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A CLASSIFICATION OF 2-DESIGNS WITH PRIME REPLICATION NUMBERS AND FLAG-TRANSITIVE AUTOMORPHISM GROUPS

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Abstract. In this article, we present a classification of 2-designs with prime replication numbers admitting flag-transitive automorphism groups. In conclusion, if G is a flag-transitive automorphism group of a 2-design $\mathcal{D}$ with prime replication number, then $G \leqslant \mathrm{A\Gamma L}_1(q)$, or $\mathcal{D}$ belongs to five infinite families of 2-designs or it is (isomorphic to) one of eight sporadic examples.

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1. Introduction

A 2-(v,k,λ) design $\mathcal{D}$ is a pair $(\mathcal{P},\mathcal{B})$ with a set $\mathcal{P}$ of v points and a set $\mathcal{B}$ of blocks such that each block is a k-subset of $\mathcal{P}$ and each two distinct points are contained in λ blocks. We say $\mathcal{D}$ is nontrivial if $2 < k < v-1$, and symmetric if $v = b$, where b is the number of blocks of $\mathcal{D}$. Each point of $\mathcal{D}$ is contained in exactly $r = bk/v$ blocks which is called the *replication number* of $\mathcal{D}$. An *automorphism* of $\mathcal{D}$ is a permutation of $\mathcal{P}$ which leaves $\mathcal{B}$ invariant. The full automorphism group $\mathrm{Aut}(\mathcal{D})$ of $\mathcal{D}$ is the group consisting of all automorphisms of $\mathcal{D}$. A *flag* of $\mathcal{D}$ is a point-block pair (α, B) such that $\alpha \in B$. For $G \leqslant \mathrm{Aut}(\mathcal{D})$, G is called *flag-transitive* if G acts transitively on the set of flags. The group G is said to be *point-primitive* if G acts primitively on $\mathcal{P}$. Further notation and definitions in both design theory and group theory are standard and can be found, for example, in [7, 9].

We know by [2, Proposition 2.1] that a flag-transitive automorphism group of a 2-design with prime replication number is a point-primitive group of almost simple or affine type (see also [23]). In [2], such 2-designs admitting a flag-transitive almost simple automorphism group have been studied (see also [1, 18, 19, 21, 22]). In this paper, we study the case where G is of affine type, and present a classification of 2-designs with prime replication number and flag-transitive automorphism groups:

TABLE 1. Some nontrivial 2-design with prime replication number, almost simple type.

Line	v	b	r	k	λ	X	$G_\alpha\cap X$	G	Design	References
1	6	10	5	3	2	Alt_5	D_{10}	Alt_5	-	[6,20]
2	7	7	3	3	1	$\mathrm{PSL}_2(7)$	Sym_4	$\mathrm{PSL}_2(7)$	$\mathrm{PG}(2,2)$	[3,6,17]
3	8	14	7	4	3	$\mathrm{PSL}_2(7)$	7:3	$\mathrm{PSL}_2(7)$	-	[6,20]
4	11	11	5	5	2	$\mathrm{PSL}_2(11)$	Alt_5	$\mathrm{PSL}_2(11)$	Paley	[3,6,17]
5	12	22	11	6	5	M_{11}	$\mathrm{PSL}_2(11)$	M_{11}	-	[6,19]
6	15	15	7	7	3	Alt_7	$\mathrm{PSL}_2(7)$	Alt_7	$\mathrm{PG}_2(3,2)$	[3,6,22]
7	15	35	7	3	1	Alt_7	$\mathrm{PSL}_2(7)$	Alt_7	$\mathrm{PG}(3,2)$	[6,21]
8	15	35	7	3	1	Alt_8	$2^3{:}\mathrm{PSL}_3(2)$	Alt_8	$\mathrm{PG}(3,2)$	[6,21]

Note: The last column addresses to references in which a design with the parameters in the line has been constructed.

Theorem 1. *Let $\mathcal{D}$ be a nontrivial 2-(v,k,λ) design with prime replication number r, and let α be a point of $\mathcal{D}$. If G is a flag-transitive automorphism group of $\mathcal{D}$, then one of the following holds:*

(a) *G is an almost simple group with socle X and one of the following holds:*
  (i) *$\lambda\in\{1,2,3,5\}$ and v, k, λ, X, $G_\alpha\cap X$ and G are as in one of the lines in Table 1;*
  (ii) *$\mathcal{D}$ is the Witt-Bose-Shrikhande space $\mathrm{W}(2^n)$ with parameters $v=2^{n-1}\cdot(2^n-1)$, $b=2^{2n}-1$, $r=2^n+1$ Fermat prime, $k=2^{n-1}$ and $\lambda=1$, for $n=2^{2^m}\geqslant 16$. Moreover, $G=X=\mathrm{PSL}_2(2^n)$ and $G_\alpha\cap X=\mathrm{D}_{2(2^n+1)}$;*
  (iii) *$X=\mathrm{PSL}_n(q)$, $G_\alpha\cap X=\hat{}[q^{n-1}]{:}\mathrm{SL}_{n-1}(q)\cdot(q-1)$, $v=(q^n-1)/(q-1)$ and r is a primitive divisor of $(q^{n-1}-1)/(q-1)$ with $n\geqslant 3$.*

(b) *G is an affine type group, $G_0\leqslant \Gamma L_n(p^{\frac{d}{n}})$ and $v=p^d=q^n$ and one of the following holds:*
  (i) *$\mathcal{D}$ has $v=q=p^d$ points and $G\leqslant \mathrm{A\Gamma L}_1(q)$;*
  (ii) *$\mathcal{D}$ is an affine space $\mathrm{AG}_n(q)$ $(n\geqslant 2)$ with parameters $v=q^n$, $b=\frac{q^{n-1}(q^n-1)}{q-1}$, $r=\frac{q^n-1}{q-1}$ prime, $k=q^d$ and $\lambda=1$, and G is 2-transitive on points of $\mathcal{D}$ and the point-stabiliser G_0 contains $\mathrm{SL}_a(q^{\frac{n}{a}})$, $\mathrm{Sp}_a(q^{\frac{n}{a}})$ or $\mathrm{G}_2(q^{\frac{n}{6}})'$;*
  (iii) *$\mathcal{D}$ is a 2-design with parameters $v=p^d$, $b=\frac{p^d(p^d-1)}{p^u(p^{(u,d)}-1)}$, $r=\frac{p^d-1}{p^{(u,d)}-1}$ prime, $k=p^u$ and $\lambda=\frac{p^u-1}{p^{(u,d/n)}-1}$, where n is a divisor of d, $(u,h,d/n)=1$, and either $(u,d)<u<d/n$, or $d-d/n\leqslant u<d$ and $(u,d)<u$;*
  (iv) *$\mathcal{D}$ is a 2-design with parameters $v=2^d-1$, $b=2(2^d-1)$, $r=2^d-1$ Mersenne prime, $k=2^{d-1}$, $\lambda=2^{d-1}-1$ with $n=d\geqslant 3$ prime.*

A detailed information on the existence and constructions of the designs obtained in Theorem 1 can be found in [2, 4, 5, 16]. As mentioned above, in order to prove

TABLE 2. Some finite linear spaces admitting flag-transitive automorphism groups of affine type.

Line	v	b	r	k	λ	**Design**	**References**
1	81	9	10	9	1	Nearfield plane	[15, Example 1.2(ii)],[6, 11]
2	81	90	10	9	1	$AG_2(9)$	[15, Example 1.1(iii)],[6, 10]
3	81	1080	40	3	1	$AG(4,3)$	[15, Example 1.1(iv)],[6, 14]
4	121	132	12	11	1	$AG_2(11)$	[15, Examples 1.1(ii)-(iii)],[6, 10]
5	361	380	20	19	1	$AG_2(19)$	[15, Example 1.1(iii)],[6, 10]
6	529	552	24	23	1	$AG_2(23)$	[15, Example 1.1(ii)],[6, 10]
7	729	756	28	27	1	Hering's plane	[15, Example 1.2(iii)],[6, 12]
8	729	7371	91	9	1	Hering's design	[15, Example 1.3],[6, 13]
9	841	870	30	29	1	$AG_2(29)$	[15, Example 1.1(iii)],[6, 10]
10	3481	3540	60	59	1	$AG_2(59)$	[15, Example 1.1(iii)],[6, 10]

Note: The last column addresses to references in which a design with the parameters in the line has been constructed.

Theorem 1 in Section 2, by [2], we only need to treat the affine type case. At the time of preparing this paper, we have noticed that a classification of 2-designs with $(r,\lambda)=1$ admitting flag-transitive affine type automorphism groups has been given in [16] (see also [4,5]). Our arguments were mostly the same as [16], and so in order to complete our project started a few years ago on the classification of 2-designs with replication numbers prime admitting flag-transitive automorphism groups (see [2]), we have decided to use the main results in [4, 5, 16] and to avoid repetition in the proof of Theorem 1.

2. PROOF OF THEOREM 1

Let $\mathcal{D}$ be a 2-design with prime replication number r admitting a flag-transitive automorphism group G. Since $r(k-1)=\lambda(v-1)$ and r is prime, $\gcd(r,\lambda)=1$, and so by [8, 2.3.7(a)], we conclude that G is point-primitive. Moreover, [23, Theorem] implies that G is point-primitive of almost simple or affine type. If G is an almost simple group, then [2, Theorem 1.1] follows part (a). Therefore, we only need to consider the case where G is a primitive group of affine type. In this case, the points of $\mathcal{D}$ can be identified with the vectors in a vector space $V=V_d(p)$ of dimension d over prime field $GF(p)$. If G_0 denotes the stabiliser of the zero vector $0\in V$ in G, then G_0 is an irreducible subgroup of $GL(V)\cong GL_d(p)$. For each divisor n of d, the group $\Gamma L_n(p^{\frac{d}{n}})$ has a natural irreducible action on V. Since the group G_0 acts irreducibly on V, choose n to be minimal divisor of d such that $G_0\leqslant\Gamma L_n(p^{\frac{d}{n}})$ in this action, and write $q=p^{\frac{d}{n}}$. Thus we have $G_0\leqslant\Gamma L_n(p^{\frac{d}{n}})$ and $v=p^d=q^n$.

Suppose first that $\lambda=1$. Then the possibilities for $(\mathcal{D},G)$ can be read of from [15, Main Theorem] and therein references. In conclusion, $\mathcal{D}$ is a 2-design with parameters as in Table 2 or one of the following holds:

TABLE 3. Some 2-designs admitting affine type automorphism groups.

v	b	r	k	λ	v	b	r	k	λ	v	b	r	k	λ
9	12	8	6	5	121	484	120	30	29	729	10206	728	52	51
25	30	24	20	19	121	605	120	24	23	729	20412	728	26	25
25	50	24	12	11	121	726	120	20	19	729	132678	728	4	3
25	75	24	8	7	121	1210	120	12	11	841	4205	840	168	167
25	100	24	6	5	121	1815	120	8	7	841	5046	840	140	139
49	56	8	7	1	361	1083	360	120	119	841	8410	840	84	83
49	56	48	42	41	361	5415	360	24	23	841	16820	840	42	41
49	98	48	24	23	361	1805	360	72	71	841	29435	840	24	23
49	196	48	12	11	361	7220	360	18	17	841	35322	840	20	19
49	294	48	8	7	361	10830	360	12	11	841	58870	840	14	13
49	392	48	6	5	361	16245	360	8	7	3481	34810	3480	348	347
64	192	21	7	2	529	552	528	506	505	3481	41772	3480	290	289
81	120	40	27	13	529	1058	528	264	263	3481	69620	3480	174	173
81	162	80	40	39	529	1587	528	176	175	3481	100949	3480	120	119
81	540	80	12	11	529	4232	528	66	65	3481	504745	3480	24	23
81	810	80	7	1	529	5819	528	48	47	3481	605694	3480	20	19
81	810	80	8	7	529	6348	528	44	43	3481	1009490	3480	12	11
81	1080	80	6	5	529	11638	528	24	23	3481	1514235	3480	8	7
121	132	120	110	109	529	34914	528	8	7					
121	242	120	60	59	529	46552	528	6	5					

(1) $\mathcal{D}$ has $v = q = p^d$ points, and $G \leqslant \mathrm{A\Gamma L}_1(q)$;
(2) $\mathcal{D}$ is an affine space $\mathrm{AG}_n(q)$ and $G \leqslant \mathrm{A\Gamma L}_n(q)$ with $n \geqslant 2$, where the point-stabiliser G_0 contains $\mathrm{SL}_a(q^{\frac{n}{a}})$, $\mathrm{Sp}_a(q^{\frac{n}{a}})$ or $\mathrm{G}_2(q^{\frac{n}{6}})'$;
(3) $\mathcal{D}$ is the Lüneburg plane related to the Suzuki group $\mathrm{Sz}(q)$ with parameters $v = q^4$, $b = q^2(q^2+1)$, $r = q^2+1$ and $k = q^2$, for $q = 2^{2e+1} > 2$.

The possibilities recorded in Table 2 cannot occur as r must be prime. If $q = 2^{2e+1} > 2$, then $r = q^2+1$ is not prime, and hence case (3) does not hold. Note that in case (2), $r = (q^n-1)/(q-1)$ has to be prime. This follows part (b.i) or part (b.ii), as claimed.

Suppose now that $\lambda > 1$ and that G is not a subgroup of $\mathrm{A\Gamma L}_1(q)$. Then by [4], $\mathcal{D}$ is non-symmetric, and hence [5, Theorem 1] and [16, Theorem 1] imply that $\mathcal{D}$ is a 2-design with parameters as in Table 3 or one of the following holds:

(1) $\mathcal{D}$ is a 2-$\left(p^d, p^u, \dfrac{p^u-1}{p^{(u,d/n)}-1}\right)$ design, where n is a divisor of d and $u < d$ satisfying $(u,h,d/n) = 1$, and either $(u,d) < u < d/n$ or $d - d/n \leqslant u < d$ and $(u,d) < u$;
(2) $\mathcal{D}$ is a 2-$(p^d, p^u\omega, p^u\omega - 1)$ design, where $\omega = \dfrac{p^{(u,d/n)}-1}{\theta}$ and $(p,u,\theta) \neq (2,1,1)$ and $(\dfrac{p^d-1}{\omega}, p^u\omega - 1) = 1$ and either $0 \leqslant u < d/n$, or $d - d/n \leqslant u < d$.

Since r is prime, the possibilities in Table 3 can be ruled out. Moreover, in case (1), $r = \frac{p^d-1}{p^{(u,d)}-1}$ needs to be prime, and this follows part (b.iii). Now if $\mathcal{D}$ is a 2-design as in case (2), then $r = p^d - 1$ must be prime, which requires $p = 2$, and hence $r = 2^d - 1$ is a Mersenne prime and d is prime. This together with the facts that $n \mid d$ and $n \neq 1$ implies that $n = d$. Then $\omega = (2^{(u,d/n)} - 1)/\theta = (2-1)/\theta$, and so $\omega = \theta = 1$. Note that either $0 \leqslant u < d/n$, or $d - d/n \leqslant u < d$. In the former case, we have that $u = 0$, and so $k = 2^u\omega = 1$, which is a contradiction. Therefore, $d - d/n \leqslant u < d$, and since $n = d$, it follows that $u = d-1$. Since $(p,u,\theta) \neq (2,1,1)$, we conclude that $u \geqslant 2$, or equivalently, $d \geqslant 3$. Therefore, $\mathcal{D}$ is a 2-design with parameters $(2^d - 1, 2^{d+1} - 2, 2^d - 1, 2^{d-1}, 2^{d-1} - 1)$ with $d \geqslant 3$ prime, this is part (b.iv). This completes the proof of Theorem 1.

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