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GLOBAL EXISTENCE AND EXPONENTIAL DECAY OF SOLUTIONS FOR HIGHER-ORDER PARABOLIC EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. This paper deals with the initial boundary value problem for a higher-order parabolic equation with logarithmic source term

$$u_t + (-\Delta)^m u = u^{r-2} u \ln|u|.$$

By employing the potential wells technique we show the global existence of the weak solution. Also, we obtain the exponential decay for the weak solutions.

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1. INTRODUCTION

In this article, we deal with the following higher-order parabolic equation with logarithmic source term

$$\begin{cases} u_t + Pu = u^{r-2} u \ln|u|, & x \in \Omega, \quad t > 0, \\ D^\gamma u(x,t) = 0, \quad |\gamma| \leq m-1, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $P = (-\Delta)^m$, $m \geq 1$ a positive integer, $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bound domain with smooth boundary $\partial\Omega$, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is multi-index, γ_i $(i = 1, 2, \ldots, n)$ are positive integers, $|\gamma| = \gamma_1 + \gamma_2 + \ldots + \gamma_n$, $D^\gamma = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \partial x_2^{\gamma_2} \ldots \partial x_n^{\gamma_n}}$ are derivative operators, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, and r satisfies

$$\begin{cases} 2 \leq r \leq +\infty, & n = 1, 2, \\ 2 \leq r \leq \frac{2n}{n-2}, & n \geq 3. \end{cases}$$

When $m = 1$, equation (1.1) becomes a heat equation as follows

$$u_t - \Delta u = u^{r-2} u \ln|u|,$$

where $2 \leq r$, which case was considered by many authors [1, 4, 10]. In the case of $r = 2$, Chen et al. [1] obtained under some suitable conditions for the global existence, decay estimate and blow-up at $+\infty$ of weak solutions, via the logarithmic Sobolev inequality and potential well technique. In the case of $2 < k$, Peng and Zhou [10] studied the existence of the unique global weak solutions and blow-up in the finite time of weak solutions, via potential well technique and energy technique.

When $m = 2$, Li and Liu [7] established the equation

$$u_t + \Delta^2 u = u^{r-2} u \ln |u|,$$

where $2 < r$. They studied the existence of global solutions, by using potential well technique. In addition, they also studied result of decay and finite time blow-up for weak solutions.

Nhan and Truong [9] studied the following nonlinear pseudo-parabolic equation

$$u_t - \Delta u_t - div\left(|\nabla u|^{r-2} \nabla u\right) = |u|^{r-2} u \log |u|,$$

where $2 < r$. They obtained results as regard the existence or non-existence of global solutions, by using the potential well technique and a logarithmic Sobolev inequality. Also, He et al. [5] proved the decay and the finite time blow-up for weak solutions of the equation, by using the potential well technique and concave technique.

Recently many other authors investigated higher-order hyperbolic and parabolic type equation [2, 3, 6, 11–15]. Ishige et al. [6] studied the Cauchy problem for nonlinear higher-order heat equation as follows

$$u_t + (-\Delta)^m u = |u|^r.$$

They obtained existence of solutions of the Cauchy problem by introducing a new majorizing kernel. In addition, they studied the local existence of solutions under the different conditions.

Xiao and Li [13] considered the initial boundary value problem for nonlinear higher-order heat equations of

$$u_t + (-\Delta)^m u_t + (-\Delta)^m u = f(u).$$

They established the existence of a weak solution to the static problem, by using the potential well technique.

The remainder of our work is organized as follows. In Section 2, some important Lemmas are given. In Section 3, the main result is proved.

2. Preliminaries

Let $\|u\|_{H^m(\Omega)} = \left(\sum_{|\gamma| \le m} \|D^\gamma u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$ denote $H^m(\Omega)$ norm, let $H_0^m(\Omega)$ denote the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. Let $\|.\|_r$ and $\|.\|$ denote the usual $L^r(\Omega)$ norm and $L^2(\Omega)$ norm.

For $u \in H_0^m(\Omega)\backslash\{0\}$, we define the energy functional

$$J(u) = \frac{1}{2}\left\|P^{\frac{1}{2}}u\right\|^2 - \frac{1}{r}\int_\Omega |u|^r \ln|u|\,dx + \frac{1}{r^2}\|u\|_r^r, \tag{2.1}$$

and Nehari functional

$$I(u) = \left\|P^{\frac{1}{2}}u\right\|^2 - \int_\Omega |u|^r \ln|u|\,dx. \tag{2.2}$$

By (2.1) and (2.2), we obtain

$$J(u) = \frac{1}{r}I(u) + \left(\frac{1}{2}-\frac{1}{r}\right)\left\|P^{\frac{1}{2}}u\right\|^2 + \frac{1}{r^2}\|u\|_r^r. \tag{2.3}$$

Further, let

$$d = \inf_{u\in\mathcal{N}} J(u), \tag{2.4}$$

denote the potential depth, where $\mathcal{N}$ is the Nehari manifold, which is defined by

$$\mathcal{N} = \{u \in H_0^m(\Omega)\backslash\{0\} : I(u) = 0\}.$$

Lemma 1. *Let k be a number with $2 \le k < +\infty$, $n \le 2m$ and $2 \le k \le \frac{2n}{n-2m}$, $n > 2m$. Then there is a constant C depending*

$$\|u\|_k \le C\left\|P^{\frac{1}{2}}u\right\|, \quad \forall u \in H_0^m(\Omega).$$

Lemma 2. *$J(t)$ is a nonincreasing function for $t \ge 0$ and*

$$J'(u) = -\int_\Omega u_t^2 dx \le 0. \tag{2.5}$$

Proof. Multiplying equation (1.1) by u_t and integrating on Ω, we get

$$\int_\Omega u_t^2 dx + \int_\Omega Puu_t dx = \int_\Omega u^{r-1}u_t \ln|u|\,dx.$$

By straightforward calculation, we obtain

$$\int_\Omega u_t^2 dx + \frac{1}{2}\frac{d}{dt}\left\|P^{\frac{1}{2}}u\right\|^2 = \frac{1}{r}\frac{d}{dt}\int_\Omega |u|^r \ln|u|\,dx - \frac{1}{r^2}\frac{d}{dt}\|u\|_r^r,$$

which yields that

$$\frac{1}{2}\frac{d}{dt}\left\|P^{\frac{1}{2}}u\right\|^2 - \frac{1}{r}\frac{d}{dt}\int_\Omega |u|^r \ln|u|\,dx + \frac{1}{r^2}\frac{d}{dt}\|u\|_r^r = -\int_\Omega u_t^2 dx.$$

Thus, we get

$$\frac{d}{dt}\left(\frac{1}{2}\left\|P^{\frac{1}{2}}u\right\|^2 - \frac{1}{r}\int_\Omega |u|^r \ln|u|\,dx + \frac{1}{r^2}\|u\|_r^r\right) = -\int_\Omega u_t^2 dx. \tag{2.6}$$

By (2.1) and (2.6), we obtain

$$\frac{d}{dt}J(u) = -\int_{\Omega} u_t^2 dx. \tag{2.7}$$

Moreover, integrating (2.7) with respect to t on $[0,t]$, we arrive at

$$J(u(t)) + \int_0^t \|u_s(\tau)\|^2 d\tau = J(u_0). \tag{2.8}$$

□

Lemma 3. *Let $u \in H_0^m(\Omega)\backslash\{0\}$ and $j(\lambda) = J(\lambda u)$. Then we get*

(i) $\lim_{\lambda\to 0^+} j(\lambda) = 0$ *and* $\lim_{\lambda\to+\infty} j(\lambda) = -\infty$,

(ii) *there is a unique* $\lambda^* > 0$ *such that* $j'(\lambda^*) = 0$,

(iii) $j(\lambda)$ *is decreasing on* $(\lambda^*,+\infty)$, *increasing on* $(0,\lambda^*)$ *and taking the maximum at* λ^*,

(iv) $I(\lambda u) < 0$ *for* $\lambda \in (\lambda^*,+\infty)$, $I(\lambda u) > 0$ *for* $\lambda \in (0,\lambda^*)$ *and* $I(\lambda^* u) = 0$.

Proof. By the definition of j, for $u \in H_0^1(\Omega)\backslash\{0\}$, we get

$$j(\lambda) = \frac{\lambda^2}{2}\left\|P^{\frac{1}{2}}u\right\|^2 - \frac{\lambda^r}{r}\int_{\Omega}|u|^r \ln|u|\,dx - \frac{\lambda^r}{r}\ln\lambda\,\|u\|_r^r + \frac{\lambda^r}{r^2}\|u\|_r^r. \tag{2.9}$$

By (2.9), we have

$$\begin{aligned}\frac{d}{d\lambda}j(\lambda) &= \lambda\left\|P^{\frac{1}{2}}u\right\|^2 - \lambda^{r-1}\int_{\Omega}|u|^r \ln|u|\,dx - \lambda^{r-1}\ln\lambda\,\|u\|_r^r - \frac{\lambda^{r-1}}{r}\|u\|_r^r + \frac{\lambda^{r-1}}{r}\|u\|_r^r \\ &= \lambda\left(\left\|P^{\frac{1}{2}}u\right\|^2 - \lambda^{r-2}\int_{\Omega}|u|^r \ln|u|\,dx - \lambda^{r-2}\ln\lambda\,\|u\|_r^r\right).\end{aligned}$$

Let $\phi(\lambda) = \lambda^{-1}j'(\lambda)$, thus we obtain

$$\phi(\lambda) = \left\|P^{\frac{1}{2}}u\right\|^2 - \lambda^{r-2}\int_{\Omega}|u|^r \ln|u|\,dx - \lambda^{r-2}\ln\lambda\,\|u\|_r^r.$$

Then

$$\phi'(\lambda) = -(r-2)\lambda^{r-3}\int_{\Omega}|u|^r \ln|u|\,dx - (r-2)\lambda^{r-3}\ln\lambda\,\|u\|_r^r - \lambda^{r-3}\|u\|_r^r,$$

which yields that there exists a $\lambda^* > 0$ such that $\phi'(\lambda) < 0$ on $(\lambda^*,+\infty)$, $\phi'(\lambda) > 0$ on $(0,\lambda^*)$ and $\phi'(\lambda) = 0$. Thus, $\phi(\lambda)$ is decreasing on $(\lambda^*,+\infty)$, increasing on $(0,\lambda^*)$. Since $\lim_{\lambda\to 0^+}\phi(\lambda) > 0$, $\lim_{\lambda\to+\infty}\phi(\lambda) = -\infty$, there exists a unique $\lambda^* > 0$ such that $\phi(\lambda^*) = 0$, i.e., $j'(\lambda^*) = 0$. Then, $j'(\lambda) = \lambda\phi(\lambda)$ is negative on $(\lambda^*,+\infty)$, positive on $(0,\lambda^*)$. Thus, $j(\lambda)$ is decreasing on $(\lambda^*,+\infty)$, increasing on $(0,\lambda^*)$ and taking the maximum at λ^*. By (2.2), we get

$$I(\lambda u) = \left\|P^{\frac{1}{2}}(\lambda u)\right\|^2 - \int_{\Omega}|\lambda u|^r \ln|\lambda u|\,dx$$

$$= \lambda^2 \left\| P^{\frac{1}{2}} u \right\|^2 - \lambda^r \int_\Omega |u|^r \ln |u| \, dx - \lambda^r \ln \lambda \|u\|_r^r$$
$$= \lambda j'(\lambda).$$

So, $I(\lambda u) < 0$ for $\lambda \in (\lambda^*, +\infty)$, $I(\lambda u) > 0$ for $\lambda \in (0, \lambda^*)$ and $I(\lambda^* u) = 0$. Therefore, the proof is completed. □

Lemma 4 ([8])**.** *Let μ be a constant and $g : R^+ \to R^+$ be a nonincreasing function such that*

$$\int_t^{+\infty} g^{1+\mu}(\tau) d\tau \le \frac{1}{\zeta} g^\mu(0) g(t), \text{ for all } t \ge 0.$$

Hence

(i) $g(t) \le g(0) \left(\frac{1+\mu}{1+\zeta\mu t} \right)^{\frac{1}{\mu}}$, $\forall t \ge 0$, *whenever* $\mu > 0$,

(ii) $g(t) \le g(0) e^{1-\zeta t}$, $\forall t \ge 0$, *whenever* $\mu = 0$.

3. Main results

As in [9], we consider the following notations:

$$\mathcal{W}_1 = \{u \in H_0^1(\Omega) \backslash \{0\} : J(u) < d\}, \qquad \mathcal{W}_2 = \{u \in H_0^1(\Omega) \backslash \{0\} : J(u) = d\},$$
$$\mathcal{W}_1^+ = \{u \in \mathcal{W}_1 : I(u) > 0\}, \qquad \mathcal{W}_2^+ = \{u \in \mathcal{W}_2 : I(u) > 0\},$$
$$\mathcal{W}_1^- = \{u \in \mathcal{W}_1 : I(u) < 0\}, \qquad \mathcal{W}_2^- = \{u \in \mathcal{W}_2 : I(u) < 0\},$$
$$\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2, \qquad \mathcal{W}^+ = \mathcal{W}_1^+ \cup \mathcal{W}_2^+, \quad \mathcal{W}^- = \mathcal{W}_1^- \cup \mathcal{W}_2^-.$$

We refer to $\mathcal{W}$ as the potential well and d as the depth of the well.

Definition 1 (Weak Solution)**.** We say that function $u(t)$ is a weak solution of problem (1.1) on $\Omega \times [0,T]$, if $u \in L^\infty(0,T; H_0^m(\Omega))$ with $u_t \in L^2(0,T; H_0^m(\Omega))$ and implies the initial condition $u(0) = u_0 \in H_0^m(\Omega) \backslash \{0\}$, and the following equality

$$(u_t, w) + \left(P^{\frac{1}{2}} u, P^{\frac{1}{2}} w \right) = \left(|u|^{r-2} u \ln |u|, w \right),$$

for all $w \in H_0^m(\Omega)$ holds for a.e. $t \in [0,T]$, and $(.,.)$ means the inner product $(.,.)_{L^2(\Omega)}$, that is

$$(\eta, \xi) = \int_\Omega \eta(x) \xi(x) dx.$$

Definition 2 (Maximal Existence Time)**.** Suppose that $u(t)$ is a weak solutions of problem (1.1). We define the following the maximal existence time $T_{\max}$

$$T_{\max} = \sup\{T > 0 : u(t) \text{ exists on } [0,T]\}.$$

Then

(a) If $T_{\max} = \infty$, we say that $u(t)$ is global;
(b) If $T_{\max} < \infty$, we say that $u(t)$ blows up in finite time.

Theorem 1 (Global Existence)**.** *Let $u_0 \in \mathcal{W}^+$. Then problem (1.1) admits a global weak solution. We get $u(t) \in \mathcal{W}^+$ holds for any $t \in [0,+\infty)$, and the energy estimate*

$$J(u(t)) + \int_0^t \|u_s(s)\|^2 ds \le J(u_0), \quad t \in [0,+\infty).$$

Also, the solution decays exponential provided $u_0 \in \mathcal{W}_1^+$.

Proof. We will the investigate the following two cases:

Firstly, we address the case of the initial data $u_0 \in \mathcal{W}_1^+$.

The Faedo-Galerkin's methods is used. In the space $H_0^m(\Omega)$, we take a bases $\{w_j\}_{j=1}^\infty$ and define the finite orthogonal space

$$V_s = span\{w_1, w_2, ..., w_s\}.$$

Let u_{0s} be an element of V_s such that

$$u_{0s} = \sum_{j=1}^{s} a_{sj} w_j \to u_0, \text{ in } H_0^m(\Omega), \tag{3.1}$$

as $s \to \infty$. We construct the following approximate solution $u_s(x,t)$ of problem (1.1)

$$u_s(x,t) = \sum_{j=1}^{s} a_{sj}(t) w_j(x), \tag{3.2}$$

where the coefficients a_{sj} $(1 \le j \le s)$ imply the ODEs

$$\int_\Omega u_{st} w_i dx + \int_\Omega P u_s w_i dx = \int_\Omega |u_s|^{r-2} u_s \ln|u_s| w_i dx, \tag{3.3}$$

for $i \in \{1,2,...,s\}$, with the initial condition

$$a_{sj}(0) = a_{sj}, \quad j \in \{1,2,...,s\}. \tag{3.4}$$

We multiply both sides of (3.3) by a'_{si}, sum for $i = 1,...,s$ and integrating with respect to time variable on $[0,t]$, we get

$$J(u_s(t)) + \int_0^t \|u_{s\tau}(\tau)\|^2 d\tau \le J(u_{0s}), \quad 0 \le t \le T_{\max}, \tag{3.5}$$

where $T_{\max}$ is the maximal existence time of solution $u_s(t)$. We shall prove that $T_{\max} = +\infty$. From (3.1), (3.5) and the continuity of J, we obtain

$$J(u_s(0)) \to J(u_{0s}), \text{ as } s \to \infty. \tag{3.6}$$

Thanks to $J(u_0) < d$ and the continuity of functional J, it follows from (3.6) that

$$J(u_{0s}) < d, \quad \text{for sufficiently large } m.$$

And therefore, from (3.5), we get

$$J(u_s(t)) + \int_0^t \|u_{s\tau}(\tau)\|^2 d\tau < d, \quad 0 \le t \le T_{\max}, \tag{3.7}$$

for sufficiently large s. Next, we will study

$$u_s(t) \in \mathcal{W}_1^+, \quad t \in [0, T_{\max}), \tag{3.8}$$

for sufficiently large s. We assume that (3.8) does not process and think that there exists a sufficiently small time t_0 such that $u_s(t_0) \notin \mathcal{W}_1^+$. Then, by continuity of $u_s(t_0) \in \partial \mathcal{W}_1^+$. So, we get

$$J(u_s(t_0)) = d, \tag{3.9}$$

and

$$I(u_s(t_0)) = 0. \tag{3.10}$$

Nevertheless, by definition of d, we see that (3.9) could not consist by (3.7) while if (3.10) holds then, we get

$$d = \inf_{u \in \mathcal{N}} J(u) \leq J(u_s(t_0)),$$

which also contradicts with (3.7). Moreover, we have (3.8), i.e., $I(u_s(t)) > 0$, and $J(u_s(t)) < d$, for all $t \in [0, T_{\max})$, for sufficiently large s. Then, from (2.3), we obtain

$$\begin{aligned} d &> J(u_s(t)) \\ &= \frac{1}{r} I(u_s(t)) + \left(\frac{1}{2} - \frac{1}{r}\right) \left\| P^{\frac{1}{2}} u_s(t) \right\|^2 + \frac{1}{r^2} \|u_s(t)\|_r^r \\ &\geq \left(\frac{1}{2} - \frac{1}{r}\right) \left\| P^{\frac{1}{2}} u_s(t) \right\|^2 + \frac{1}{r^2} \|u_s(t)\|_r^r, \end{aligned}$$

which gives

$$\|u_s(t)\|_r^r < r^2 d, \tag{3.11}$$

and

$$\left\| P^{\frac{1}{2}} u_s(t) \right\|^2 < \frac{2r}{r-2} d. \tag{3.12}$$

Since $u_s(x,t) \in \mathcal{W}_1^+$ for s large enough, it follows from (2.3) that $J(u_s) \geq 0$ for s large enough. So, by (3.7) it follows for s large enough

$$\int_0^t \|u_{s\tau}(\tau)\|^2 \, d\tau < d. \tag{3.13}$$

By (3.12), we know that

$$T_{\max} = +\infty.$$

It follows from (3.11) and (3.13) that there exist a function $u \in H_0^m(\Omega)$ and a subsequence of $\{u_s\}_{j=1}^{\infty}$ is indicated by $\{u_s\}_{j=1}^{\infty}$ such that

$$u_s \to u \ \text{ weakly* in } L^\infty(0, \infty; H_0^m(\Omega)), \tag{3.14}$$

$$u_{st} \to u_t \ \text{ weakly in } L^2(0, \infty; L^2(\Omega)). \tag{3.15}$$

By (3.14), (3.15) and the Aubin-Lions compactness theorem, we obtain

$$u_s \to u \ \text{ strongly in } C([0, +\infty]; L^2(\Omega)).$$

This yields that

$$|u_s|^{r-2} u_s \ln |u_s| \to |u|^{r-2} u \ln |u| \quad \text{a.e. } (x,t) \in \Omega \times (0,+\infty). \tag{3.16}$$

Moreover, since

$$\alpha^{r-1} \ln \alpha = -(e(r-1))^{-1} \quad \text{for } \alpha > 1,$$

and

$$\ln \alpha = 2 \ln \left(\alpha^{\frac{1}{2}}\right) \le 2\alpha^{\frac{1}{2}} \quad \text{for } \alpha > 0.$$

By (3.11), we have

$$\begin{aligned}
\int_{\Omega} \left(|u_s(t)|^{r-1} \ln |u_s(t)|\right)^{\frac{2r}{2r-1}} dx &= \int_{\Omega_1} \left(|u_s(t)|^{r-1} \ln |u_s(t)|\right)^{\frac{2r}{2r-1}} dx \\
&\quad + \int_{\Omega_2} \left(|u_s(t)|^{r-1} \ln |u_s(t)|\right)^{\frac{2r}{2r-1}} dx \\
&\le [e(r-1)]^{-\frac{2r}{2r-1}} |\Omega| + 2^{\frac{2r}{2r-1}} \int_{\Omega_2} |u_s(t)|^{\frac{2r\left(r-1+\frac{1}{2}\right)}{2r-1}} dx \\
&= [e(r-1)]^{-\frac{2r}{2r-1}} |\Omega| + 2^{\frac{2r}{2r-1}} \int_{\Omega_2} |u_s(t)|^{r} dx \\
&\le C_d := [e(r-1)]^{-\frac{2r}{2r-1}} |\Omega| + 2^{\frac{2r}{2r-1}} r^2 d,
\end{aligned} \tag{3.17}$$

where

$$\Omega_1 = \{x \in \Omega : |u_s(t)| \le 1\}, \text{ and } \Omega_2 = \{x \in \Omega : |u_s(t)| \ge 1\}.$$

Hence, it follows from (3.16) and (3.17) that

$$|u_s|^{r-2} u_s \ln |u_s| \to |u|^{r-2} u \ln |u| \quad \text{weakly* in } L^\infty\left(0,+\infty; L^{\frac{2r}{2r-1}}(\Omega)\right).$$

Then integrating (3.3) respect to t for $0 \le t < \infty$, we obtain

$$(u_t, w_i) + \left(P^{\frac{1}{2}} u, P^{\frac{1}{2}} w_i\right) = \left(|u|^{r-2} u \ln |u|, w_i\right).$$

On the other hand, there exists a global weak solution $u_0 \in \mathcal{W}_1^+$ of problem (1.1).

Now we address the case of the initial data $u_0 \in \mathcal{W}_2^+$.

First we can choose a sequence $\{\omega_s\}_{s=1}^{\infty} \subset (0,1)$ and $\lim_{s\to\infty} \omega_s = 1$. Next, we investigate the following problem:

$$\begin{cases} u_t + Pu = u^{r-2} u \ln |u|, & x \in \Omega, \quad t > 0, \\ D^\gamma u(x,t) = 0, \quad |\gamma| \le m-1, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_{0s}(x), & x \in \Omega, \end{cases} \tag{3.18}$$

where $u_{0s} = \omega_s u_0$. By $I(u_0) > 0$ and Lemma 3, it is clear that there exists a $\lambda^* > 1$. Also, $J(u_{0s}) = J(\omega_s u_0) < J(u_0) = d$ and $I(u_{0s}) = I(\omega_s u_0) > 0$ hold. So, we have $u_0 \in \mathcal{W}_2^+$. Similarly to the previous situation, it is clear that problem (3.18) implies that, for all $s > 0$, there exists a global u_s which implies $u_s \in L^\infty(0,\infty; H_0^m(\Omega))$,

$u_{st} \in L^2(0,\infty;L^2(\Omega))$, $u_s(0) = u_{0s} = \omega_s u_0 \to u_0$ strongly in $H_0^m(\Omega)$, and the following equality

$$\int_\Omega u_{st} w dx + \int_\Omega P u_s w dx = \int_\Omega |u_s|^{r-2} u_s \ln|u_s| w dx, \tag{3.19}$$

with any $w \in H_0^m(\Omega)$ holds for a.e. $0 \le t < \infty$. Also, we get

$$u_s(t) \in \mathcal{W}_2^+, \;\; t \in [0,\infty),$$

and

$$J(u_s(t)) + \int_0^t \|u_{s\tau}(\tau)\|^2 d\tau \le J(u_{0s}) < d.$$

On the other hand, we can deduce (3.12), (3.13) and (3.17) for each s. Also, there exist u and a subsequence still denoted by $\{u_s\}$, such that, as $s \longrightarrow \infty$,

$$u_s \to u \;\text{ weakly* in } L^\infty(0,\infty;H_0^m(\Omega)),$$

$$u_{st} \to u_t \;\text{ weakly in } L^2(0,\infty;L^2(\Omega)),$$

$$|u_s|^{r-2} u_s \ln|u_s| \to |u|^{r-2} u \ln|u| \;\text{ weakly* in } L^\infty(0,+\infty;L^{\frac{2r}{2r-1}}(\Omega)).$$

Then integrating (3.19) respect to t for $0 \le t < \infty$, we obtain

$$(u_t, w) + \left(P^{\frac{1}{2}} u, P^{\frac{1}{2}} w\right) = \left(|u|^{r-2} u \ln|u|, w\right).$$

Therefore, there exists a global weak solution $u_0 \in \mathcal{W}_2^+$ of problem (1.1).

Decay estimates

Thanks to $u_0 \in \mathcal{W}_1^+$, we deduce from (2.3) that

$$\begin{aligned} J(u_0) &> J(u(t)) \\ &= \frac{1}{r} I(u(t)) + \left(\frac{1}{2} - \frac{1}{r}\right) \left\|P^{\frac{1}{2}} u(t)\right\|^2 + \frac{1}{r^2} \|u(t)\|_r^r \\ &\ge \left(\frac{1}{2} - \frac{1}{r}\right) \left\|P^{\frac{1}{2}} u(t)\right\|^2 + \frac{1}{r^2} \|u(t)\|_r^r. \end{aligned} \tag{3.20}$$

From Lemma 2, (2.4) and $I(u(t)) > 0$, there exists a $\lambda^* > 1$ such that $I(\lambda^* u(t)) = 0$. We get

$$\begin{aligned} d &\le J(\lambda^* u(t)) \\ &= (\lambda^*)^r \left((\lambda^*)^{2-r} \left(\frac{1}{2} - \frac{1}{r}\right) \left\|P^{\frac{1}{2}} u(t)\right\|^2 + \frac{1}{r^2} \|u(t)\|_r^r \right) \\ &\le (\lambda^*)^r \left(\left(\frac{1}{2} - \frac{1}{r}\right) \left\|P^{\frac{1}{2}} u(t)\right\|^2 + \frac{1}{r^2} \|u(t)\|_r^r \right). \end{aligned} \tag{3.21}$$

Using (3.20) and (3.21), we get

$$d \le (\lambda^*)^r J(u_0),$$

which yields that

$$\lambda^* \geq \left(\frac{d}{J(u_0)}\right)^{\frac{1}{r}}. \tag{3.22}$$

By (2.2), we get

$$0 = I(\lambda^* u(t)) = (\lambda^*)^r I(u(t)) + \left[(\lambda^*)^2 - (\lambda^*)^r\right] \left\|P^{\frac{1}{2}}u(t)\right\|^2 - (\lambda^*)^r \ln(\lambda^*) \left\|u(t)\right\|_r^r. \tag{3.23}$$

From (3.22), (3.23) and Lemma 1, we obtain

$$\begin{aligned} I(u(t)) &\geq \left[1 - (\lambda^*)^{2-r}\right] \left\|P^{\frac{1}{2}}u(t)\right\|^2 \\ &\geq \left[1 - \left(\frac{d}{J(u_0)}\right)^{\frac{2-r}{r}}\right] \left\|P^{\frac{1}{2}}u(t)\right\|^2 \\ &\geq C_1 \left\|u(t)\right\|^2, \end{aligned} \tag{3.24}$$

where C_1 is constant. Integrating the $I(u(\tau))$ with respect to τ over (t,T), we obtain

$$\int_t^T I(u(\tau))d\tau = -\int_t^T \int_\Omega u_\tau(\tau)u(\tau)dxd\tau \leq \frac{C_2}{2}\left\|u(t)\right\|^2. \tag{3.25}$$

where C_2 is constant. From (3.24) and (3.25), we have

$$\int_t^T C_1 \left\|u(t)\right\|^2 ds \leq \frac{C_2}{2}\left\|u(t)\right\|^2, \text{ for all } t \in [0,T]. \tag{3.26}$$

Let $T \to +\infty$ in (3.26), we can have

$$\int_t^\infty \left\|u(t)\right\|^2 ds \leq C_3 \left\|u(t)\right\|^2,$$

where $C_3 = \frac{C_2}{2C_1}$. By Lemma 4, we have

$$\left\|u(t)\right\|^2 \leq \left\|u(0)\right\|^2 e^{1-\frac{t}{C_3}}, \quad t \in [0,\infty).$$

□

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