Miskolc Mathematical Notes
Vol. 24 (2023), No. 1, pp. 335–341
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.3896

SOME PROPERTIES OF $\oplus - g-$SUPPLEMENTED MODULES

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Received 30 June, 2021

Abstract. In this work $\oplus - g-$supplemented modules are defined and some properties of these modules are investigated. It is proved that the finite direct sum of $\oplus - g-$supplemented modules is also $\oplus - g-$supplemented. Let M be a distributive and $\oplus - g-$supplemented $R-$module. Then every factor module and homomorphic image of M are $\oplus - g-$supplemented. Let M be a $\oplus - g-$supplemented $R-$module with *SSP* property. Then every direct summand of M is $\oplus - g-$supplemented.

2010 *Mathematics Subject Classification:* 16D10; 16D80

Keywords: Essential Submodules, Small Submodules, Supplemented Modules, $g-$Supplemented Modules

1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R -module. We will denote a submodule N of M by $N \leq M$. Let M be an R -module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small submodule* of M and denoted by $N \ll M$. Let M be an R -module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a *direct summand* of M and it is denoted by $M = N \oplus K$. For any $R-$module M, we have $M = M \oplus 0$. The intersection of all maximal submodules of M is called the *radical* of M and denoted by $RadM$. A submodule N of an R -module M is called an *essential submodule* and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$. Let M be an R -module and K be a submodule of M. K is called a *generalized small* (or briefly, $g-$*small*) *submodule* of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, then we write $K \ll_g M$ (in [10], it is called an *e-small submodule* of M and denoted by $K \ll_e M$). It is clear that every small submodule is a generalized small submodule but the converse is not true in general. Let M be an R -module. M is called a *hollow module* if every proper submodule of M is small in M. M is called a *generalized hollow* (or briefly, $g-$*hollow*) *module* if every proper submodule of M is g-small in M. Here it is clear that every hollow module is generalized hollow.

The converse of this statement is not always true. M is called a *local module* if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. Let U and V be submodules of M. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is called a *supplemented module* if every submodule of M has a supplement in M. If every submodule of M has a supplement that is a direct summand in M, then M is called a $\oplus-$*supplemented module*. Let M be an $R-$module and $U,V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. If every submodule of M has a g-supplement in M, then M is called a *g-supplemented module*. A module M is said to have the *Summand Sum Property* (*SSP*) if the sum of two direct summands of M is again a direct summand of M (see [1]). We say that a module M has $(D3)$ property if $M_1 \cap M_2$ is a direct summand of M for every direct summands M_1 and M_2 of M with $M = M_1 + M_2$ (see[7]). The intersection of all essential maximal submodules of an $R-$module M is called the *generalized radical* of M and denoted by $Rad_g M$ (in [10], it is denoted by $Rad_e M$). If M have no essential maximal submodules, then we denote $Rad_g M = M$.

More information about supplemented modules are in [2, 9]. More results about $\oplus-$supplemented modules are in [3, 4, 7]. More information about g-supplemented modules are in [5].

Now we will give some important properties of generalized small submodules.

Lemma 1. *Let M be an R -module and K, $N \leq M$. Consider the following conditions.*

(1) *If $K \leq N$ and N is a generalized small submodule of M, then K is a generalized small submodule of M.*
(2) *If K is contained in N and a generalized small submodule of N, then K is a generalized small submodule in submodules of M which contains N.*
(3) *Let T be an $R-$module and $f : M \to T$ be an R-module homomorphism. If $K \ll_g M$, then $f(K) \ll_g T$.*
(4) *If $K \ll_g L$ and $N \ll_g T$ with L, $T \leq M$, then $K + N \ll_g L + T$.*

Proof. See [5, Lemma 1]. □

Corollary 1. *Let M be an R -module and $K \leq N \leq M$. If $N \ll_g M$, then $N/K \ll_g M/K$.* [5]

Corollary 2. *Let M be an R -module, $K \ll_g M$ and $L \leq M$. Then $(K+L)/L \ll_g M/L$.* [5]

2. $\oplus - g-$SUPPLEMENTED MODULES

Definition 1. Let M be an $R-$module. If every submodule of M has a g-supplement that is a direct summand of M, then M is called a $\oplus - g-$supplemented module. (See also [8])

Clearly we can see that every $\oplus - g-$supplemented module is $g-$supplemented. We also clearly can see that every $\oplus-$supplemented and every generalized hollow modules are $\oplus - g-$supplemented.

Lemma 2. *Let M be an $R-$module, V be a supplement of U in M and $X,Y \leq V$. Then X is a g-supplement of Y in V if and only if X is a g-supplement of $U+Y$ in M.*

Proof. $(\Longrightarrow)$Let $M = U+Y+T$ with $T \trianglelefteq X$. Since V is a supplement of U in M and $Y+T \leq V$, $V = Y+T$ and since X is a g-supplement of Y in V, then $T = X$. Hence X is a g-supplement of $U+Y$ in M.

$(\Longleftarrow)$ Let $V = Y+T$ with $T \trianglelefteq X$. Since V is a supplement of U in M, $M = U+V = U+Y+T$ and since X is a g-supplement of $U+Y$ in M, then $T = X$. Hence X is a g-supplement of Y in V. □

Corollary 3. *Let $M = M_1 \oplus M_2$ and $X,Y \leq M_2$. Then X is a g-supplement of Y in M_2 if and only if X is a g-supplement of M_1+Y in M.*

Proof. Clear from Lemma 2. □

Lemma 3. *Let M be an $R-$module and $M = M_1 \oplus M_2$. If M_1 and M_2 are $\oplus - g-$supplemented, then M is also $\oplus - g-$supplemented.*

Proof. Let U be any submodule of M. Since M_2 is $\oplus - g-$supplemented, $(M_1+U) \cap M_2$ has a g-supplement X that is a direct summand of M_2. Since X is a g-supplement of $(M_1+U)\cap M_2$ in M_2, $M_2 = (M_1+U)\cap M_2 + X$ and $(M_1+U)\cap X = (M_1+U)\cap M_2 \cap X \ll_g X$. By $M_2 = (M_1+U)\cap M_2+X$, $M = M_1 \oplus M_2 = M_1 + (M_1+U)\cap M_2 + X = M_1+U+X$. Since M_1 is $\oplus - g-$supplemented, $(U+X)\cap M_1$ has a g-supplement Y that is a direct summand of M_1. Since Y is a g-supplement of $(U+X)\cap M_1$ in M_1, $M_1 = (U+X)\cap M_1+Y$ and $(U+X)\cap Y = (U+X)\cap M_1 \cap Y \ll_g Y$. By $M_1 = (U+X)\cap M_1+Y$, $M = M_1+U+X = (U+X)\cap M_1+Y+U+X = U+X+Y$. Since $(M_1+U)\cap X \ll_g X$ and $(U+X)\cap Y \ll_g Y$, by Lemma 1, $U\cap(X+Y) \leq (U+Y)\cap X + (U+X)\cap Y \leq (M_1+U)\cap X + (U+X)\cap Y \ll_g X+Y$. Hence $X+Y$ is a g-supplement of U in M. Since X is a direct summand of M_2 and Y is a direct summand of M_1, $X+Y$ is a direct summand of $M = M_1 \oplus M_2$. Hence M is $\oplus - g-$supplemented. □

Corollary 4. *Let M be an $R-$module and $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$. If M_i is $\oplus - g-$supplemented for every $i = 1,2,\ldots,n$, then M is also $\oplus - g-$supplemented.*

Proof. Clear from Lemma 3. □

Proposition 1. *Let M be a $\oplus-g-$supplemented module. If every g-supplement submodule in M is a direct summand of M, then every direct summand of M is $\oplus-g-$supplemented.*

Proof. Let N be a direct summand of M and $M=N\oplus K$ with $K\leq M$. Since M is g-supplemented, by [5, Theorem 2], M/K is g-supplemented. By $\frac{M}{K}=\frac{N\oplus K}{K}\simeq\frac{N}{N\cap K}=\frac{N}{0}\cong N$, N is also g-supplemented. Let $X\leq N$ and Y be a g-supplement of X in N. Since $M=N\oplus K$, by Corollary 3, $Y$is a g-supplement of $K+X$ in M. Since every g-supplement submodule in M is a direct summand of M, Y is a direct summand of M. By $Y\leq N$, Y is also a direct summand of N. Hence N is $\oplus-g-$supplemented. □

Lemma 4. *Let M be a $\oplus-g-$supplemented $R-$module and $K\leq M$. If $\frac{X+K}{K}$ is a direct summand of $\frac{M}{K}$ for every direct summand X of M, then $\frac{M}{K}$ is $\oplus-g-$supplemented.*

Proof. Let U/K be any submodule of M/K. Since M is $\oplus-g-$supplemented, U has a g-supplement X in M that is a direct summand in M. Since X is a g-supplement of U in M and $K\leq U$, by [5, Lemma 4], $\frac{X+K}{K}$ is a g-supplement of U/K in M/K. Since X is a direct summand of M, by hypothesis, $\frac{X+K}{K}$ is a direct summand of M/K. Hence M/K is $\oplus-g-$supplemented. □

Lemma 5. *Let M be a distributive and $\oplus-g-$supplemented $R-$module. Then every factor module of M is $\oplus-g-$supplemented.*

Proof. Let $K\leq M$ and X be a direct summand of M. Since X is a direct summand of M, there exists $Y\leq M$ such that $M=X\oplus Y$. Since $M=X\oplus Y$, $\frac{M}{K}=\frac{X+K}{K}+\frac{Y+K}{K}$. Since M is distributive, $(X+K)\cap(Y+K)=K$ and $\frac{X+K}{K}\cap\frac{Y+K}{K}=\frac{(X+K)\cap(Y+K)}{K}=\frac{K}{K}=0$. Hence $\frac{M}{K}=\frac{X+K}{K}\oplus\frac{Y+K}{K}$ and by Lemma 4, M/K is $\oplus-g-$supplemented. □

Corollary 5. *Let M be a distributive and $\oplus-g-$supplemented $R-$module. Then every homomorphic image of M is $\oplus-g-$supplemented.*

Proof. Clear from Lemma 5. □

Lemma 6. *Let M be a $\oplus-g-$supplemented $R-$module with $(D3)$ property. Then every direct summand of M is $\oplus-g-$supplemented.*

Proof. Let K be any direct summand of M. Then there exists $T\leq M$ such that $M=K\oplus T$. Let $U\leq K$. Since M is $\oplus-g-$supplemented, U has a g-supplement X that is a direct summand in M. Here $M=U+X$ and $U\cap X\ll_g X$. Since $U\leq K$, $M=U+X=K+X$ and since M has $(D3)$ property, $K\cap X$ is a direct summand of M. Then there exists $Y\leq M$ such that $M=(K\cap X)\oplus Y$. Here $K=(K\cap X)\oplus(K\cap Y)$. Since $M=U+X$ and $U\leq K$, by Modular Law, $K=U+(K\cap X)$. Let $\pi:M\longrightarrow K\cap X$ be a canonical projection. Since $U\cap X\ll_g X\leq M$, by Lemma 1, $U\cap K\cap X=U\cap X=\pi(U\cap X)\ll_g K\cap X$. Hence K is $\oplus-g-$supplemented. □

Corollary 6. *Let M be a $\oplus-g-$supplemented $R-$module with $(D3)$ property. Then M/X is $\oplus-g-$supplemented for every direct summand X of M.*

Proof. Let X be any direct summand of M. Then there exists $Y \leq M$ such that $M = X \oplus Y$. By Lemma 6, Y is $\oplus-g-$supplemented. Then by $\frac{M}{X} = \frac{X+Y}{X} \cong \frac{Y}{X\cap Y} = \frac{Y}{0} \cong Y$, M/X is also $\oplus-g-$supplemented. □

Corollary 7. *Let M be a $\oplus-g-$supplemented $R-$module with $(D3)$ property and $f : M \longrightarrow N$ be an $R-$module epimorphism with N is an $R-$module and $Ker(f)$ is a direct summand of M. Then N is $\oplus-g-$supplemented.*

Proof. Clear from Corollary 6, since $M/Ker(f) \cong Im(f) = N$. □

Lemma 7. *Let M be a $\oplus-g-$supplemented $R-$module, $K \leq M$ and $K = (K\cap M_1) \oplus (K\cap M_2)$ for every $M_1, M_2 \leq M$ with $M = M_1 \oplus M_2$. Then M/K is $\oplus-g-$supplemented.*

Proof. Let $U/K \leq M/K$. Since M is $\oplus-g-$supplemented, U has a g-supplement V that is a direct summand in M. Here there exists $X \leq M$ such that $M = V \oplus X$. By hypothesis, $K = (K\cap V) \oplus (K\cap X)$. Since V is a g-supplement of U in M and $K \leq U$, by [5, Lemma 4], $\frac{V+K}{K}$ is a g-supplement of U/K in M/K. Since $M = V \oplus X$, $\frac{M}{K} = \frac{V+K}{K} + \frac{X+K}{K}$. Here $\frac{V+K}{K} \cap \frac{X+K}{K} = \frac{(V+K)\cap(X+K)}{K} = \frac{(V+K)\cap X+K}{K} = \frac{(V+K\cap V+K\cap X)\cap X+K}{K} = \frac{(V+K\cap X)\cap X+K}{K} = \frac{V\cap X+K\cap X+K}{K} = \frac{0+K}{K} = \frac{K}{K} = 0$. Hence $\frac{M}{K} = \frac{V+K}{K} \oplus \frac{X+K}{K}$. Thus M/K is $\oplus-g-$supplemented. □

Corollary 8. *Let M be a $\oplus-g-$supplemented $R-$module, $f : M \longrightarrow N$ be and $R-$module epimorphism with N be an $R-$module and $Ker(f) = (Ker(f)\cap M_1) \oplus (Ker(f)\cap M_2)$ for every $M_1, M_2 \leq M$ with $M = M_1 \oplus M_2$. Then N is $\oplus-g-$supplemented.*

Proof. Clear from Lemma 7, since $M/Ker(f) \cong Im(f) = N$. □

Proposition 2. *Let M be a $\oplus-g-$supplemented $R-$module. Then there exist $M_1, M_2 \leq M$ such that $M = M_1 \oplus M_2$, $Rad_g M_1 \ll_g M_1$ and $Rad_g M_2 = M_2$.*

Proof. Since M is $\oplus-g-$supplemented, $Rad_g M$ has a g-supplement M_1 in M such that M_1 is a direct summand of M. Since M_1 is a direct summand of M, there exists $M_2 \leq M$ such that $M = M_1 \oplus M_2$. Since M_1 is a g-supplement of $Rad_g M$ in M, $M = Rad_g M + M_1$ and $M_1 \cap Rad_g M \ll_g M_1$. Since $M = M_1 \oplus M_2$, by [6, Lemma 4], $Rad_g M = Rad_g M_1 \oplus Rad_g M_2$. Hence $Rad_g M_1 = M_1 \cap Rad_g M \ll_g M_1$. Since $Rad_g M = Rad_g M_1 \oplus Rad_g M_2$, $M = Rad_g M + M_1 = Rad_g M_1 + Rad_g M_2 + M_1 = M_1 \oplus Rad_g M_2$. Hence $M_2 = M_2 \cap M = M_2 \cap (M_1 \oplus Rad_g M_2) = (M_2 \cap M_1) \oplus Rad_g M_2 = 0 \oplus Rad_g M_2 = Rad_g M_2$. □

Proposition 3. *Let M be a $\oplus-g-$supplemented $R-$module. Then there exist $M_1, M_2 \leq M$ such that $M = M_1 \oplus M_2$, $RadM_1 \ll_g M_1$ and $RadM_2 = M_2$.*

Proof. We can also prove this similar to proof of the previous Proposition. But we prove by different way. Since M is $\oplus-g-$supplemented, $RadM$ has a g-supplement

M_1 in M such that M_1 is a direct summand of M. Since M_1 is a direct summand of M, there exists $M_2 \leq M$ such that $M = M_1 \oplus M_2$. Since M_1 is a g-supplement of $RadM$ in M, $M = RadM + M_1$ and $RadM_1 = M_1 \cap RadM \ll_g M_1$. Assume that X be a maximal submodule of M_2. Since $\frac{M}{M_1+X} = \frac{M_1+M_2}{M_1+X} \cong \frac{M_2}{M_2 \cap (M_1+X)} = \frac{M_2}{M_2 \cap M_1 + X} = \frac{M_2}{X}$, $M_1 + X$ is a maximal submodule of M. Then $M = RadM + M_1 \leq M_1 + X$. This is a contradiction. Hence $RadM_2 = M_2$. □

Lemma 8. *Let M be a $\oplus - g-$supplemented $R-$module with SSP property. Then M/K is $\oplus - g-$supplemented for every direct summand K of M.*

Proof. Let K be any direct summand of M and $U/K \leq M/K$. Since M is $\oplus - g-$supplemented, U has a g-supplement V in M such that V is a direct summand of M. By [5, Lemma 4], $\frac{V+K}{K}$ is a g-supplement of U/K in M/K. Since K and V are direct summands of M and M has *SSP* property, $K + V$ is also a direct summand of M. Hence there exists $T \leq M$ such that $M = (K + V) \oplus T$. Since $M = (K + V) \oplus T$, $\frac{M}{K} = \frac{K+V+T}{K} = \frac{V+K}{K} + \frac{T+K}{K}$. Since $(V + K) \cap T = 0$, $\frac{V+K}{K} \cap \frac{T+K}{K} = \frac{(V+K)\cap(T+K)}{K} = \frac{(V+K)\cap T+K}{K} = \frac{0+K}{K} = 0$. Hence $\frac{M}{K} = \frac{V+K}{K} \oplus \frac{T+K}{K}$ and M/K is $\oplus - g-$supplemented. □

Corollary 9. *Let M be a $\oplus - g-$supplemented $R-$module with SSP property. Then every direct summand of M is $\oplus - g-$supplemented.*

Proof. Let T be any direct summand of M. Then there exists a submodule K of M such that $M = T \oplus K$. By Lemma 8, M/K is $\oplus - g-$supplemented. Since $\frac{M}{K} = \frac{T+K}{K} \cong \frac{T}{T\cap K} = \frac{T}{0} \cong T$, T is also $\oplus - g-$supplemented. □

Remark 1. Let M be an $R-$module which has only four proper submodules 0, A, B, C with $C \leq A$, $C \leq B$, $A \nleq B$ and $B \nleq A$. Then M is g-supplemented but not $\oplus - g-$supplemented.

ACKNOWLEDGEMENT

This research was in part supported by grants from Ondokuz Mayıs University (Project No: PYO.EGF.1901.17.003).

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