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A NEW APPROACH TO STATISTICAL RIEMANN-STIELTJES INTEGRALS

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Abstract. This paper aims to introduce various concepts of statistical Riemann-Stieltjes sum via the deferred Nörlund summability mean for the sequence of functions as well as the sequence of distribution functions. We first establish some fundamental limit theorems for statistical Riemann-Stieltjes sum in the sequence space. Then over the probability sequence space, we establish some more advanced results. Finally, over both the spaces we establish some inclusion theorems via our proposed mean in association with statistical Riemann-Stieltjes integral.

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1. Introduction and Motivation

Let $[a,b] \subset \mathbb{R}$, and for all $k \in \mathbb{N}$ there is a function $g_k : [a,b] \to \mathbb{R}$ and it is called a sequence (g_k) of functions on $[a,b]$.

We now define the Riemann-Stieltjes sum of a sequence (g_k) of functions along with a tagged partition $\dot{\mathcal{P}}$ which is given by

$$\delta(g_k; \dot{\mathcal{P}}) := \sum_{i=1}^{k} g_i(\gamma_i)[\alpha(r_i) - \alpha(r_{i-1})],$$

where α is a monotonic increasing function.

Let $\mathcal{F}$ be a real-valued monotone transformation on $I \subset \mathbb{R}$,

$$\mathcal{F} : I \to \mathcal{F}(I).$$

We next assume that (h_k) is a real-valued sequence of step functions on the interval I,

$$h_k(t) = \sum_{i=1}^{k} c_i 1(t \in I_i),$$

where c_i's are constants and I_i is a partition with $I = \cup_{i=1}^{n} I_i$.

We now recall the Riemann-Stieltjes integral of $(h_k(u))$ sequence of functions as

$$\int h_k(u)d\mathcal{F}(u) = \sum_{i=1}^{k} c_i|\mathcal{F}(I_i)|.$$

Note 1. Let $|\mathcal{F}(I)| = \mathcal{F}(\max(I_i)) - \mathcal{F}(\min(I_i))$. If $I_i = (a_i, b_i)$, then

$$| \mathcal{F}(I_i) |= \mathcal{F}(b_i) - \mathcal{F}(a_i),$$

and the integral value of $(h_k(u))$ is

$$\int_I h_k(u)d\mathcal{F}(u) = \sum_{i=1}^{k} c_i(\mathcal{F}(b_i) - \mathcal{F}(a_i)).$$

Accordingly, we make the following definition.

Definition 1. Let $\mathcal{F}$ be an increasing function and (g_k) be a sequence of functions defined over the interval I. The given sequence (g_k) of functions is Riemann-Stieltjes integrable with respect to $\mathcal{F}$, if for every $\varepsilon > 0$, there exists a sequence of step functions h'_k and h''_k such that

$$\int_I h''_k(u)d\mathcal{F}(u) - \int_I h'_k(u)d\mathcal{F}(u) < \varepsilon \quad (h'_k < g_k < h''_k)$$

and

$$\int_I g_k(u)d\mathcal{F}(u) = \sup \int_I h_k(u)d\mathcal{F}(u),$$

where $h_k < g_k$ and h_k is a sequence of step functions.

Note 2. The Riemann integral is a special case of Riemann-Stieltjes integral, when

$$\mathcal{F}(x) = x$$

is the identity transformation. Thus,

$$\int g_k(t)dF(t) = \int g_k(t)dt \qquad (\text{because}\, \mathcal{F}(t) = t).$$

The study of *convergence* on sequence space is one of the most important and fascinating aspects in the domain of real and functional analysis. The gradual improvement in this study leads to the development of *statistical convergence* which is more general than the usual convergence. The credit for independently defining this beautiful concept goes to both Fast [3] and Steinhaus [20]. Nowadays, this potential notion of statistical convergence has been a field of interest of many researchers and becoming an active research area in various fields of pure and applied Mathematics. In particular, it is very much useful in the study of Machine Learning, Soft Computing, Number Theory, Measure theory, Probability theory, and so on. For some recent research works in this direction, see [1], [4], [5], [6], [7], [8], [9], [12], [14], [19] and [21].

Suppose $\mathcal{Y} \subseteq \mathbb{N}$, and let $\mathcal{Y}_k = \{\zeta : \zeta \leq k \quad \text{and} \quad \zeta \in \mathcal{Y}\}$. Then the natural density $d(\mathcal{Y})$ of $\mathcal{Y}$ is defined by

$$d(\mathcal{Y}) = \lim_{k\to\infty} \frac{|\mathcal{Y}_k|}{k} = \rho,$$

where the number ρ is real and finite, and $|\mathcal{Y}_k|$ is the cardinality of $\mathcal{Y}_k$.

A given sequence (y_k) is statistically convergent to a if, for each $\varepsilon > 0$,

$$\mathcal{Y}_\varepsilon = \{\zeta : \zeta \in \mathbb{N} \quad \text{and} \quad |y_\zeta - a| \geq \varepsilon\}$$

has zero natural density (see [3] and [20]). Thus, for each $\varepsilon > 0$,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0.$$

We write

$$\text{stat} \lim_{k\to\infty} y_k = a.$$

In the year 2002, Móricz [11] studied and introduced the notion of statistical Cesàro summability. Subsequently, Karakaya and Chishti [10] introduced the idea of weighted statistical convergence. Recently, Srivastava *et al.* [16] introduced and studied the notion of statistical probability convergence via the deferred Nörlund mean and its applications to approximation theorems, and in the same year Srivastava *et al.* [14] also studied a result on deferred weighted statistical probability convergence together with its applications to approximation theorems. Furthermore, Jena and Paikray [5] also proved a result on the product of deferred Cesàro and deferred weighted statistical probability convergence and its applications to Korovkin-type theorems. Subsequently, Srivastava *et al.* [18] established another result on statistical product convergence of martingale sequences and its applications to Korovkin-type approximation theorems. Recently, Jena and Paikray [6] demonstrated statistical convergence of martingale difference sequence via deferred weighted mean and proved a Korovkin-type theorem. Moreover, such concepts have been generalized in many aspects. In this direction, the interested readers may refer, [17], [15] and [13].

We now make the following definition of statistical Riemann-Stieltjes integrablility for sequence of functions.

Definition 2. Let $\mathcal{F}$ be an increasing function, and let (g_k) be a sequence of functions defined over the interval I. The given sequence (g_k) of functions is statistical Riemann-Stieltjes integrable to g with respect to $\mathcal{F}$, if for every $\varepsilon > 0$, there exists a sequence of step functions h'_k and h''_k such that

$$\int_I h''_k(u)d\mathcal{F}(u) - \int_I h'_k(u)d\mathcal{F}(u) < \varepsilon \quad (h'_k < g_k < h''_k)$$

and the set

$$\mathcal{Y}_\varepsilon = \left\{\zeta : \zeta \in \mathbb{N} \quad \text{and} \quad \left|\int_I g_k(u)d\mathcal{F}(u) - g\right| \geq \varepsilon\right\}$$

has zero natural density (see [3] and [20]). Thus, for each $\varepsilon > 0$,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0.$$

We write

$$\mathrm{stat}_{\mathrm{RS}} \lim_{k\to\infty} \int_I g_k(u) d\mathcal{F}(u) = g.$$

By making use of Definitions 1 and 2, we setup an example (below), every Riemann-Stieltjes integrability implies statistical Riemann-Stieltjes integrability. However, the converse is not true.

Example 1. Let $g_k : [0,1] \to \mathbb{R}$ be a sequence of functions defined by

$$g_k(x) = \begin{cases} \frac{1}{2} & (x \in \mathbb{Q} \cap [0,1];\ k = j^2,\ j \in \mathbb{N}) \\ \\ \frac{n}{n+1} & (\text{otherwise}). \end{cases} \tag{1.1}$$

It is easy to see that the sequence (g_k) of functions is statistically Riemann-Stieltjes integrable to 1 over $[0,1]$, but not Riemann-Stieltjes integrable (in the ordinary sense) over $[0,1]$.

Next, in view of Definition 2, we establish the following theorem.

Theorem 1. *Let $\mathcal{F}$ be a bounded and increasing function on I, and let the sequence of functions (g_k) be bounded and uniformly convergent on I. Then (g_k) is statistically Riemann-Stieltjes integrable on I.*

Proof. Given $\mathcal{F}$ be a bounded and increasing function on $I = [a,b]$, that is,

$$-\infty < \ell = \mathcal{F}(a) = \inf_I \mathcal{F} \leq \sup_I \mathcal{F} = \mathcal{F}(b) = L < +\infty.$$

Suppose that I is finite. Since (g_k) is uniformly convergent to a function g and also bounded on I, so for each $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ with $k \geq N(\varepsilon)$ such that

$$\|g_k(t) - g(t)\| < \varepsilon \qquad (\forall\, t \in I).$$

Next, let I_i be a partition with $I = \cup_{i=1}^n I_i$ and $|\mathcal{F}(I_i)| \leq \delta$ $(\delta > 0)$, and let

$$m_i = \inf_{t\in I_i} g_k(t) \quad \text{and} \quad M_i = \sup_{t\in I_i} g_k(t),$$

where $M_i - m_i < \varepsilon$.

Further, the sequence of step functions are

$$h_k'(t) = \sum_{i=1}^k m_i 1\{t \in I_i\} \quad \text{and} \quad h_k''(t) = \sum_{i=1}^k M_i 1\{t \in I_i\}$$

with $h_k' < g_k < h_k''$.

Thus,

$$\int_I h'_k(u)d\mathcal{F}(u) = \sum_{i=1}^{n} m_i|\mathcal{F}(I_i)| \le \sum_{i=1}^{n} M_i|\mathcal{F}(I_i)| = \int_I h''_k(u)d\mathcal{F}(u),$$

which implies that

$$\begin{aligned}\int_I h''_k(u)d\mathcal{F}(u) - \int_I h'_k(u)d\mathcal{F}(u) &\le \sum_{i=1}^{n}(M_i - m_i)|\mathcal{F}(I_i)| \\ &\le \varepsilon \sum_{i=1}^{n} |\mathcal{F}(I_i)| = \varepsilon|F(I)|,\end{aligned}$$

where $\mathcal{F}(I_i)$'s are disjoint.

Thus, for each $\varepsilon > 0$, (h'_k) and (h''_k) are sequences of step functions, and we get (g_k) is statistically Riemann-Stieltjes integrable on I.

Next, for I is infinite, and $\mathcal{F}$ being increasing and continuous (piecewise), for every $\varepsilon > 0$ there exists a finite interval $\tilde{I}$ with $\tilde{I} \subseteq I$ such that

$$\max(\sup_I \mathcal{F} - \sup_{\tilde{I}} \mathcal{F}; \inf_{\tilde{I}} \mathcal{F} - \inf_I \mathcal{F}) < \varepsilon.$$

Also, since (g_k) is bounded

$$\sup_{I\setminus\tilde{I}} |g_k| \le \mathcal{G}.$$

This implies

$$\int_{I\setminus\tilde{I}} \mathcal{G}d\mathcal{F}(u) - \int_{I\setminus\tilde{I}} -\mathcal{G}d\mathcal{F}(u) \le 2\mathcal{G}\varepsilon,$$

assuming finite number of points of I and $\tilde{I}$. Successively, we choose the sequences of step functions $\tilde{h}'_k = (-\mathcal{G}, h'_k, -\mathcal{G})$ and $\tilde{h}''_k = (-\mathcal{G}, h'{}_k, -\mathcal{G})$ such that $\tilde{h}'_k < (g_k) < \tilde{h}''_k$, that is, (g_k) is bounded over I, and as such

$$\begin{aligned}\int_I \tilde{h}''_k(u)d\mathcal{F}(u) - \int_I \tilde{h}'_k(u)d\mathcal{F}(u) &= \int_{\tilde{I}} h''_k(u)d\mathcal{F}(u) - \int_{\tilde{I}} h'_k(u)d\mathcal{F}(u) \\ &\quad + \int_{I\setminus\tilde{I}} \mathcal{G}d\mathcal{F}(u) - \int_{I\setminus\tilde{I}} -\mathcal{G}d\mathcal{F}(u) \\ &\le \varepsilon|\mathcal{F}(\tilde{I})| + 2\mathcal{G}\varepsilon.\end{aligned}$$

Hence, (g_k) is statistically Riemann-Stieltjes integrable over I. □

We next discuss the following two special cases of statistically Riemann-Stieltjes integrals.

Corollary 1. *Let $\mathcal{F}$ be an increasing step function on I such that*

$$\mathcal{F}(t) = \sum_{i=1}^{k} c_i 1\{t \le t_i\},$$

with $\min(I) = t_0 < t_1 < \cdots < t_k = \max(I)$, *and* $c_i > 0$. *Then* (g_k) *is uniformly convergent, and*

$$\mathrm{stat}_{\mathrm{RS}} \lim_{k\to\infty} \int_I g_k(t) d\mathcal{F}(t) = \sum_{i=1}^{k} g_k(\lambda_i) c_i.$$

Proof. In view of statistical Riemann-Stieltjes sums

$$\mathrm{stat}_{\mathrm{RS}} \sum_{i=1}^{k} g_k(\gamma_i)[\mathcal{F}(t_i) - \mathcal{F}(t_{i-1})]$$

with

$$\mathcal{F}(t_i) - \mathcal{F}(t_{i-1}) = \begin{cases} c_k & (t_{i-1} < \lambda_k < t_i; t_i - t_{i-1} < \varepsilon) \\ 0 & (\lambda_k \notin (t_i - t_{i-1}) \forall\, k), \end{cases} \tag{1.2}$$

we have,

$$\mathrm{stat}_{\mathrm{RS}} \sum_{i=1}^{k} g_k(\gamma_i)[\mathcal{F}(t_i) - \mathcal{F}(t_{i-1})] = \sum_{i=1}^{k} g_k(\gamma_i) c_i.$$

Next, (g_k) is uniformly convergent with $\gamma_i \to \lambda_i$, that is $g_k(\gamma_i) \to g_k(\lambda_i)$. Thus,

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{k} g_k(\gamma_i)[\mathcal{F}(t_i) - \mathcal{F}(t_{i-1})] &= \lim_{n\to\infty} \sum_{i=1}^{k} g_k(\gamma_i) c_i \\ &= \lim_{n\to\infty} \sum_{i=1}^{k} g_k(t_i) c_i. \end{aligned}$$

□

Corollary 2. *Let* $\mathcal{F}$ *be differentiable with* $\mathcal{F} = f$ *(continuous). Then* (g_k) *is integrable, and*

$$\mathrm{stat}_{\mathrm{RS}} \int_I g_k(t) d\mathcal{F}(t) = \mathrm{stat}_{\mathrm{RS}} \int_I g_k(t) f(t) dt.$$

Proof. In view of the statistical Riemann-Stieltjes sums, we have

$$\mathrm{stat}_{\mathrm{RS}} \sum_{i=1}^{k} g_k(\gamma_i)[\mathcal{F}(t_i) - \mathcal{F}(t_{i-1})]. \tag{1.3}$$

Following mean value theorem,

$$\mathcal{F}(t_i) - \mathcal{F}(t_{i-1}) = f(\mu_i)(t_i - t_{i-1})$$

for some $\mu_i \in (t_{i-1} - t_i)$. Consequently from (1.3), we obtain

$$\mathrm{stat}_{\mathrm{RS}} \sum_{i=1}^{k} g_k(\gamma_i)[\mathcal{F}(t_i) - \mathcal{F}(t_{i-1})] = \mathrm{stat}_{\mathrm{RS}} \sum_{i=1}^{k} g_k(\gamma_i) f(\mu_i)(t_i - t_{i-1}),$$

which ends the proof of the Corollary 2. □

Motivated chiefly essentially by the above-mentioned investigations and developments, we introduce various concepts of statistical Riemann-Stieltjes sum via the deferred Nörlund summability mean for the sequence of functions as well as the sequence of distribution functions. We first establish some fundamental limit theorems for statistical Riemann-Stieltjes sum in the sequence space. Then over the probability sequence space, we establish some more advance results. Finally, over both the spaces we establish some inclusion theorems via our proposed mean in association with statistical Riemann-Stieltjes integral.

2. Riemann-Stieltjes Integrability Over a Probability Space

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space, and let $(X_n)_{n\in\mathbb{N}}$ be the sequence of random variables defined on Ω, such that

$$\{\omega \in \Omega : X_n(\omega) \leq x\} \in \mathbb{F}$$

for each $x \in \mathbb{R}$. Also, it is known as *measurability*.

Accordingly, we can define the sequence of distribution functions $(F_n(x))$ of (X_n), that is,

$$F_n(x) = \mathbb{P}\{\omega \in \Omega : X_n(\omega) \leq x\}.$$

Definition 3. Let $(X_n)_{n\in\mathbb{N}}$ be the sequence of random variables with sequence of distribution functions $(F_n(x))$. Then the *expectation* $E(X_n)$ of (X_n) is

$$E(X_n) = \int x dF_n(x).$$

We next present the statistical version of Definition 3.

Definition 4. Let $E(X_n)$ be the expectation of (X_n). Then, for each $\varepsilon > 0$, we define the statistical expectation $E(X_n)$, that is,

$$\mathcal{Y}_\varepsilon = \{\zeta : \zeta \in \mathbb{N} \quad \text{and} \quad |E(X_n) - h| \geq \varepsilon\}$$

has zero natural density (see [3] and [20]). Thus,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0.$$

We write

$$\text{stat}_{\text{SE}} \lim_{k\to\infty} E(X_n) = h.$$

We interpret the expectation $E(X_n)$ in the form of Riemann-Stieltjes integral, that is,

$$E(X_n;\xi) = \sum_{i=1}^{n} \xi_i [F_n(x_i) - F_n(x_{i-1})] \tag{2.1}$$

where $\xi \in (x_{i-1}, x_i]$.

We next present below the statistical version of (2.1).

Definition 5. Let $F_n(x)$ be a sequence of distribution function, and let $E(X_n)$ be the expectation of (X_n). Then, for each $\varepsilon > 0$, we define the statistical Riemann-Stieltjes integral of $E(X_n)$, that is,

$$\mathcal{Y}_\varepsilon = \{\zeta : \zeta \in \mathbb{N} \quad \text{and} \quad |E(X_n;\xi) - h| \geq \varepsilon\}$$

has zero natural density (see [3] and [20]). Thus,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0.$$

We write

$$\text{stat}_{\text{ERS}} \lim_{k\to\infty} E(X_n;\xi) = h.$$

We now easily capable to adopt the following propositions from the earlier established Corollaries 1 and 2, and Definition 5.

Proposition 1. *If $(\mathcal{F}_i)$ is a sequence of step functions with jump at (x_i), then*

$$\text{stat}_{\text{ERS}} E(X_n;\xi) = \text{stat}_{\text{ERS}} \sum_{i=1}^{n} x_i[\mathcal{F}_n(x_i) - \mathcal{F}_n(x_{i-1})].$$

Proposition 2. *If $(\mathcal{F}_i)$ is differentiable with $\mathcal{F}_i' = f_i$, then*

$$\text{stat}_{\text{ERS}} E(X_n) = \text{stat}_{\text{ERS}} \int x f_n(x) dx.$$

Let f be a function on $\mathbb{R}$ with

$$\int f(x) dF_n(x) < \infty$$

such that

$$\{\omega : f(X_n(\omega)) \leq u\} \in \mathcal{F}, \quad u \in \mathbb{R}^2.$$

We can define the distribution function F of Y,

$$F_n(y) = \mathbb{P}\{\omega : Y_n(\omega) \leq y\}.$$

Then the expectation of (Y_n) is

$$E(Y_n) = \int y dF_n(y)$$

(exists and finite).

We next define below the statistical version of $E(Y_n)$.

Definition 6. Let $E(Y_n)$ be the expectation of (Y_n). Then, for each $\varepsilon > 0$, we define the statistical expectation of $E(Y_n)$, that is,

$$\mathcal{Y}_\varepsilon = \{\zeta : \zeta \in \mathbb{N} \quad \text{and} \quad |E(Y_n) - h| \geq \varepsilon\}$$

has zero natural density (see [3] and [20]). Thus,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0.$$

We write

$$\mathrm{stat}_{\mathrm{SE}'} \lim_{k\to\infty} E(Y_n) = h.$$

We now reform the expectation $E(Y_n)$ in the form of Riemann-Stieltjes integral,

$$E(Y_n;\xi) = \sum_{i=1}^{n} \xi_i [F_n(y_i) - F_n(y_{i-1})], \tag{2.2}$$

where $\xi \in (y_{i-1}, y_i]$.

We next present the statistical version of (2.2).

Definition 7. Let $(F_n(y))$ be a sequence of distribution function, and let $E(Y_n)$ be the expectation of (Y_n). Then, for each $\varepsilon > 0$, we define the statistical Riemann-Stieltjes integral of $E(Y_n)$, that is,

$$\mathcal{Y}_\varepsilon = \{\zeta : \zeta \in \mathbb{N} \quad \text{and} \quad |E(Y_n;\xi) - h| \geq \varepsilon\}$$

has zero natural density (see [3] and [20]). Thus,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0.$$

We write

$$\mathrm{stat}_{\mathrm{E}'\mathrm{RS}} \lim_{k\to\infty} E(Y_n;\xi) = h.$$

In view of Definition 7, we establish the following theorem.

Theorem 2. *Let X_n is a sequence of random variables, and let f be a function on $\mathbb{R}$. Then the sequence of random variables $(Y_n) = f(X_n)$ has the statistical expectation*

$$\mathrm{stat}_{\mathrm{SE}'} E(Y_n) = \mathrm{stat}_{\mathrm{SE}'} \int f(x) dF_n(x).$$

Proof. Following the Riemann-Stieltjes integral,

$$\begin{aligned}
\mathrm{stat}_{\mathrm{E}'\mathrm{RS}} \sum_{i=1}^{n} \xi_i [F_n(y_i) - F_n(y_{i-1}, y_i)] &= \mathrm{stat}_{\mathrm{E}'\mathrm{RS}} \sum_{i=1}^{n} \xi_i \mathbb{P}(Y_n \in (y_{i-1}, y_i]) \\
&= \mathrm{stat}_{\mathrm{E}'\mathrm{RS}} \sum_{i=1}^{n} \xi_i \mathbb{P}(f(X_n) \in (y_{i-1}, y_i]) \\
&= \mathrm{stat}_{\mathrm{E}'\mathrm{RS}} \sum_{i=1}^{n} \xi_i \mathbb{P}(X_n \in f^{-1}(y_{i-1}, y_i]),
\end{aligned}$$

where $\xi_i \in (y_{i-1}, y_i)]$. Also, recall that

$$\begin{aligned}
\xi_i \in (y_{i-1}, y_i)] &\Longleftrightarrow \eta_i = f^{-1}(\xi_i) \in f^{-1}\{(y_{i-1}, y_i)]\} \\
&\Longleftrightarrow f(\eta_i) \in (y_{i-1}, y_i)].
\end{aligned}$$

Consequently, we have

$$\mathrm{stat}_{\mathrm{SE}'} \sum_{i=1}^{n} f(\eta_i)\mathbb{P}(X_n \in f^{-1}(y_{i-1}, y_i]) \tag{2.3}$$

with $\eta_i \in f^{-1}\{(y_{i-1}, y_i)\}$.

Furthermore, if the intervals $(y_{i-1}, y_i]$ form a partition, then the intervals $(x_{i-1}, x_i] = f^{-1}\{(y_{i-1}, y_i]\}$ also form a partition.

Thus, (2.3) can be written as

$$\mathrm{stat}_{\mathrm{SE}'} \sum_{i=1}^{n} f(\eta_i)\mathbb{P}(X_n \in f(x_{i-1}, x_i]), \tag{2.4}$$

where $\eta_i \in (x_{i-1}, x_i)$, which is the Riemann-Stieltjes sum. □

3. Statistical Riemann-Stieltjes Integrability via Deferred Nörlund Mean

Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$ with $\phi_k < \varphi_k$ and $\lim_{k\to\infty} \varphi_k = +\infty$, and let (p_i) be a sequence of non-negative real numbers with

$$P_k = \sum_{i=\phi_k+1}^{\varphi_k} p_{\varphi_k - i}.$$

Accordingly, we define the deferred Nörlund summability mean for the Riemann-Stieltjes sum $\delta(g_k; \dot{\mathcal{P}})$ of a sequence of functions with tagged partition $\dot{\mathcal{P}}$ of the form

$$z_k = \frac{1}{P_k} \sum_{\rho=\phi_k+1}^{\varphi_k} p_{\varphi_k - \rho}\ \delta(g_\rho; \dot{\mathcal{P}}). \tag{3.1}$$

We now present the notions of statistical Riemann-Stieltjes integrability and statistical Riemann-Stieltjes summability of a sequence of functions via the deferred Nörlund mean.

Definition 8. Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$, and let (p_k) be a sequence of non-negative real numbers. Also let $\mathcal{F}$ be an increasing function, and let (g_k) be a sequence of functions defined over the interval I. The given sequence (g_k) of functions is deferred Nörlund statistically Riemann-Stieltjes integrable to g with respect to $\mathcal{F}$, if for every $\varepsilon > 0$, there exists a sequence of step functions h'_k and h''_k such that

$$\int_I h''_k(u)\,d\mathcal{F}(u) - \int_I h'_k(u)\,d\mathcal{F}(u) < \varepsilon \quad (h'_k < g_k < h''_k)$$

and the set

$$\{\zeta : \zeta \leq P_k \quad \text{and} \quad p_{\varphi_k - \zeta}|\delta(g_\zeta; \dot{\mathcal{P}}) - g| \geq \varepsilon\}$$

has zero natural density. This implies that,

$$\lim_{k\to\infty} \frac{|\{\zeta : \zeta \leq P_k \quad \text{and} \quad p_{\varphi_k - \zeta}|\delta(g_\zeta; \dot{\mathcal{P}}) - g| \geq \varepsilon\}|}{P_k} = 0.4$$

We write

$$\mathrm{DNRS}_{\mathrm{stat}} \lim_{k\to\infty} \delta(g_k;\dot{\mathcal{P}}) = g.$$

Definition 9. Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$, and let (p_k) be a sequence of non-negative real numbers. Also let $\mathcal{F}$ be an increasing function, and let (g_k) be a sequence of functions defined over the interval I. The given sequence (g_k) of functions is statistically deferred Nörlund Riemann-Stieltjes summable to g with respect to $\mathcal{F}$, if for every $\varepsilon > 0$, there exists a sequence of step functions h'_k and h''_k such that

$$\int_I h''_k(u)d\mathcal{F}(u) - \int_I h'_k(u)d\mathcal{F}(u) < \varepsilon \quad (h'_k < g_k < h''_k),$$

and the set

$$\{\zeta : \zeta \leq \mathbb{N} \quad \text{and} \quad |z_\zeta - g| \geq \varepsilon\}$$

has zero natural density. This implies that,

$$\lim_{k\to\infty} \frac{|\{\zeta : \zeta \leq P_k \quad \text{and} \quad |z_\zeta - g| \geq \varepsilon\}|}{P_k} = 0.$$

We write

$$\mathrm{stat}_{\mathrm{DNRS}} \lim_{k\to\infty} z_k = g.$$

We now establish an inclusion theorem between these two new potentially useful notions.

Theorem 3. *Let (ϕ_k) and (φ_k) be sequences of non-negative integers and let (p_k) be a sequence of non-negative real numbers. If a sequence $(g_k)_{k\in\mathbb{N}}$ of functions is deferred Nörlund statistically Riemann-Stieltjes integrable to a function g on $[a,b]$, then it is statistically deferred Nörlund Riemann-Stieltjes summable to the same function g on $[a,b]$, but not conversely.*

Proof. Suppose $(g_k)_{k\in\mathbb{N}}$ is deferred Nörlund statistically Riemann-Stieltjes integrable to a function g on $[a,b]$, then by Definition 8, we have

$$\lim_{k\to\infty} \frac{|\{\zeta : \zeta \leq P_k \quad \text{and} \quad p_{\varphi_k-\zeta}|\delta(g_\zeta;\dot{\mathcal{P}}) - g| \geq \varepsilon\}|}{P_k} = 0.$$

Now assuming two sets as follows:

$$\mathcal{J}_\varepsilon = \{\zeta : \zeta \leq P_k \quad \text{and} \quad p_{\varphi_k-\zeta}|\delta(g_\zeta;\dot{\mathcal{P}}) - g| \geq \varepsilon\}$$

and

$$\mathcal{J}_\varepsilon^c = \{\zeta : \zeta \leq P_k \quad \text{and} \quad p_{\varphi_k-\zeta}|\delta(g_\zeta;\dot{\mathcal{P}}) - g| \geq \varepsilon\},$$

we have

$$|z_k - g| = \left| \frac{1}{P_k} \sum_{\rho=\phi_k+1}^{\varphi_k} p_{\varphi_k-\rho}\delta(g_\rho;\dot{\mathcal{P}}) - g \right|$$

$$\leq \left| \frac{1}{P_k} \sum_{\rho=\phi_k+1}^{\varphi_k} p_{\varphi_k-\rho} \left[\delta(g_\rho; \dot{\mathcal{P}}) - g \right] \right| + \left| \frac{1}{P_k} \sum_{\rho=\phi_k+1}^{\varphi_k} p_{\varphi_k-\rho} g - g \right|$$
$$\leq \frac{1}{P_k} \sum_{\substack{\rho=\phi_k+1 \\ (\zeta \in \mathcal{J}_\varepsilon)}}^{\varphi_k} p_{\varphi_k-\rho} \left| \delta(g_\rho; \dot{\mathcal{P}}) - g \right| + \frac{1}{P_k} \sum_{\substack{\rho=\phi_k+1 \\ (\zeta \in \mathcal{J}_\varepsilon^c)}}^{\varphi_k} p_{\varphi_k-\rho} \left| \delta(g_\rho; \dot{\mathcal{P}}) - g \right|$$
$$+ |g| \left| \frac{1}{P_k} \sum_{\rho=\phi_k+1}^{\varphi_k} p_{\varphi_k-\rho} - 1 \right|$$
$$\leq \frac{1}{P_k} |\mathcal{J}_\varepsilon| + \frac{1}{P_k} |\mathcal{J}_\varepsilon^c| = 0.$$

This implies that

$$|z_k - g| < \varepsilon.$$

Thus, the sequence of functions (g_k) is statistically deferred Nörlund Riemann-Stieltjes summable to the function g on $[a,b]$.

Next, in view of the non-validity of the converse statement, we present the following example. □

Example 2. Let $\phi_k = 2k$, $\varphi_k = 4k$ and $p_{\varphi_k - \zeta} = 1$ and let $g_k : [0,1] \to \mathbb{R}$ be a sequence of functions of the form

$$g_k(x) = \begin{cases} 0 & (x \in \mathbb{Q} \cap [0,1];\ k = 2m : m \in \mathbb{N}) \\ \\ 1 & (x \in \mathbb{R} - \mathbb{Q} \cap [0,1];\ k = 2m+1 : m \in \mathbb{N}). \end{cases} \tag{3.2}$$

The given sequence (g_k) of functions trivially indicates that, it is neither Riemann-Stieltjes integrable nor deferred Nörlund statistically Riemann-Stieltjes integrable. However, as per our proposed mean (3.1), it is easy to see that, the sequence (g_k) of functions has deferred Nörlund Riemann-Stieltjes sum $\frac{1}{2}$ under the tagged partition $\dot{\mathcal{P}}$. Therefore, the sequence (g_k) of functions is statistically deferred Nörlund Riemann-Stieltjes summable to $\frac{1}{2}$ over $[0,1]$ but it is not deferred Nörlund statistically Riemann-Stieltjes integrable.

We now present the notions of statistical Riemann-Stieltjes integrability and statistical Riemann-Stieltjes summability of a sequence of random variables via the deferred Nörlund mean.

Definition 10. Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$, and let (p_k) be a sequence of non-negative real numbers. Let $F_n(x)$ be a sequence of distribution function, and let $E(X_n)$ be the expectation of (X_n). Then, for each $\varepsilon > 0$, $E(X_n)$ is deferred Nörlund statistically Riemann-Stieltjes integrable to the function g, if

$$\{\zeta : \zeta \leq P_k \quad \text{and} \quad p_{\varphi_k - \zeta} |\delta(E(X_\zeta; \zeta); \dot{\mathcal{P}}) - g| \geq \varepsilon\}$$

has zero natural density. This implies that,

$$\lim_{k\to\infty} \frac{|\{\zeta : \zeta \leq P_k \quad \text{and} \quad p_{\varphi_k-\zeta}|\delta(E(X_\zeta;\zeta);\dot{\mathcal{P}}) - g| \geq \varepsilon\}|}{P_k} = 0.$$

We write

$$\mathrm{DNERS_{stat}} \lim_{k\to\infty} \delta(E(X_k;k);\dot{\mathcal{P}}) = g.$$

Definition 11. Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$, and let (p_k) be a sequence of non-negative real numbers. Let $F_n(x)$ be a sequence of distribution function, and let $E(X_n)$ be the expectation of (X_n). Then, for each $\varepsilon > 0$, $E(X_n)$ is statistically deferred Nörlund Riemann-Stieltjes summable to g, if

$$\{\zeta : \zeta \leq \mathbb{N} \quad \text{and} \quad |z_n(E(X_\zeta;\zeta) - g| \geq \varepsilon\}$$

has zero natural density. This implies that,

$$\lim_{k\to\infty} \frac{|\{\zeta : \zeta \leq \mathbb{N} \quad \text{and} \quad |z_\zeta(E(X_\zeta;\zeta) - g| \geq \varepsilon\}|}{P_k} = 0.$$

We write

$$\mathrm{stat_{DNERS}} \lim_{k\to\infty} z_k(E(X_k;k) = g.$$

We now establish an inclusion theorem between these two new potentially useful notions.

Theorem 4. *Let (ϕ_k) and (φ_k) be sequences of non-negative integers and let (p_k) be a sequence of non-negative real numbers. If $(F_n(x))$ be the sequence of distribution functions, and let $E(X_n)$ be the expectation of (X_n), then $E(X_n)$ is deferred Nörlund statistically Riemann-Stieltjes integrable to a function g on $[a,b]$ implies, it is statistically deferred Nörlund Riemann-Stieltjes summable to the same function g on $[a,b]$, but not conversely.*

Proof. The proof of Theorem 4 follows in the similar lines from the proof of the Theorem 3. Thus, we choose skip the details. □

4. CONCLUSION

In this study, we have introduced the notion of statistical Riemann-Stieltjes sum on the sequence space via the deferred Nörlund summability mean and established some fundamental limit theorems. Next, considering the probability space, we also established some basic new results based on the Riemann-Stieltjes integral for the sequence of distribution functions. Finally, over both the spaces we established some inclusion theorems via our proposed deferred Nörlund summability means associated with statistical Riemann-Stieltjes integral for the sequence of functions as well as the sequence of distribution functions.

Many researchers have considered different summability means on the sequence spaces to prove several approximation results. A list of some articles has been mentioned in the references. Further combining the existing ideas and direction of the sequence spaces associated with our proposed mean, many new Korvokin-type approximation theorems can be proved under different settings of algebraic and trigonometric functions.

Influenced by a recently-published article by Srivastava [14], we draw the awareness of the interested reader's toward the prospect of establishing some Korvokin-type approximation theorems over the Banach space as well as the probability space. Furthermore, in view of a recent result of Das *et al.* [2], the attention of the curious readers is also drawn for further researches towards fuzzy approximation theorems.

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