,\\
\Delta &= z_1+\alpha^d z_2\beta\alpha+z_1\beta^d\alpha,\\
\Lambda &= \beta^d\alpha+(\beta^d)^2\alpha^2+\beta\alpha^d+\beta\alpha^d z_1\beta\alpha+\beta\alpha z_2\beta^d\beta\alpha\\
&\quad+\beta\alpha^d z_2\beta\alpha^2+\beta\alpha z_2\beta^d\alpha^2,\\
\Xi &= \beta^d+(\beta^d)^2\alpha+\beta\alpha z_2+\beta\alpha^d z_2\beta\alpha+\beta\alpha z_2\beta^d\alpha,\\
\alpha &= b^2+ab, \beta=ba+a^2,\\
\alpha^d &= (b^d)^2+(b^d)^4ab, \beta^d=(a^d)^2+b(a^d)^3+bab(a^d)^5,\\
z_1 &= \sum_{i=0}^{\infty}(\alpha^d)^{i+2}\beta^i\beta^\pi+\sum_{i=0}^{\infty}\alpha^i\alpha^\pi(\beta^d)^{i+2}-\alpha^d\beta^d,\\
z_2 &= \alpha^d z_1+z_1\beta^d.
\end{aligned}$$

Proof. Let $\alpha=b^2+ab$ and $\beta=ba+a^2$. Since $ab^3=0$, it follows by Lemma 2 that α has g-Drazin inverse and $\alpha^d=(b^d)^2+(b^d)^4ab$. Since $a^2ba=0$, similarly, we have $\beta^d=(a^d)^2+b(a^d)^3+bab(a^d)^5$. Set
$$M=\begin{pmatrix} b^2+ab & 1\\ (a+b)a(a+b)b & ba+a^2\end{pmatrix}.$$
Then $M=A+B$, where
$$A=\begin{pmatrix}\alpha & 1\\ 0 & \beta\end{pmatrix} \text{ and } B=\begin{pmatrix} 0 & 0\\ \beta\alpha & 0\end{pmatrix}.$$
In view of Lemma 1, $A^d=\begin{pmatrix}\alpha^d & z_1\\ 0 & \beta^d\end{pmatrix}$, where
$$z_1=\sum_{i=0}^{\infty}(\alpha^d)^{i+2}\beta^i\beta^\pi+\sum_{i=0}^{\infty}\alpha^i\alpha^\pi(\beta^d)^{i+2}-\alpha^d\beta^d.$$
Let $z_2=\alpha^d z_1+z_1\beta^d$. Then
$$(A^d)^2=\begin{pmatrix}(\alpha^d)^2 & z_2\\ 0 & (\beta^d)^2\end{pmatrix}.$$
Clearly $B^2=0$, and so $B^d=0$. We compute that
$$(A^d)^2B=\begin{pmatrix} z_2\beta\alpha & 0\\ \beta^d\alpha & 0\end{pmatrix},$$

$$
\begin{aligned}
(A^d)^3BA &= \begin{pmatrix} \alpha^d z_2\beta\alpha^2 + z_1\beta^d\alpha^2 & \alpha^d z_2\beta\alpha + z_1\beta^d\alpha \\ (\beta^d)^2\alpha^2 & (\beta^d)^2\alpha \end{pmatrix}, \\
B(A^d)^2 &= \begin{pmatrix} 0 & 0 \\ \beta\alpha^d & \beta\alpha z_2 \end{pmatrix}, \\
B(A^d)^3B &= \begin{pmatrix} 0 & 0 \\ \beta\alpha^d z_1\beta\alpha + \beta\alpha z_2\beta^d\beta\alpha & 0 \end{pmatrix}, \\
B(A^d)^4BA &= \begin{pmatrix} 0 & 0 \\ \beta\alpha^d z_2\beta\alpha^2 + \beta\alpha z_2\beta^d\alpha^2 & \beta\alpha^d z_2\beta\alpha + \beta\alpha z_2\beta^d\alpha \end{pmatrix}.
\end{aligned}
$$

We easily check that $ABAB = 0, ABA^2 = 0$ and $B^2 = 0$. In view of Corollary 1, we have

$$
M^d = A^d + (A^d)^2B + (A^d)^3BA + B(A^d)^2 + B(A^d)^3B + B(A^d)^4BA = \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},
$$

where

$$
\begin{aligned}
\Gamma &= \alpha^d + z_2\beta\alpha + \alpha^d z_2\beta\alpha^2 + z_1\beta^d\alpha^2, \\
\Delta &= z_1 + \alpha^d z_2\beta\alpha + z_1\beta^d\alpha, \\
\Lambda &= \beta^d\alpha + (\beta^d)^2\alpha^2 + \beta\alpha^d + \beta\alpha^d z_1\beta\alpha + \beta\alpha z_2\beta^d\beta\alpha \\
&\quad + \beta\alpha^d z_2\beta\alpha^2 + \beta\alpha z_2\beta^d\alpha^2, \\
\Xi &= \beta^d + (\beta^d)^2\alpha + \beta\alpha z_2 + \beta\alpha^d z_2\beta\alpha + \beta\alpha z_2\beta^d\alpha,
\end{aligned}
$$

Let $N = \left(\begin{pmatrix} 1 \\ a \end{pmatrix}(b,1)\right)^2$. Then $N = \begin{pmatrix} b^2+ab & a+b \\ ab^2+a^2b & a^2+ab \end{pmatrix}$. As in the proof of [3, Theorem 2.4], we have

$$
M = \begin{pmatrix} 1 & 0 \\ 0 & a+b \end{pmatrix}\begin{pmatrix} (a+b)b & 1 \\ a(a+b)b & a \end{pmatrix}, \quad N = \begin{pmatrix} (a+b)b & 1 \\ a(a+b)b & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & a+b \end{pmatrix}.
$$

By using Cline's formula (see [5, Theorem 2.9]), N has g-Drazin inverse and

$$
N^d = \begin{pmatrix} (a+b)b & 1 \\ a(a+b)b & a \end{pmatrix}[M^d]^2\begin{pmatrix} 1 & 0 \\ 0 & a+b \end{pmatrix}.
$$

In view of [5, Theorem 2.7], $Q := \begin{pmatrix} 1 \\ a \end{pmatrix}(b,1)$ has g-Drazin inverse and $(Q^d)^2 = (Q^2)^d = N^d$. By using Cline's formula again,

$$
\begin{aligned}
(a+b)^d &= (b,1)N^d\begin{pmatrix} 1 \\ a \end{pmatrix} \\
&= (b,1)\begin{pmatrix} (a+b)b & 1 \\ a(a+b)b & a \end{pmatrix}[M^d]^2\begin{pmatrix} 1 & 0 \\ 0 & a+b \end{pmatrix}\begin{pmatrix} 1 \\ a \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= ((a+b)^2b, a+b)[M^d]^2 \begin{pmatrix} 1 \\ a^2+ba \end{pmatrix} \\
&= \big(((a+b)^2b, a+b)M^d\big)\big(M^d \begin{pmatrix} 1 \\ a^2+ba \end{pmatrix}\big) \\
&= \big[(a+b)^2b\Gamma + (a+b)\Lambda\big]\big[\Gamma + \Delta(a+b)a\big] \\
&\quad + \big[(a+b)^2b\Delta + (a+b)\Xi\big]\big[\Lambda + \Xi(a+b)a\big].
\end{aligned}
$$

This completes the proof. □

Corollary 2. *Let $a, b \in \mathcal{A}^d$. If $a^2ba = 0, abab = 0$ and $b^2 = 0$, then*

$$
\begin{aligned}
(a+b)^d = &\big[(a+b)ab\Gamma + (a+b)\Lambda\big]\big[\Gamma + \Delta(a+b)a\big] \\
&+ \big[(a+b)ab\Delta + (a+b)\Xi\big]\big[\Lambda + \Xi(a+b)a\big],
\end{aligned}
$$

where

$$
\begin{aligned}
&\Gamma = z_2\beta\alpha + z_1\beta^d\alpha^2, \quad \Delta = z_1 + z_1\beta^d\alpha, \\
&\Lambda = \beta^d\alpha + (\beta^d)^2\alpha^2 + \beta\alpha z_2\beta^d\beta\alpha + \beta\alpha z_2\beta^d\alpha^2, \\
&\Xi = \beta^d + (\beta^d)^2\alpha + \beta\alpha z_2 + \beta\alpha z_2\beta^d\alpha, \\
&\alpha = ab, \quad \beta = ba + a^2, \quad \alpha^d = 0, \quad \beta^d = (a^d)^2 + b(a^d)^3 + bab(a^d)^5, \\
&z_1 = (\beta^d)^2 + ab(a^d)^6, \quad z_2 = (\beta^d)^3 + ab(a^d)^8.
\end{aligned}
$$

Proof. This is obvious by Theorem 2. □

For complex matrices, Drain and g-Drazin inverses coincide with each other. Thus the preceding theorems give alternative formulas for the Drazin inverse of the sum of two block complex matrices provided in [7, Corollary 2.]. The following example illustrates that Theorem 2 is not a trivial generalization of [9, Theorem 2.1].

Example 1. Let

$$
a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in M_4(\mathbb{C}).
$$

Then $a, b \in M_4(\mathbb{C})^d$. It is clear that $a^2ba = 0, abab = 0, ab^2a = 0$ and $ab^3 = 0$, but $aba = ab^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0.$

3. Block operator matrices

In [1], Bu and Zhang considered a class of block complex matrices with zero generalized Schur complement. The goal of this section is to provide explicit representations for the g-Drazin inverse of the operator matrix M given by $(*)$. We now derive.

Theorem 3. *Let $A \in \mathcal{L}(X)$ have g-Drazin inverse, $D \in \mathcal{L}(Y)$ and M be given by $(*)$. Let $W = AA^d + A^dBCA^d$. If AW has g-Drazin inverse, $AA^{\pi}BCA = 0, CA^{\pi}BCA = 0, AA^{\pi}BCB = 0, CA^{\pi}\ BCB = 0$ and $D = CA^dB$, then M has g-Drazin inverse. In this case*

$$M^d = P^d + (P^d)^2Q + (P^d)^3QP + Q(P^d)^2 + Q(P^d)^3Q + Q(P^d)^4QP,$$

where $P = \begin{pmatrix} A & AA^dB \\ C & CA^dB \end{pmatrix}$ *and* $Q = \begin{pmatrix} 0 & A^{\pi}B \\ 0 & 0 \end{pmatrix}$ *and*

$$P^d = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^dB \end{pmatrix} + \sum_{i=1}^{\infty} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^{i+2} \begin{pmatrix} A^{i+1}A^{\pi} + AA^dBCA^{i-1}A^{\pi}, 0 \end{pmatrix}.$$

Proof. One easily checks that

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A & AA^dB \\ C & CA^dB \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & A^{\pi}B \\ 0 & 0 \end{pmatrix}.$$

By hypothesis, we have $PQPQ = 0, PQP^2 = 0, Q^2 = 0$. By virtue of Corollary 1, we have $M^d = P^d + (P^d)^2Q + (P^d)^3QP + Q(P^d)^2 + Q(P^d)^3Q + Q(P^d)^4QP$. Moreover we have

$$P = P_1 + P_2, P_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix}, \quad P_2 = \begin{pmatrix} AA^{\pi} & 0 \\ CA^{\pi} & 0 \end{pmatrix}$$

and $P_2P_1 = 0$. By virtue of Lemma 1, P_2 has g-Drazin inverse. Obviously, we have

$$P_1 = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} \begin{pmatrix} A, AA^dB \end{pmatrix}.$$

By hypothesis, we see that

$$\begin{pmatrix} A, AA^dB \end{pmatrix} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} = AW$$

has g-Drazin inverse. By using Cline's formula, we see that

$$P_1^d = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^dB \end{pmatrix}.$$

For any $i \in \mathbb{N}$, we compute that

$$(P_1^d)^i = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^{i+1} \begin{pmatrix} A, AA^dB \end{pmatrix} \text{ and } P_2^i = \begin{pmatrix} A^iA^\pi & 0 \\ CA^{i-1}A^\pi & 0 \end{pmatrix}.$$

According to Lemma 2, P has g-Drazin inverse and

$$\begin{aligned} P^d &= \sum_{i=0}^{\infty} P_1^i P_1^\pi (P_2^d)^{i+1} + \sum_{i=0}^{\infty} (P_1^d)^{i+1} P_2^i P_2^\pi \\ &= \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^dB \end{pmatrix} \\ &+ \sum_{i=1}^{\infty} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^{i+2} \begin{pmatrix} A^{i+1}A^\pi + AA^dBCA^{i-1}A^\pi, 0 \end{pmatrix}. \end{aligned}$$

This completes the proof. □

Corollary 3. *Let $A \in \mathcal{L}(X)$ have g-Drazin inverse, $D \in \mathcal{L}(Y)$ and M be given by $(*)$. Let $W = AA^d + A^dBCA^d$. If AW has g-Drazin inverse, $A^\pi BC = 0$ and $D = CA^dB$, then M has g-Drazin inverse. In this case*

$$M^d = P^d + (P^d)^2 Q,$$

where

$$P = \begin{pmatrix} A & AA^dB \\ C & CA^dB \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} P^d &= \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^dB \end{pmatrix} \\ &+ \sum_{i=1}^{\infty} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^{i+2} \begin{pmatrix} A^{i+1}A^\pi + BCA^{i-1}A^\pi, 0 \end{pmatrix}. \end{aligned}$$

Proof. Since $QP = 0$, we obtain the result by Theorem 3. □

The following example illustrates that Theorem 3 is a nontrivial generalization of [9, Theorem 3.3].

Example 2. Let A, B, C and D be the linear operators on $\mathbb{C}^2$ given by 2×2 matrices over $\mathbb{C}$.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } D = 0.$$

Set $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then M can be seen as the linear operator acting on $\mathbb{C}^2 \times \mathbb{C}^2$. We check that

$$AA^\pi BCA = 0, CA^\pi BCA = 0, AA^\pi BCB = 0, CA^\pi BCB = 0$$

and $D = CA^dB$, while $CA^{\pi}BC = \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} \neq 0$.

Finally, we split the block operator matrix M in an other way and extend [1, Theorem 4.2] as follows.

Theorem 4. *Let $A \in \mathcal{L}(X)$ have g-Drazin inverse, $D \in \mathcal{L}(Y)$ and M be given by $(*)$. Let $W = AA^d + A^dBCA^d$. If AW has g-Drazin inverse, $BCA^{\pi}A^2 = 0$, $BCA^{\pi}AB = 0$, $BCA^{\pi}BCA^d = 0$ and $D = CA^dB$, then M has g-Drazin inverse. In this case*

$$M^d = P^d + (P^d)^2Q + (P^d)^3QP + Q(P^d)^2 + Q(P^d)^3Q + Q(P^d)^4QP,$$

where

$$P = \begin{pmatrix} A & B \\ CAA^d & CA^dB \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 \\ CA^{\pi} & 0 \end{pmatrix}$$

and

$$\begin{aligned} P^d = & \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^dB \end{pmatrix} \\ & + \sum_{i=1}^{\infty} \begin{pmatrix} A^{i-1}A^{\pi}BCA^d \\ 0 \end{pmatrix} [(AW)^d]^{i+2} \begin{pmatrix} A, AA^dB \end{pmatrix}. \end{aligned}$$

Proof. Clearly we have

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A & B \\ CAA^d & CA^dB \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0 & 0 \\ CA^{\pi} & 0 \end{pmatrix}.$$

Then we check that $PQPQ = 0, PQP^2 = 0, Q^2 = 0$. In view of Corollary 1, we have

$$M^d = P^d + (P^d)^2Q + (P^d)^3QP + Q(P^d)^2 + Q(P^d)^3Q + Q(P^d)^4QP.$$

Moreover we have

$$P = P_1 + P_2, P_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix}, \quad P_2 = \begin{pmatrix} AA^{\pi} & A^{\pi}B \\ 0 & 0 \end{pmatrix}$$

and $P_1P_2 = 0$. In light of Lemma 2, P_2 has g-Drazin inverse. As in the proof of Theorem 3, we have

$$P_1^d = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^dB \end{pmatrix}.$$

For any $i \in \mathbb{N}$, we compute that

$$(P_1^d)^i = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^{i+1} \begin{pmatrix} A, AA^dB \end{pmatrix} \text{ and } P_2^i = \begin{pmatrix} A^iA^{\pi} & A^{i-1}A^{\pi}B \\ 0 & 0 \end{pmatrix}.$$

According to Lemma 2, P has g-Drazin inverse and

$$\begin{aligned} P^d &= \sum_{i=0}^{\infty} P_2^i P_2^{\pi} (P_1^d)^{i+1} + \sum_{i=0}^{\infty} (P_2^d)^{i+1} P_1^i P_1^{\pi} \\ &= \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^d B \end{pmatrix} \\ &\quad + \sum_{i=1}^{\infty} \begin{pmatrix} A^{i-1}A^{\pi}BCA^d \\ 0 \end{pmatrix} [(AW)^d]^{i+2} \begin{pmatrix} A, AA^d B \end{pmatrix}, \end{aligned}$$

and we are through. □

As an immediate consequence of Theorem 4, we have

Corollary 4. *Let $A \in \mathcal{L}(X)$ have g-Drazin inverse, $D \in \mathcal{L}(Y)$ and M be given by $(*)$. Let $W = AA^d + A^d BCA^d$. If AW has g-Drazin inverse, $BCA^{\pi} = 0$ and $D = CA^d B$, then M has g-Drazin inverse. In this case,*

$$M^d = P^d + Q(P^d)^2,$$

where

$$\begin{aligned} Q &= \begin{pmatrix} 0 & 0 \\ CA^{\pi} & 0 \end{pmatrix}, \\ P^d &= \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^d B \end{pmatrix} \\ &\quad + \sum_{i=1}^{\infty} \begin{pmatrix} A^{i-1}A^{\pi}BCA^d \\ 0 \end{pmatrix} [(AW)^d]^{i+2} \begin{pmatrix} A, AA^d B \end{pmatrix}. \end{aligned}$$

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