Mapping bijectively $\sigma$-algebras onto power sets

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Dedicated to the memory of my Father

Abstract. As an application of the so-called "optimal measure" we attempt to seek sets whose power sets are equinumerous with $\sigma$-algebras, which seems to be new information about $\sigma$-algebras.

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1. Introduction

Some new information about $\sigma$-algebras is investigated, consisting of mapping bijectively $\sigma$-algebras onto power sets. Such $\sigma$-algebras, in fact, form a rather broad class. A special grouping of the so-called optimal measures is used in our investigation (for more about optimal measures cf. [1-4]. We provide constructively a bijective mapping that will serve the purpose. In the proof we first characterize set-inclusion as well as some asymptotic behaviors of sequences of measurable sets. Without loss of generality we shall restrict ourselves to infinite $\sigma$-algebras, since the opposite case can be easily done.

Throughout this communication $(\Omega, \mathcal{F})$ will stand for an arbitrary measurable space, with both $\Omega$ and $\mathcal{F}$ being infinite sets (where, as usual, the elements of $\mathcal{F}$ are referred to as measurable sets).

By an optimal measure we mean a set function $p^* : \mathcal{F} \rightarrow [0, 1]$ which fulfills the following axioms:

P1. $p^*(\emptyset) = 0$ and $p^*(\Omega) = 1$.
P2. $p^*(B \cup E) = p^*(B) \lor p^*(E)$ for all measurable sets $B$ and $E$ (where $\lor$ stands for the maximum).
P3. $p^* \left( \bigcap_{n=1}^{\infty} E_n \right) = \lim_{n \to \infty} p^*(E_n) = \bigwedge_{n=1}^{\infty} p^*(E_n)$, for every decreasing sequence of measurable sets $(E_n)$, where $\bigwedge$ stands for the minimum.

In [2] we have obtained the following results for all optimal measures $p^*$. 
By \((p^*)\)-atom we mean a measurable set \(H\), \(p^*(H) > 0\) such that whenever \(B \in \mathcal{F}\), \(B \subset H\), then \(p^*(B) = p^*(H)\) or \(p^*(B) = 0\).

A \((p^*)\)-atom \(H\) is decomposable if there exists a subatom \(B \subset H\) such that \(p^*(B) = p^*(H) = p^*(H \setminus B)\). If no such subatom exists, we shall say that \(H\) is indecomposable.

**Fundamental Optimal Measure Theorem.** Let \((\Omega, \mathcal{F})\) be a measurable space and \(p^*\) an optimal measure on it. Then there exists a collection \(\mathcal{H}(p^*) = \{H_n : n \in J\}\) of disjoint indecomposable \((p^*)\)-atoms, where \(J\) is some countable (i.e. finite or countably infinite) index-set such that for any measurable set \(B\), with \(p^*(B) > 0\), we have that
\[
p^*(B) = \max \left\{ p^*(B \bigcap H_n) : n \in J \right\}.
\]
Moreover, the only limit point of the set \(\{p^*(H_n) : n \in J\}\) is 0 provided that \(J\) is a countably infinite set. (\(\mathcal{H}(p^*)\) is referred to as a \((p^*)\)-generating countable system.)

**NOTATIONS.**
1. \(\mathcal{P}\) will denote the set of all optimal measures defined on \((\Omega, \mathcal{F})\).
2. \(\mathcal{P}_\infty\) is the set of all optimal measures whose generating systems are countably infinite.
3. For every \(A \in \mathcal{F}\), we write \(\overline{A}\) for the complement of \(A\).
4. \(\mathbb{N}\) stands for the set of counting numbers (or positive integers).
5. \(A \subset B\) means set \(A\) is a proper subset of set \(B\).
6. \(A \subseteq B\) means set \(A\) is a subset of set \(B\).
7. \(\mathcal{P}(A)\) stands for the power set of set \(A\).

2. Main results

**Definition 2.1.** We say that an optimal measure \(p^* \in \mathcal{P}_\infty\) is of order-one if there is a unique indecomposable \((p^*)\)-atom \(H\) such that \(p^*(H) = 1\). (Any such atom will be referred to as an order-one-atom and the set of all order-one optimal measures will be denoted by \(\mathcal{P}_{\infty}^1\).)

**Example 1.** Fix a sequence \((\omega_n) \subset \Omega\) and define \(p_0^* \in \mathcal{P}_\infty\) by
\[
p_0^*(B) = \max \left\{ \frac{1}{n} : \omega_n \in B \right\}.
\]
Then \(p_0^* \in \mathcal{P}_{\infty}^1\).

In fact, via the Structure Theorem, there is an indecomposable \(p_0^*\)-atom \(H\) such that \(p_0^*(H) = 1\). This is possible if and only if \(\omega_1 \in H\). We note that there is no other indecomposable \(p_0^*\)-atom \(H^*\) with \(H^* \cap H = \emptyset\) such that \(p_0^*(H^*) = 1\), otherwise necessarily it would ensue that \(\omega_1 \in H^*\), which is absurd. Hence \(p_0^* \in \mathcal{P}_{\infty}^1\).

**FURTHER NOTATIONS.**

If \(H\) is the order-one-atom of some \(p^* \in \mathcal{P}_\infty\), we write \(p = \left\{ q^* \in \mathcal{P}_\infty : q^*(H) = 1 \right\}\).

We then refer to the elements of the class \(p\) as representing members of the class, and
call $H$ the unitary atom of the class. (If the unitary atom of a class is the order-one-atom of a representing member, we shall speak of representation.)

We further denote by $\mathcal{P}_1^\infty$ the set of all $p$ classes.

If $A$ is a nonempty measurable set and $p \in \mathcal{P}_1^\infty$, the identity $p(A) = 1$ (resp. the inequality $p(A) < 1$) will simply mean that $p^*(A) = 1$ (resp. $p^*(A) < 1$) for any representing member $p^* \in p$. We shall also write $p(A) = 0$ to mean that $p^*(A) = 0$ whenever $p^* \in p$.

Write $\forall$ for the set of all unitary atoms on the measurable space $(\Omega, \mathcal{F})$.

**Lemma 2.1.** Let $A, B \in \mathcal{F}$ and $p \in \mathcal{P}_1^\infty$ be arbitrary. In order that $p(A \cap B) = 1$ it is necessary and sufficient that $p(A) = 1$ and $p(B) = 1$.

**Proof.** As the necessity is obvious, we only have to show the sufficiency. In fact, assume that $p(A) = 1$ and $p(B) = 1$. Let $H$ be the unitary atom of class $p$, and let $p^*$ denote an arbitrary but fixed representing member in the class. Without loss of generality we may assume that $p^*$ is a representation of $p$ (i.e. $H$ is the order-one atom of $p^*$). Then $p^*(H) = 1$. Clearly, $p^*(A \cap H) = 1$ and $p^*(B \cap H) = 1$. Hence $p^*(A \cap H \cap B) = 0$. It is enough to prove that both identities $p^*(A \cap H \cap B) = 0$ and $p^*(A \cap H \cap B) = 0$ are valid. On the contrary, assume that at least one of these identities fails to hold: $p^*(A \cap H \cap B) = 0$, say. Then $p^*(A \cap H \cap B) = 1$. Now, since $p^*(H \cap B) = 1$, it ensues that either $p^*(A \cap H \cap B) = 1$ or $p^*(A \cap H \cap B) = 1$. Then combining each of these last identities with $p^*(A \cap H \cap B) = 1$, we have that $p^*(A \cap H \cap B) = 1$ and $p^*(A \cap H \cap B) = 1$, or $p^*(A \cap H \cap B) = 1$ and $p^*(A \cap H \cap B) = 1$. This violates that $H$ is an order-one-atom (because the sets $A \cap H \cap B, A \cap H \cap B$ and $A \cap H \cap B$ are pairwise disjoint). **q.e.d.**

**Remark 2.0.** Let $p \in \mathcal{P}_1^\infty$ be arbitrary. Then the identity $p(\emptyset) = 0$ holds.

**Remark 2.1.** Let $A \in \mathcal{F}$ and $p \in \mathcal{P}_1^\infty$ be arbitrary. Then the identities $p(A) = 1$ and $p(A) = 1$ cannot hold simultaneously, i.e., for no representing member $p^*$ of class $p$ the identities $p^*(A) = 1$ and $p^*(A) = 1$ hold at the same time.

In fact, assume the contrary. Then **Lemma 2.1** would imply that

$$p(A) = p(\overline{A}) = 1 = p(A \cap \overline{A}) = p(\emptyset) = 0$$

which is absurd, indeed. **q.e.d.**

**Definition 2.2.** For any $A \in \mathcal{F}$ define the set $\Delta(A)$ by

1. $\Delta(A) \subseteq \mathcal{P}_1^\infty$.
2. If $p \in \Delta(A)$, then $p(A) = 1$.

**Remark 2.2.** Let $A \in \mathcal{F}$. Then $\Delta(A) = \emptyset$ if and only if $A = \emptyset$. 


Remark 2.3. If $H$ is a unitary atom (with $p$ its corresponding class), then $\Delta (H) = \{p\}$.

Let $A \in \mathcal{F}$ and denote by $\nabla_A$ the set of all unitary atoms $H$ such that $p(A) = 1$, where $\Delta (H) = \{p\}$. It is clear that $\nabla_A \cap \nabla_A = \emptyset$ and $\nabla_A \cup \nabla_A = \nabla$. From this observation the following lemma is straightforward:

Lemma 2.2. For every set $A \in \mathcal{F}$, we have that $\Delta (\overline{A}) = \overline{\Delta (A)}$.

Proposition 2.3. Let $A, B \in \mathcal{F}$ be arbitrary. Then

1. $\Delta (\Omega) = P_\infty^1$.
2. $\Delta (A \cap B) = \Delta (A) \cap \Delta (B)$.
3. $\Delta (A \cup B) = \Delta (A) \cup \Delta (B)$.

Proof. Part 1 is an easy task. Let us show Part 2. In fact, let $p \in \Delta (A \cap B)$. Then $p(A \cap B) = 1$. Hence Lemma 2.1 implies that $p(A) = 1$ and $p(B) = 1$, so that $p \in \Delta (A)$ and $p \in \Delta (B)$, i.e. $p \in \Delta (A) \cap \Delta (B)$. Consequently $\Delta (A \cap B) \subseteq \Delta (A) \cap \Delta (B)$. To show the reverse inclusion, pick an arbitrary $p \in \Delta (A) \cap \Delta (B)$. Then $p(A) = 1$ and $p(B) = 1$. Via Lemma 2.1, we have that $p(A \cap B) = 1$, i.e. $p \in \Delta (A \cap B)$. So $\Delta (A) \cap \Delta (B) \subseteq \Delta (A \cap B)$.

To end the proof, let us show the third part. In fact, let $A$ and $B \in \mathcal{F}$ be arbitrary. Then making use of the second part of this proposition, it ensues that $\Delta (\overline{A \cap B}) = \Delta (\overline{A}) \cap \Delta (\overline{B})$. By applying Lemma 2.2 and De Morgan identities, we obtain that

$$\Delta (A \cup B) = \overline{\Delta (A \cup B)} = \overline{\overline{A \cap B}} = \overline{\Delta (A) \cap \Delta (B)}$$

$$= \Delta (\overline{A}) \cup \Delta (\overline{B}) = \overline{\Delta (A) \cup \Delta (B)} = \Delta (A) \cup \Delta (B).$$

This was to be proven. q.e.d.

Lemma 2.4. Let $A$ and $B \in \mathcal{F}$ be arbitrary nonempty sets. In order that $A \subseteq B$, it is necessary and sufficient that $\Delta (A) \subseteq \Delta (B)$.

Proof. As the necessity is trivial, we need only show the sufficiency. In fact, assume that $A \setminus B$ is not an empty set. Then because of Remark 2.2, $\Delta (A \setminus B)$ is neither empty. Fix some $p \in \Delta (A \setminus B)$, i.e. $p(A \setminus B) = 1$. This implies that $p(B) < 1$. (Otherwise we would obtain via Lemma 2.1 that $1 = p((A \setminus B) \cap B) = p(\emptyset) = 0$, which is absurd.) Then $p(A) = 1$ and $p(B) < 1$, i.e. $p \in \Delta (A) \setminus \Delta (B)$. So the set $\Delta (A) \setminus \Delta (B)$ is not empty. q.e.d.

Lemma 2.5. Let $A$ and $B \in \mathcal{F}$ be arbitrary nonempty sets. In order that $A \cap B = \emptyset$, it is necessary and sufficient that $\Delta (A) \cap \Delta (B) = \emptyset$.

(The proof follows from Proposition 2.3/2 and Remark 2.2.)

Lemma 2.6. Let $A$ and $B \in \mathcal{F}$ be arbitrary nonempty sets. In order that $A = B$ it is necessary and sufficient that $\Delta (A) = \Delta (B)$. 
We simply note that

\[
\Delta (A \setminus B) = \Delta (A) \setminus \Delta (B)
\]

which completes the proof.

**Lemma 2.7.** Let \( A \) and \( B \in \mathcal{F} \) be arbitrary nonempty sets. Then \( \Delta (A \setminus B) = \Delta (A) \setminus \Delta (B) \).

**Proof.** We simply note that Proposition 2.3/2 and Lemma 2.2 entail that

\[
\Delta (A \setminus B) = \Delta (A \cap \overline{B}) = \Delta (A \setminus (\overline{B})) = \Delta (A \setminus B),
\]

which completes the proof. q.e.d.

**Proposition 2.8.** Let \( (A_n) \subseteq \mathcal{F} \) and \( A \in \mathcal{F} \) be arbitrary. Then \( (A_n) \) converges increasingly to \( A \) if and only if \( (\Delta (A_n)) \) converges increasingly to \( \Delta (A) \).

**Proof.** Assume that \( (A_n) \) converges increasingly to \( A \). Then by applying repeatedly Lemma 2.4, we have for every \( n \in \mathbb{N} \) that

\[
\Delta (A_n) \subseteq \Delta (A_{n+1}) \subseteq \Delta (A).
\]

We need to prove that \( \Delta (A) = \bigcup_{n=1}^{\infty} \Delta (A_n) \). To do this, it will be enough to show that \( \Delta (A) \subseteq \bigcup_{n=1}^{\infty} \Delta (A_n) \) and \( \bigcup_{n=1}^{\infty} \Delta (A_n) \subseteq \Delta (A) \). In fact, we note that the second inclusion is trivial. To prove the first one, let us pick an arbitrary class \( p \in \Delta (A) \) and fix any representing member \( p^* \) of class \( p \). We note that following the proof of Lemma 0.1 (cf. [1], page 134), there can be found a positive integer \( n_0 \) such that

\[
1 = p^* (A) = p^* \left( \bigcup_{k=1}^{\infty} A_k \right) = p^* (A_n), \text{ whenever } n \geq n_0. \text{ Hence } p \in \bigcup_{n=n_0}^{\infty} \Delta (A_n) \subseteq \bigcup_{n=1}^{\infty} \Delta (A_n), \text{ i.e.}
\]

\[
\Delta (A) \subseteq \bigcup_{n=n_0}^{\infty} \Delta (A_n) \subseteq \bigcup_{n=1}^{\infty} \Delta (A_n).
\]

Conversely, assume that sequence \( (\Delta (A_n)) \) converges increasingly to \( \Delta (A) \). Then for every \( n \in \mathbb{N} \) we have that \( \Delta (A_n) \subseteq \Delta (A_{n+1}) \subseteq \Delta (A) \), so that \( A_n \subseteq A_{n+1} \subseteq A \) (because of Lemma 2.4). Hence \( \bigcup_{n=1}^{\infty} A_n \subseteq A \). Now, suppose that set \( A \setminus \bigcup_{n=1}^{\infty} A_n \) is not empty. Then via Remark 2.2 and Axiom 3 there can be found some \( p \in \mathcal{P}_\infty^\infty \) and some representing member \( p^* \) of class \( p \) such that

\[
1 = p^* \left( A \setminus \bigcup_{n=1}^{\infty} A_n \right) = p^* \left( \bigcap_{n=1}^{\infty} A \cap \overline{A_n} \right) = \bigcup_{n=1}^{\infty} p^* (A \cap \overline{A_n}),
\]
since sequence \( (\overline{A_n}) \) is a decreasing sequence. Consequently \( 1 = p^\ast (A \cap \overline{A_n}) \) for all \( n \in \mathbb{N} \). But Lemma 2.1 yields that \( p^\ast (A) = 1 \) and \( p^\ast (\overline{A_n}) = 1 \) for all \( n \in \mathbb{N} \). Hence Axiom 3 entails that

\[
1 = \bigwedge_{n=1}^{\infty} p^\ast (\overline{A_n}) = p^\ast \left( \bigcap_{n=1}^{\infty} \overline{A_n} \right) = p^\ast (\overline{A}).
\]

Nevertheless, this contradicts Remark 2.1 q.e.d.

**Proposition 2.9.** Let \( (A_n) \subset \mathcal{F} \) and \( A \in \mathcal{F} \) be arbitrary. Then \( (A_n) \) converges decreasingly to \( A \) if and only if \( (\Delta (A_n)) \) converges decreasingly to \( \Delta (A) \).

**Proof.** Assume that \( (A_n) \) converges decreasingly to \( A \). Then by applying repeatedly Lemma 2.4, we have for every \( n \in \mathbb{N} \) that

\[
\Delta (A) \subset \Delta (A_{n+1}) \subset \Delta (A_n).
\]

We need to prove that \( \Delta (A) = \bigcap_{n=1}^{\infty} \Delta (A_n) \). To do this, it will be enough to show that \( \Delta (A) \subseteq \bigcap_{n=1}^{\infty} \Delta (A_n) \) and \( \bigcap_{n=1}^{\infty} \Delta (A_n) \subseteq \Delta (A) \). In fact, we note that the first inclusion is trivial. To prove the second inclusion let us pick some \( p \in \bigcap_{n=1}^{\infty} \Delta (A_n) \).

Then \( p \in \Delta (A_n) \) for all \( n \in \mathbb{N} \). Hence \( p (A_n) = 1 \) for all \( n \in \mathbb{N} \). If we fix any representing member \( p^\ast \) in class \( p \), we then obtain via Axiom 3 that

\[
p^\ast (A) = p^\ast \left( \bigcap_{n=1}^{\infty} A_n \right) = \bigwedge_{n=1}^{\infty} p^\ast (A_n) = 1,
\]

implying that \( p (A) = 1 \), i.e. \( p \in \Delta (A) \). Consequently, \( \bigcap_{n=1}^{\infty} \Delta (A_n) \subseteq \Delta (A) \).

Conversely, assume that sequence \( (\Delta (A_n)) \) converges decreasingly to \( \Delta (A) \). Then for every \( n \in \mathbb{N} \) we obtain that \( \Delta (A) \subset \Delta (A_{n+1}) \subset \Delta (A_n) \) so that \( A \subset A_{n+1} \subset A_n \), \( n \in \mathbb{N} \) (by Lemma 2.4). Hence \( A \subseteq \bigcap_{n=1}^{\infty} A_n \). To show the reverse inclusion let us assume that set \( \left( \bigcap_{n=1}^{\infty} A_n \right) \setminus A \) is not empty. Then via Remark 2.2 and Axiom 3 there can be found some \( p \in \mathcal{P}_{\ast}^\ast \) such that for every representing member \( p^\ast \) of class \( p \)

\[
1 = p^\ast \left( \bigcap_{n=1}^{\infty} A_n \right) \setminus A = p^\ast \left( \bigcap_{n=1}^{\infty} A_n \cap \overline{A} \right) \supseteq \bigwedge_{n=1}^{\infty} p^\ast (A_n \cap \overline{A}),
\]

since \( (A_n) \) is a decreasing sequence. Consequently, \( 1 = p^\ast (A_n \cap \overline{A}) \) for all \( n \in \mathbb{N} \). Hence Lemma 2.1 yields that \( p (\overline{A}) = 1 \) and \( p (A_n) = 1 \) for all \( n \in \mathbb{N} \). But then \( p \in \Delta (A_n) \) for all \( n \in \mathbb{N} \) and hence \( p \in \bigcap_{n=1}^{\infty} \Delta (A_n) = \Delta (A) \). Nevertheless, this
is absurd since $p \in \Delta(A) = \overline{\Delta(A)}$. We can thus conclude on the validity of the proposition. q.e.d.

**Theorem 2.10.** Let $(A_n) \subset \mathcal{F}$ and $A \in \mathcal{F}$ be arbitrary. In order that $(A_n)$ converge to $A$, it is necessary and sufficient that $(\Delta(A_n))$ converge to $\Delta(A)$.

**Proof.** For every counting number $n \in \mathbb{N}$ write $E_n = \cap_{k=n}^\infty A_k$ and $B_n = \cup_{k=n}^\infty A_k$. It is clear that sequence $(B_n)$ converges decreasingly to $\limsup_{n \to \infty} A_n$ and sequence $(E_n)$ converges increasingly to $\liminf_{n \to \infty} A_n$. Consequently, by applying Theorems 2.8 and 2.9 to these sequences, we can conclude on the validity of the theorem. q.e.d.

**Definition 2.3.** A mapping $\Delta : \mathcal{F} \to \mathcal{P}(\mathcal{P}_\infty^1)$ is said to be powering if it is defined by:

$$\Delta(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \{p \in \mathcal{P}_\infty^1 : p(A) = 1\} & \text{if } A \neq \emptyset \end{cases}$$

**Remark 2.3.** If $H$ is the unitary atom of a class $p \in \mathcal{P}_\infty^1$, then $\Delta(H) = \{p\}$.

The following result can easily be derived from Lemma 2.6 and Remark 2.2.

**Proposition 2.11.** If $\Delta : \mathcal{F} \to \mathcal{P}(\mathcal{P}_\infty^1)$ is a powering mapping, then it is an injection.

**Definition 2.4.** If $\Gamma \subseteq \mathcal{P}_\infty^1$ is a nonempty set, then the collection $C$ of all the unitary atoms of the classes $p \in \Gamma$ will be called unitary-atomic collection of $\Gamma$.

**Postulate of powering.** If $\Gamma \in \mathcal{P}(\mathcal{P}_\infty^1) \setminus \{\emptyset\}$ and $C$ denotes the governing-atomic collection of $\Gamma$, then $\bigcup C$ is measurable and $\Delta(\bigcup C) \subseteq \Gamma$.

**Theorem 2.12.** The powering mapping $\Delta : \mathcal{F} \to \mathcal{P}(\mathcal{P}_\infty^1)$ is surjective if and only if the postulate of powering is valid.

**Proof.** Assume that *Postulate of powering* is valid. Let $\Gamma \in \mathcal{P}(\mathcal{P}_\infty^1)$ be arbitrarily fixed. We note that if $\Gamma = \emptyset$, then there is nothing to be proven. Suppose that $\Gamma$ is a nonempty subset of $\mathcal{P}_\infty^1$, and denote by $C$ its corresponding governing-atomic collection. Then $\bigcup C$ is measurable and $\Delta(\bigcup C) \subseteq \Gamma$ (by the postulate). Let us show that $\Gamma \subseteq \Delta(\bigcup C)$. In fact, pick any class $p \in \Gamma$ and $p^*$ any representing member of $p$, with $H$ the unitary atom of $p$. Since $H \subseteq \bigcup C$, it ensues from Lemma 2.2 that $\Delta(H) \subseteq \Delta(\bigcup C)$. But, via Remark 2.3 we have that $\{p\} = \Delta(H)$ and $p \in \Delta(\bigcup C)$, i.e. $\Gamma \subseteq \Delta(\bigcup C)$. Therefore $\Gamma = \Delta(\bigcup C)$.

To prove the converse biconditional, let us assume that the powering mapping $\Delta$ is a surjection. We note that $\Delta$ is a bijection, since it is also an injection (by Proposition 2.11). Let $\Gamma \in \mathcal{P}(\mathcal{P}_\infty^1) \setminus \{\emptyset\}$ be arbitrary and write $C$ for the corresponding unitary-atomic collection. Obviously we have that $\Gamma = \bigcup \{\Delta(H) : H \in C\}$ is a subset of $\mathcal{P}_\infty^1$. 
Then via the bijective property it ensues that $\Delta^{-1}(\Gamma) \subseteq \mathcal{F}$. Clearly $\Delta(H) \subseteq \Gamma$ for every $H \in \mathcal{C}$. By Lemma 2.2 together with the bijective property, we obtain that

$$H = \Delta^{-1}(\Delta(H)) \subseteq \Delta^{-1}(\Gamma)$$

whenever $H \in \mathcal{C}$. Consequently the inclusion $\bigcup \mathcal{C} \subseteq \Delta^{-1}(\Gamma)$ follows. Now let us show that if $\omega \in \Delta^{-1}(\Gamma)$, then there is some $H \in \mathcal{C}$ such that $\omega \in H$. Assume on the contrary that there can be found some $\omega_1 \in \Delta^{-1}(\Gamma)$ such that $\omega_1 \notin H$ for all $H \in \mathcal{C}$. We can thus define an optimal measure $q^* : \mathcal{F} \to [0, 1]$ so that

$$q^*(B) \begin{cases} = 1 & \text{if } \omega_1 \in B \\ < 1 & \text{if } \omega_1 \notin B. \end{cases}$$

(See Example 1) Then there is a unique indecomposable $q^*$-atom (to be denoted by $\tilde{H}$) such that $q^*(\tilde{H}) = 1$. It is clear that $\omega_1 \in \tilde{H}$ and $q^*(\Delta^{-1}(\Gamma)) = 1$. We further note that

$$\bigcup \{\Delta(H) : H \in \mathcal{C}\} = \Gamma = \Delta(\Delta^{-1}(\Gamma)) = \{p \in \mathcal{P}_1^\infty : p(\Delta^{-1}(\Gamma)) = 1\}.$$  

From this fact and the identity $q^*(\Delta^{-1}(\Gamma)) = 1$, there must exist some class $p_0 \in \mathcal{P}_1^\infty$ with $p_0(\Delta^{-1}(\Gamma)) = 1$, such that $q^*(\tilde{H} \cap H \cap \Delta^{-1}(\Gamma)) = 1$, where $H$ is the unitary atom of class $p_0$. Nevertheless, this is possible only if $\omega_1 \in H$, which is absurd, since we have supposed that $\omega_1 \notin H$ for all $H \in \mathcal{C}$. Therefore, if $\omega \in \Delta^{-1}(\Gamma)$, then there is some $H \in \mathcal{C}$ such that $\omega \in H$. It ensues that $\omega \in \bigcup \mathcal{C}$ for all $\omega \in \Delta^{-1}(\Gamma)$, as $H \subseteq \bigcup \mathcal{C}$ whenever $H \in \mathcal{C}$. Thus $\Delta^{-1}(\Gamma) \subseteq \bigcup \mathcal{C}$. Therefore, $\bigcup \mathcal{C} = \Delta^{-1}(\Gamma)$, which leads to the postulate. q.e.d.

Theorem 2.12 entails that an infinite $\sigma$-algebra is equinumerous with a power set if and only if Postulate 1 is valid. This suggests that there are infinite $\sigma$-algebras that are not equinumerous with infinite power sets.

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