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SOME INEQUALITIES OF ANTI-INVARIANT RIEMANNIAN SUBMERSIONS IN COMPLEX SPACE FORMS

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Abstract. The aim of the present paper is to analyze sharp type inequalities including the scalar and Ricci curvatures of anti-invariant Riemannian submersions in complex space forms.

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1. Introduction

B.-Y. Chen revealed the intrinsic and extrinsic invariants who established an inequality including Ricci curvature and squared mean curvature of a submanifold in a real space form $R^n(c)$ in 1999 (see [4]). In 2005 by B.-Y. Chen, a generalization of this inequality was proved for arbitrary submanifolds in an arbitrary Riemannian manifold (see [5]). Subsequently, this inequality has been comprehensively examined for different ambient spaces by some authors who are achieved some results (see [3, 7, 16, 19, 22, 25]).

A C^∞-submersion φ can be defined according to the following conditions: a (pseudo)-Riemannian submersion [1, 8, 12, 17, 20, 21], an almost Hermitian submersion [23], a quaternionic submersion [13] , a slant submersion [11], a Clairaut Submersion [10], an anti-invariant submersion [6], conformal anti-invariant submersion [2], a semi-invariant submersion [18], etc. As far as we know, Riemannian submersions were presented by B. O'Neill [17] and A. Gray [8] in 1960s, independently. Especially, by utilizing the notion of almost Hermitian submersions, B. Watson [23] presented some differential geometric features among fibers, base manifolds, and total manifolds. Subsequently, many results occur on this topic.

The main goal of the current paper is to study sharp type inequalities including the scalar and Ricci curvatures of anti-invariant Riemannian submersions in complex space forms. The structure of the paper is as follows: After recalling some basic definitions and formulas in the second part, we investigate several inequalities including the Ricci and the scalar curvatures on $\ker\varphi_*$ and $(\ker\varphi_*)^\perp$ distributions of

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anti-invariant Riemannian submersions in complex space forms and then, we obtain Chen-Ricci inequalities on $\ker\varphi_*$ and $(\ker\varphi_*)^\perp$ of anti-invariant Riemannian submersions in complex space forms.

2. PRELIMINARIES

Let (B_1, g_1) be an almost Hermitian manifold. This implies [24] that B_1 admits a tensor field J of type $(1,1)$ on B_1 such that $\forall\, Z_1, Z_2 \in \chi(B_1)$, we obtain

$$J^2 = -I, \quad g_1(JZ_1, Z_2) + g_1(Z_1, JZ_2) = 0. \tag{2.1}$$

An almost Hermitian manifold B_1 is called Kaehler manifold if

$$(\nabla^1_{Z_1})Z_2 = 0, \quad \forall Z_1, Z_2 \in \chi(B_1),$$

here ∇^1 is the Levi-Civita connection on B_1. If $\{Z_1, JZ_1\}$ spans a plane section, the sectional curvature $F_{B_1}(Z_1) = K_{B_1}(Z_1 \wedge JZ_1)$ of span$\{Z_1, JZ_1\}$ is called a sectional curvature. The Riemannian-Christoffel curvature tensor of a Kaehler manifold [24] $B_1(\nu)$ of constant holomorphic sectional curvature ν satisfies

$$\begin{aligned} R_{B_1}(Z_1, Z_2, Z_3, Z_4) = \frac{\nu}{4}\{& g_1(Z_1, Z_4)g_1(Z_2, Z_3) - g_1(Z_1, Z_3)g_1(Z_2, Z_4) \\ &+ g_1(JZ_2, Z_3)g_1(JZ_1, Z_4) - g_1(JZ_1, Z_3)g_1(JZ_2, Z_4) \\ &+ 2g_1(Z_1, JZ_2)g_1(JZ_3, Z_4)\} \end{aligned} \tag{2.2}$$

for all $Z_1, Z_2, Z_3, Z_4 \in \chi(B_1)$.

Let (B_1, g_1) and (B_2, g_2) be Riemannian manifolds. A Riemannian submersion is a smooth map $\varphi\colon B_1 \to B_2$ which is onto and satisfies the following conditions:

(i) $\varphi_{*p}\colon T_pB_1 \to T_{\varphi(p)}B_2$ is onto for all $p \in B_1$;
(ii) the fibres $\varphi_x^{-1}, x \in B_2$, are Riemannian submanifolds of B_1;
(iii) φ_{*p} preserves the length of the horizontal vectors.

The vectors tangent to the fibres are called vertical and those normal to the fibres are called horizontal. The tangent bundle of B_1 splits as the Whitney sum of two distributions, the vertical one $\ker\varphi_*$ and the orthogonal complementary distribution $(\ker\varphi_*)^\perp$ called horizontal, and we denote by h and v the horizontal and vertical projections, respectively. A horizontal vector field Z_1 on B_1 is called as basic if Z_1 is φ-related to a vector field Z_{1*} on B_2 [17]. A Riemannian submersion $\varphi\colon B_1 \to B_2$ specifies two $(1,2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ on B_1, by the formulae [17]:

$$\mathcal{T}(Z_1, Z_2) = \mathcal{T}_{Z_1}Z_2 = h\nabla^1_{vZ_1}vZ_2 + v\nabla^1_{vZ_1}hZ_2$$

and

$$\mathcal{A}(Z_1, Z_2) = \mathcal{A}_{Z_1}Z_2 = v\nabla^1_{hZ_1}hZ_2 + h\nabla^1_{hZ_1}vZ_2$$

for all $Z_1, Z_2 \in \chi(B_1)$.

Lemma 1 (Lemma 4 in [17]). *Let* $\varphi\colon (B_1,g_2)\to(B_2,g_2)$ *be a Riemannian submersion. Then we have:*

$$\mathcal{A}_{Z_1}Z_2 = -\mathcal{A}_{Z_2}Z_1, \quad Z_1,Z_2\in\chi((\ker\varphi_*)^{\perp}); \tag{2.3}$$

$$\mathcal{T}_{F_1}F_2 = \mathcal{T}_{F_2}F_1, \quad F_1,F_2\in\chi(\ker\varphi_*); \tag{2.4}$$

$$g_1(\mathcal{T}_{F_1}Z_2,Z_3) = -g_1(\mathcal{T}_{F_1}Z_3,Z_2), \quad F_1\in\chi(\ker\varphi_*), \quad Z_2,Z_3\in\chi(B_1);$$

$$g_1(\mathcal{A}_{Z_1}Z_2,Z_3) = -g_1(\mathcal{A}_{Z_1}Z_3,Z_2), \quad Z_1\in\chi((\ker\varphi_*)^{\perp}), \quad Z_2,Z_3\in\chi(B_1).$$

Let $R^{B_1}, R^{B_2}, R^{\ker\varphi_*}$ and $R^{(\ker\varphi_*)^{\perp}}$ stand for the Riemannian curvature tensors of Riemannian manifolds B_1,B_2, the vertical distribution $\ker\varphi_*$ and the horizontal distribution $(\ker\varphi_*)^{\perp}$, respectively.

Lemma 2 (Theorem 2 in [17]). *Let* $\varphi\colon (B_1,g_2)\to(B_2,g_2)$ *be a Riemannian submersion. Then we have:*

$$\begin{aligned} R^{B_1}(F_1,F_2,F_3,F_4) &= R^{\ker\varphi_*}(F_1,F_2,F_3,F_4) + g_1(\mathcal{T}_{F_1}F_4,\mathcal{T}_{F_2}F_3) \\ &\quad - g_1(\mathcal{T}_{F_2}F_4,\mathcal{T}_{F_1}F_3), \end{aligned} \tag{2.5}$$

$$\begin{aligned} R^{B_1}(Z_1,Z_2,Z_3,Z_4) &= R^{(\ker\varphi_*)^{\perp}}(Z_1,Z_2,Z_3,Z_4) - 2g_1(\mathcal{A}_{Z_1}Z_2,\mathcal{A}_{Z_3}Z_4) \\ &\quad + g_1(\mathcal{A}_{Z_2}Z_3,\mathcal{A}_{Z_1}Z_4) - g_1(\mathcal{A}_{Z_1}Z_3,\mathcal{A}_{Z_2}Z_4), \end{aligned} \tag{2.6}$$

$$\begin{aligned} R^{B_1}(Z_1,F_1,Z_2,F_2) &= g_1((\nabla^1_{Z_1}\mathcal{T})(F_1,F_2),Z_2) + g_1((\nabla^1_{F_1}\mathcal{A})(Z_1,Z_2),F_2) \\ &\quad - g_1(\mathcal{T}_{F_1}Z_1,\mathcal{T}_{F_2}Z_2) + g_1(\mathcal{A}_{Z_2}F_2,\mathcal{A}_{Z_1}F_1) \end{aligned} \tag{2.7}$$

for all $Z_1,Z_2,Z_3,Z_4\in\chi((\ker\varphi_*)^{\perp})$ *and* $F_1,F_2,F_3,F_4\in\chi(\ker\varphi_*)$.

Further, the $\mathcal{H}$ mean curvature of every fibre of φ Riemannian submersion is defined

$$\mathcal{H} = \frac{1}{s}\mathcal{N}, \quad \mathcal{N} = \sum_{p=1}^{s}\mathcal{T}_{E_p}E_p, \tag{2.8}$$

where $\{E_1,E_2,\dots,E_s\}$ forms an orthonormal basis for the vertical distribution $\ker\varphi_*$. Also, φ has totally geodesic fibres if $\mathcal{T}=0$ on $\ker\varphi_*$ and $(\ker\varphi_*)^{\perp}$.

Definition 1 (Definition 3.1 in [6]). Let (B_1,g_1,J) be a Kaehler manifold and (B_2,g_2) be a Riemannian manifold. $\varphi\colon(B_1,g_1,J)\to(B_2,g_2)$ is called anti-invariant, if $\ker\varphi_*$ is anti-invariant with respect to J, i.e. $J(\ker\varphi_*)\subseteq(\ker\varphi_*)^{\perp}$.

From above definition, we get $J(\ker\varphi_*)\cap(\ker\varphi_*)^{\perp}\neq\{0\}$. We denote the complementary orthogonal distribution to $J(\ker\varphi_*)$ in $(\ker\varphi_*)^{\perp}$ by η. Then we obtain

$$(\ker\varphi_*)^{\perp} = J(\ker\varphi_*)\oplus\eta.$$

It is straightforward to show that η is an invariant distribution of $(\ker\varphi_*)^{\perp}$ under the endomorphism J. So, for $Z_1\in\chi(\ker\varphi_*)^{\perp}$, we can state

$$JZ_1 = \alpha Z_1 + \beta Z_1, \tag{2.9}$$

here $\alpha Z_1 \in \chi(\ker \varphi_*)$ and $\beta Z_1 \in \chi(\eta)$. Using (2.1) and (2.9), we have

$$\beta^2 Z_1 = -Z_1 - J\alpha Z_1. \tag{2.10}$$

Example 1. Let B_1 be a 4-dimensional Euclidean space given by $B_1 = \{(x,y,z,w) \in \mathcal{R}^4 : z \in \mathcal{R} - \{k\frac{\pi}{2}, k\pi\}, k \in \mathcal{Z} \text{ and } x \neq 0\}$. We define the Kaehler structure (J, g_1) on B_1 given by

$$g_1 = (dx)^2 + (dy)^2) + (dz)^2 + (dw)^2 \quad \text{and} \quad J(b_1,b_2,b_3,b_4) = (-b_4, b_3, -b_2, b_1).$$

Let B_2 be $\{(x,v) \in \mathcal{R}^2 : x \neq 0\}$. We choose the Riemannian metric g_2 on B_2 in the following form

$$g_2 = e^{-2x}((dx)^2 + (dv)^2).$$

Now we define the map $\varphi\colon (B_1, g_1, J) \to (B_2, g_2)$ by

$$\varphi(x,y,z,w) = (e^x \cos z, e^x \sin z).$$

Then the kernel of φ_* is

$$\ker \varphi_* = \operatorname{Span}\{F_1 = -e^x \cos z \frac{\partial}{\partial y} - e^x \sin z \frac{\partial}{\partial w}, F_2 = e^x \sin z \frac{\partial}{\partial y} - e^x \cos z \frac{\partial}{\partial w}\},$$

and the horizontal distribution is spanned by

$$(\ker \varphi_*)^\perp = \operatorname{Span}\{Z_1 = e^x \cos z \frac{\partial}{\partial x} - e^x \sin z \frac{\partial}{\partial z}, Z_2 = e^x \sin z \frac{\partial}{\partial x} + e^x \cos z \frac{\partial}{\partial z}\}.$$

Thus, φ is a Riemannnian submersion. Moreover, $JF_1 = Z_2$ and $JF_2 = Z_1$ imply that $(\ker \varphi_*)^\perp = J(\ker \varphi_*)$. Hence φ ia an anti-invariant Riemannnian submersion.

3. Basic Inequalities

First we give the following result. Since φ is an anti-invariant Riemannian submersion, and using (2.2) and (2.5) we have:

Lemma 3. *$(B_1(\upsilon), g_1)$ and (B_2, g_2) denote a complex space form and a Riemannian manifold and let $\varphi\colon (B_1(\upsilon), g_1) \to (B_2, g_2)$ be an anti-invariant Riemannian submersion. Then any for $F_1, F_2, F_3, F_4 \in \chi(\ker \varphi_*)$ we obtain*

$$\begin{aligned} R^{\ker \varphi_*}(F_1,F_2,F_3,F_4) &= \frac{\nu}{4}\{g_1(F_1,F_4)g_1(F_2,F_3) - g_1(F_1,F_3)g_1(F_2,F_4)\} \\ &\quad - g_1(\mathcal{T}_{F_1}F_4, \mathcal{T}_{F_2}F_3) + g_1(\mathcal{T}_{F_2}F_4, \mathcal{T}_{F_1}F_3), \\ K^{\ker \varphi_*}(F_1,F_2) &= \frac{\nu}{4}\{g_1^2(F_1,F_2) - \|F_1\|^2\|F_2\|^2\} - \|\mathcal{T}_{F_1}F_2\|^2 \\ &\quad + g_1(\mathcal{T}_{F_2}F_2, \mathcal{T}_{F_1}F_1), \end{aligned} \tag{3.1}$$

here $K^{\ker \varphi_*}$ is a bi-sectional curvature of $\ker \varphi_*$.

Let $\varphi\colon B_1(\nu) \to B_2$ be an anti-invariant Riemannian submersion. For every node $k \in B_1$, let $\{E_1,\dots,E_s,e_1,\dots,e_m\}$ be an orthonormal basis of $T_kB_1(\nu)$ such that $\ker\varphi_* = \operatorname{span}\{E_1,\dots,E_s\}$, $(\ker\varphi_*)^\perp = \operatorname{span}\{e_1,\dots,e_m\}$.

Now, if we take $F_4 = F_1$ and $F_2 = F_3 = E_i$, $i = 1,2,\dots,s$ in (3.1), and using (2.8) then we arrive at

$$Ric^{\ker\varphi_*}(F_1) = \frac{\nu}{4}(s-1)g_1(F_1,F_1) - sg_1(\mathcal{T}_{F_1}F_1,\mathcal{H}) + \sum_{i=1}^{s} g_1(\mathcal{T}_{F_i}F_1,\mathcal{T}_{F_1}F_i). \tag{3.2}$$

From here, we get:

Theorem 1. *Let $\varphi\colon (B_1(\nu),g_1) \to (B_2,g_2)$ be an anti-invariant Riemannian submersion. Then we have*

$$Ric^{\ker\varphi_*}(F_1) \geq \frac{\nu}{4}(s-1)g_1(F_1,F_1) - sg_1(\mathcal{T}_{F_1}F_1,\mathcal{H}).$$

For a unit vertical vector $F_1 \in \chi(\ker\varphi_*)$, the equality status of the inequality holds if and only if every fibre is totally geodesic.

Taking $F_1 = E_j$, $j = 1,\dots,s$ in (3.2) and using (2.4), then we obtain

$$2\rho^{\ker\varphi_*} = \frac{\nu}{4}s(s-1) - s^2\|\mathcal{H}\|^2 + \sum_{i,j=1}^{s} g_1(\mathcal{T}_{E_i}E_j,\mathcal{T}_{E_i}E_j).$$

Therefore, we can state the following result.

Theorem 2. *Let $\varphi\colon (B_1(\nu),g_1) \to (B_2,g_2)$ be an anti-invariant Riemannian submersion. Then we have*

$$2\rho^{\ker\varphi_*} \geq \frac{\nu}{4}s(s-1) - s^2\|\mathcal{H}\|^2.$$

The equality status of the inequality satisfies if and only if every fibre is totally geodesic.

Since φ is an anti-invariant submersion, and using (2.2), (2.6), (2.9) we obtain:

Lemma 4. *Let $\varphi\colon (B_1(\nu),g_1) \to (B_2,g_2)$ be an anti-invariant Riemannian submersion. Then for $Z_1,Z_2,Z_3,Z_4 \in \chi((\ker\varphi_*)^\perp)$ we have*

$$\begin{aligned} R^{(\ker\varphi_*)^\perp}(Z_1,Z_2,Z_3,Z_4) = \frac{\nu}{4}\{&g_1(Z_1,Z_4)g_1(Z_2,Z_3) - g_1(Z_1,Z_3)g_1(Z_2,Z_4) \\ &+ g_1(\beta Z_2,Z_3)g_1(\beta Z_1,Z_4) - g_1(\beta Z_1,Z_3)g_1(\beta Z_2,Z_4) \\ &+ 2g_1(Z_1,\beta Z_2)g_1(\beta Z_3,Z_4)\} + 2g_1(\mathcal{A}_{Z_1}Z_2,\mathcal{A}_{Z_3}Z_4) \\ &- g_1(\mathcal{A}_{Z_2}Z_3,\mathcal{A}_{Z_1}Z_4) + g_1(\mathcal{A}_{Z_1}Z_3,\mathcal{A}_{Z_2}Z_4), \end{aligned} \tag{3.3}$$

$$\begin{aligned} B^{(\ker\varphi_*)^\perp}(Z_1,Z_2) = \frac{\nu}{4}\{&g_1^2(Z_1,Z_2) - \|Z_1\|^2\|Z_2\|^2 \\ &- 3g_1^2(\beta Z_1,Z_2)\} + 3\|\mathcal{A}_{Z_1}Z_2\|^2, \end{aligned}$$

here $B^{(\ker\varphi_*)^\perp}$ is a bi-sectional curvature of $(\ker\varphi_*)^\perp$.

Now, if we take $Z_4 = Z_1$ and $Z_2 = Z_3 = e_j$, $j = 1,2,\dots,m$ in (3.3), and using (2.3), (2.10) then we get

$$\begin{aligned} Ric^{(\ker\varphi_*)^\perp}(Z_1) = \frac{\nu}{4}\{(m+2)g_1(Z_1,Z_1) + 3g_1(J\alpha Z_1, Z_1)\} \\ -3\sum_{j=1}^{m} g_1(\mathcal{A}_{Z_1}e_j, \mathcal{A}_{Z_1}e_j). \end{aligned} \tag{3.4}$$

Taking $Z_1 = e_i$, $i = 1,2,\dots,m$ in (3.4), then we have:

$$2\rho^{(\ker\varphi_*)^\perp} = \frac{\nu}{4}\{m(m+2) + 3tr(J\alpha)\} - 3\sum_{i,j=1}^{m} g_1(\mathcal{A}_{e_i}e_j, \mathcal{A}_{e_i}e_j). \tag{3.5}$$

Then we write

$$2\rho^{(\ker\varphi_*)^\perp} \le \frac{\nu}{4}\{m(m+2) + 3tr(J\alpha)\}. \tag{3.6}$$

Thus, we can give:

Theorem 3. *Let $\varphi\colon (B_1(\nu), g_1) \to (B_2, g_2)$ be an anti-invariant Riemannian submersion. Then*

$$2\rho^{(\ker\varphi_*)^\perp} \le \frac{\nu}{4}\{m(m+2) + 3tr(J\alpha)\}.$$

The equality status of (3.6) satisfies if and only if $(\ker\varphi_*)^\perp$ is integrable.

4. Chen-Ricci inequalities

Let $(B_1(\nu), g_1)$ be a complex space form, (B_2, g_2) a Riemannian manifold and $\varphi\colon B_1(\nu) \to B_2$ be an anti-invariant Riemannian submersion. For every node $k \in B_1$, let $\{E_1,\dots,E_s,e_1,\dots,e_m\}$ be an orthonormal basis of $T_kB_1(\nu)$ such that $\ker\varphi_* = \operatorname{span}\{E_1,\dots,E_s\}$ and $(\ker\varphi_*)^\perp = \operatorname{span}\{e_1,\dots,e_m\}$. Let's denote $\mathcal{T}_{ij}^t$ by

$$\mathcal{T}_{ij}^t = g_1(\mathcal{T}_{E_i}E_j, e_t), \tag{4.1}$$

where $1 \le i,j \le s$ and $1 \le t \le m$. Similarly, let's denote $\mathcal{A}_{ij}^\alpha$ by

$$\mathcal{A}_{ij}^\alpha = g_1(\mathcal{A}_{ei}e_j, E_\alpha), \tag{4.2}$$

in which $1 \le i,j \le m$ and $1 \le \alpha \le s$ and we employee

$$\delta(\mathcal{N}) = \sum_{i=1}^{m}\sum_{k=1}^{s}((\nabla^1_{e_i}\mathcal{T})_{E_k}E_k, e_i). \tag{4.3}$$

Now, from (3.1), we get

$$2\rho^{\ker\varphi_*} = \frac{\nu}{4}s(s-1) - s^2\|\mathcal{H}\|^2 + \sum_{i,j=1}^{s} g_1(\mathcal{T}_{E_i}E_j, \mathcal{T}_{E_i}E_j).$$

Using (2.4) and (4.1), we arrive at

$$2\rho^{\ker\varphi_*} = \frac{\nu}{4}s(s-1) - s^2\|\mathcal{H}\|^2 + \sum_{t=1}^{m}\sum_{i,j=1}^{s}(\mathcal{T}_{ij}^t)^2. \tag{4.4}$$

From [9] we know that

$$\sum_{t=1}^{m}\sum_{i,j=1}^{s}(\mathcal{T}_{ij}^{t})^{2}=\frac{1}{2}s^{2}\|\mathcal{H}\|^{2}+\frac{1}{2}\sum_{t=1}^{m}\left[\mathcal{T}_{11}^{t}-\mathcal{T}_{22}^{t}-\cdots-\mathcal{T}_{ss}^{t}\right]^{2} \\ +2\sum_{t=1}^{m}\sum_{j=2}^{s}(\mathcal{T}_{1j}^{t})^{2}-2\sum_{t=1}^{m}\sum_{2\le i<j\le s}\left[\mathcal{T}_{ii}^{t}\mathcal{T}_{jj}^{t}-\left(\mathcal{T}_{ij}^{t}\right)^{2}\right]. \tag{4.5}$$

If we put (4.5) in (4.4), we obtain

$$2\rho^{\ker\varphi_*}=\frac{\nu}{4}s(s-1)-\frac{1}{2}s^{2}\|\mathcal{H}\|^{2}+\frac{1}{2}\sum_{t=1}^{m}\left[\mathcal{T}_{11}^{t}-\mathcal{T}_{22}^{t}-\cdots-\mathcal{T}_{ss}^{t}\right]^{2} \\ +2\sum_{t=1}^{m}\sum_{j=2}^{s}(\mathcal{T}_{1j}^{t})^{2}-2\sum_{t=1}^{m}\sum_{2\le i<j\le s}\left[\mathcal{T}_{ii}^{t}\mathcal{T}_{jj}^{t}-\left(\mathcal{T}_{ij}^{t}\right)^{2}\right].$$

From here, we have

$$2\rho^{\ker\varphi_*}\ge\frac{\nu}{4}s(s-1)-\frac{1}{2}s^{2}\|\mathcal{H}\|^{2}-2\sum_{t=1}^{m}\sum_{2\le i<j\le s}\left[\mathcal{T}_{ii}^{t}\mathcal{T}_{jj}^{t}-\left(\mathcal{T}_{ij}^{t}\right)^{2}\right]. \tag{4.6}$$

On the other hand, from (2.5), taking $F_1=F_4=E_i, F_2=F_3=E_j$ and using (4.1), we have

$$2\sum_{2\le i<j\le s}R^{B_1}(E_i,E_j,E_j,E_i)=2\sum_{2\le i<j\le s}R^{\ker\varphi_*}(E_i,E_j,E_j,E_i) \\ +2\sum_{t=1}^{m}\sum_{2\le i<j\le s}\left[\mathcal{T}_{ii}^{t}\mathcal{T}_{jj}^{t}-\left(\mathcal{T}_{ij}^{t}\right)^{2}\right].$$

From the last equality, (4.6) can be written as

$$2\rho^{\ker\varphi_*}\ge\frac{\nu}{4}s(s-1)-\frac{1}{2}s^{2}\|\mathcal{H}\|^{2}+2\sum_{2\le i<j\le s}R^{\ker\varphi_*}(E_i,E_j,E_j,E_i) \\ -2\sum_{2\le i<j\le s}R^{B_1}(E_i,E_j,E_j,E_i). \tag{4.7}$$

Furthermore, we know that

$$2\rho^{\ker\varphi_*}=2\sum_{2\le i<j\le s}R^{\ker\varphi_*}(E_i,E_j,E_j,E_i)+2\sum_{j=1}^{s}R^{\ker\varphi_*}(E_1,E_j,E_j,E_1).$$

If we put the last equality in (4.7), then we have

$$2Ric^{\ker\varphi_*}(E_1)\ge\frac{\nu}{4}s(s-1)-\frac{1}{2}s^{2}\|\mathcal{H}\|^{2}-2\sum_{2\le i<j\le s}R^{B_1}(E_i,E_j,E_j,E_i).$$

Since B_1 is a complex space form, curvature tensor R^{B_1} of B_1 provides equation (2.2), therefore we acquire

$$Ric^{\ker\varphi_*}(E_1)\ge\frac{\nu}{4}(s-1)-\frac{1}{4}s^{2}\|\mathcal{H}\|^{2}.$$

Thus, we can give the following result:

Theorem 4. *Let $\varphi\colon B_1(\nu) \to B_2$ be an anti-invariant Riemannian submersion from a complex space form $(B_1(\nu), g_1)$ onto a Riemannian manifold (B_2, g_2). Then we have*

$$Ric^{\ker\varphi_*}(E_1) \geq \frac{\nu}{4}(s-1) - \frac{1}{4}s^2\|\mathcal{H}\|^2.$$

The equality status of the inequality satisfies if and only

$$\mathcal{T}^t_{11} = \mathcal{T}^t_{22} + \cdots + \mathcal{T}^t_{ss}$$
$$\mathcal{T}^t_{1j} = 0, j = 2, \ldots, s.$$

From (3.5), we have

$$2\rho^{(\ker\varphi_*)^\perp} = \frac{\nu}{4}\{m(m+2) + 3tr(J\alpha)\} - 3\sum_{i,j=1}^{m} g_1(\mathcal{A}_{e_i}e_j, \mathcal{A}_{e_i}e_j).$$

Using (2.10) and (4.2), then we have

$$2\rho^{(\ker\varphi_*)^\perp} = \frac{\nu}{4}\{m(m+2) + 3tr(J\alpha)\} - 3\sum_{\alpha=1}^{s}\sum_{i,j=1}^{m}(\mathcal{A}^\alpha_{ij})^2. \tag{4.8}$$

From (2.3) then (4.8) turns into

$$2\rho^{(\ker\varphi_*)^\perp} = \frac{\nu}{4}\{m(m+2) + 3tr(J\alpha)\} - 6\sum_{\alpha=1}^{s}\sum_{j=2}^{m}(\mathcal{A}^\alpha_{1j})^2 - 6\sum_{\alpha=1}^{s}\sum_{2\leq i<j\leq m}(\mathcal{A}^\alpha_{ij})^2. \tag{4.9}$$

Moreover, from (2.6), taking $Z_1 = Z_4 = e_i, Z_2 = Z_3 = e_j$ and using (4.2) we obtain

$$\begin{aligned} 2\sum_{2\leq i<j\leq m} R^{B_1}(e_i, e_j, e_j, e_i) = 2\sum_{2\leq i<j\leq m} R^{(\ker\varphi_*)^\perp}(e_i, e_j, e_j, e_i) \\ + 6\sum_{\alpha=1}^{s}\sum_{2\leq i<j\leq m}(\mathcal{A}^\alpha_{ij})^2. \end{aligned} \tag{4.10}$$

If we consider (4.10) in (4.9), then we have

$$\begin{aligned} 2\rho^{(\ker\varphi_*)^\perp} = {} & \frac{\nu}{4}\{m(m+2) + 3tr(J\alpha)\} - 6\sum_{\alpha=1}^{s}\sum_{j=2}^{m}(\mathcal{A}^\alpha_{1j})^2 \\ & - 2\sum_{2\leq i<j\leq m} R^{B_1}(e_i, e_j, e_j, e_i) + 2\sum_{2\leq i<j\leq m} R^{(\ker\varphi_*)^\perp}(e_i, e_j, e_j, e_i). \end{aligned}$$

Since B_1 is a complex space form, curvature tensor R^{B_1} of B_1 satisfies (2.2), hence we get

$$2Ric^{(\ker\varphi_*)^\perp}(e_1) = \frac{\nu}{4}(2m - 2 + 6\|\beta e_1\|^2) - 6\sum_{\alpha=1}^{s}\sum_{j=2}^{m}(\mathcal{A}^\alpha_{1j})^2.$$

Then we can write

$$Ric^{(\ker\varphi_*)^\perp}(e_1) \leq \frac{\nu}{4}(m - 1 + 3\|\beta e_1\|^2).$$

Thus, we can give the following result:

Theorem 5. *Let $\varphi\colon B_1(\nu)\to B_2$ be an anti-invariant Riemannian submersion from a complex space form $(B_1(\nu),g_1)$ onto a Riemannian manifold (B_2,g_2). Then we have*

$$Ric^{(\ker\varphi_*)^\perp}(e_1)\le\frac{\nu}{4}(m-1+3\|\beta e_1\|^2),$$

the equality status of the inequality satisfies if and only

$$\mathcal{A}_{1j}=0, j=2,\dots,m.$$

Next, we can state the inequality of Chen Ricci among the $\ker\varphi_*$ and $(\ker\varphi_*)^\perp$. The ρ scalar curvature of $B_1(\nu)$ is defined as

$$\begin{aligned}2\rho&=\sum_{t=1}^{m}Ric(e_t,e_t)+\sum_{k=1}^{s}Ric(E_k,e_k),\\2\rho&=\sum_{j,k=1}^{s}R^{B_1}(E_j,E_k,E_k,E_j)+\sum_{i=1}^{m}\sum_{k=1}^{s}R^{B_1}(e_i,E_k,E_k,e_i)\\&\quad+\sum_{i,t=1}^{m}R^{B_1}(e_i,e_t,e_t,e_i)+\sum_{t=1}^{m}\sum_{j=1}^{s}R^{B_1}(E_j,e_t,e_t,E_j).\end{aligned}\tag{4.11}$$

Since $B_1(\nu)$ is a complex space form, using (4.11) and (2.2), we have

$$2\rho=\frac{\nu}{4}\{s(s-1)+m(m+2)+2sm+3tr(J\alpha)\}.\tag{4.12}$$

On the other hand, using the equations (2.5), (2.6) and (2.7), we obtain also the ρ scalar curvature of $B_1(\nu)$ as

$$\begin{aligned}2\rho&=2\rho^{\ker\varphi_*}+2\rho^{(\ker\varphi_*)^\perp}+s^2\|\mathcal{H}\|^2\\&\quad+\sum_{j,k=1}^{s}g_1(\mathcal{T}_{E_k}E_j,\mathcal{T}_{E_k}E_j)+3\sum_{i,t=1}^{m}g_1(\mathcal{A}_{e_i}e_t,\mathcal{A}_{e_i}e_t)\\&\quad-\sum_{i=1}^{m}\sum_{k=1}^{s}g_1((\nabla^1_{e_i}\mathcal{T})_{E_k}E_k,e_i)+\sum_{i=1}^{m}\sum_{k=1}^{s}\{g_1(\mathcal{T}_{E_k}e_i,\mathcal{T}_{E_k}e_i)-g_1(\mathcal{A}_{e_i}E_k,\mathcal{A}_{e_i}E_k)\}\\&\quad-\sum_{t=1}^{m}\sum_{j=1}^{s}g_1((\nabla^1_{e_t}\mathcal{T})_{E_j}E_j,e_t)+\sum_{t=1}^{m}\sum_{j=1}^{s}\left\{g_1(\mathcal{T}_{E_j}e_t,\mathcal{T}_{E_j}e_t)-g_1(\mathcal{A}_{e_t}E_j,\mathcal{A}_{e_t}E_j)\right\}.\end{aligned}$$

Using (4.3) and (4.5), we obtain

$$\begin{aligned}2\rho&=2\rho^{\ker\varphi_*}+2\rho^{(\ker\varphi_*)^\perp}+\frac{1}{2}s^2\|\mathcal{H}\|^2-\frac{1}{2}\sum_{t=1}^{m}\left[\mathcal{T}^t_{11}-\mathcal{T}^t_{22}-\dots-\mathcal{T}^t_{ss}\right]^2\\&\quad-2\sum_{t=1}^{m}\sum_{j=2}^{s}\left(\mathcal{T}^t_{1j}\right)^2+2\sum_{t=1}^{m}\sum_{2\le j<k\le s}\left[\mathcal{T}^t_{jj}\mathcal{T}^t_{kk}-\left(\mathcal{T}^t_{jk}\right)^2\right]+6\sum_{\alpha=1}^{s}\sum_{t=2}^{m}(\mathcal{A}^\alpha_{1t})^2\end{aligned}\tag{4.13}$$

$$+6\sum_{\alpha=1}^{s}\sum_{2\le i<t\le m}(\mathcal{A}_{it}^{\alpha})^2+\sum_{i=1}^{m}\sum_{k=1}^{s}\{g_1(\mathcal{T}_{E_k}e_i,\mathcal{T}_{E_k}e_i)-g_1(\mathcal{A}_{e_i}E_k,\mathcal{A}_{e_i}E_k)\}$$
$$-2\delta(\mathcal{N})+\sum_{t=1}^{m}\sum_{j=1}^{s}\left\{g_1(\mathcal{T}_{E_j}e_t,\mathcal{T}_{E_j}e_t)-g_1(\mathcal{A}_{e_t}E_j,\mathcal{A}_{e_t}E_j)\right\}.$$

Using (4.7), (4.10) and (4.12) in the (4.13) then we have

$$\frac{\nu}{4}\{sm+m+s-1+3\|\beta e_1\|^2\}=Ric^{\ker\varphi_*}(E_1)+Ric^{(\ker\varphi_*)^\perp}(e_1)+\frac{1}{4}s^2\|\mathcal{H}\|^2$$
$$-\frac{1}{4}\sum_{t=1}^{m}\left[\mathcal{T}_{11}^t-\mathcal{T}_{22}^t-\cdots-\mathcal{T}_{ss}^t\right]^2-\sum_{t=1}^{m}\sum_{j=2}^{s}\left(\mathcal{T}_{1j}^t\right)^2$$
$$+3\sum_{\alpha=1}^{s}\sum_{t=2}^{m}(\mathcal{A}_{1t}^{\alpha})^2-2\delta(\mathcal{N})+\|\mathcal{T}^V\|^2-\|\mathcal{A}^H\|^2,$$

where $\|\mathcal{T}^V\|^2=\sum_{i=1}^{m}\sum_{k=1}^{s}g_1(\mathcal{T}_{E_k}e_i,\mathcal{T}_{E_k}e_i)$, $\|\mathcal{A}^H\|^2=\sum_{i=1}^{m}\sum_{k=1}^{s}g_1(\mathcal{A}_{e_i}E_k,\mathcal{A}_{e_i}E_k)$. Since $B_1(\nu)$ is a complex space form, from (2.2), we have following result readily:

Theorem 6. *Let $\varphi\colon B_1(\nu)\to B_2$ be an anti-invariant Riemannian submersion from a complex space form $(B_1(\nu),g_1)$ onto a Riemannian manifold (B_2,g_2). Then we have*

$$\frac{\nu}{4}\{sm+m+s-1+3\|\beta e_1\|^2\}\le Ric^{\ker\varphi_*}(E_1)+Ric^{(\ker\varphi_*)^\perp}(e_1)+\frac{1}{4}s^2\|\mathcal{H}\|^2$$
$$+3\sum_{\alpha=1}^{s}\sum_{t=2}^{m}(\mathcal{A}_{1t}^{\alpha})^2-\delta(\mathcal{N})+\|\mathcal{T}^V\|^2-\|\mathcal{A}^H\|^2$$

the equality status of the inequality satisfies if and only

$$\mathcal{T}_{11}^t=\mathcal{T}_{22}^t+\cdots+\mathcal{T}_{ss}^t \qquad \mathcal{T}_{1j}^t=0,\quad j=2,\ldots,s.$$

Remark 1. Recently, Chen-Ricci inequalities were stated for Riemannian maps from complex space forms in [14]. Recall that Riemannian maps generalize the well-known concepts of isometric immersions and Riemannian submersions (see, e.g., the recent work of Lee et. al.,[15]). Therefore, a natural problem is to extend the results of this work in the general setting of anti-invariant Riemannian maps.

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