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A STUDY OF q-ANALOGUE OF THE ANALYTIC CHARACTERIZATION OF LIMAÇON FUNCTIONS

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Abstract. The investigation of Ma and Minda classes of functions associated with q-calculus has been on increase in recent times. Newly, limaçon functions, the classes of starlike and convex limaçon functions were initiated and investigated in the literature. As a result, the present article is aimed to present the q-analogue of the limaçon functions and utilize it to establish the classes of starlike and convex limaçon functions that are correlated with quantum calculus. To this end, the geometric characterization of these functions is examined. Moreover, radius, subordination and some other related results with these novel classes are verified. Overall, some consequences of our investigations are also illustrated.

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1. INTRODUCTION AND PRELIMINARIES

The study of univalent functions is a significant aspect of Complex analysis. Due to the complexity and difficulty involved in analyzing the geometric properties of these functions as a whole, numerous subclasses of them have been emerging in the literature. To unify and classify this concept of the subfamily of univalent functions, Ma and Minda [7] gave an encyclopedic classification of those univalent functions that map the open unit disc U onto convex and starlike domains. As a result, these functions are scattered in the literature in different directions and perspectives. For more details on Ma and Minda functions, see [5, 8, 9, 12–14, 16, 19, 20] and the references therein.

In recent years, the novel associated area of Complex analysis showed up and created several interesting outcomes with potential utilities. The most fascinating aspect of it is the introduction of q-calculus (also known as quantum calculus). Jackson was the first mathematician who demonstrated the concept of q-integration and differentiation in a systematic way, see [2, 3]. The development of this concept in Complex analysis is traced to the work of Ismail et al. [1]. As such, many subclasses of Ma and Minda classes correlated with the idea of quantum calculus have been illustrated in various distinct directions (see [6, 10, 15, 21, 22]). For example, Srivastava et al. [23] introduce certain higher-order q-derivatives, which were used to study new subclasses of multivalent q-starlike functions that are associated with the Janowski functions and involving certain conic domains. Also in [6], Bilal et al., utilized the idea of q-calculus to study the class of starlike functions which mapped U onto a conic domains of Janowski type. They obtained some useful results, such as, the sufficient conditions, closure results, the Fekete-Szegö type inequalities and distortion theorems. q-Srivastava-Attiya operator was implemented by Srivastava et al. [24] to define and investigate classes of analytic-bi univalent function, which are subordinate to Horadam polynomials. In this direction, the second and third coefficients of functions associated with these classes were derived.

Srivastava [6] uncovered that the purported (p,q)-calculus extension of the classical Quantum calculus is irrelevant, trivial, unimportant and inconsequential since the p-parameter remained insignificant, redundant and contributed nothing to the so-called generalization.

Recently, Kanas and Masih [8] initiated analytic characterization of limaçon domains. The geometric properties of this function were studied and used to present convex and starlike limaçon classes denoted by $CV_L(s)$ and $ST_L(s)$, respectively. Furthermore, Afis et al. [16] continued with the investigation of these classes and proved many interesting results associated with them.

Motivated with these new work, our interest in this paper is to present and study q-analogue of the analytic classification of the limaçon functions. Besides, the classes q-starlike limaçon (denoted by $ST_{\mathcal{L}_q}(s)$) and q-convex limaçon (depicted by $CV_{\mathcal{L}_q}(s)$) are introduced. Moreover, radius, subordination and some other related results with these novel classes are verified.

To put our findings in a clear perspectives, the following preliminaries and definitions are presented.

Let $\mathcal{A}$ denotes the class of normalized analytic functions $f(\zeta)$ of the form

$$f(\zeta)=\zeta+\sum_{n=2}^{\infty}a_n\zeta^n, \quad \zeta\in U. \tag{1.1}$$

The subclass of $\mathcal{A}$ which are univalent in U is depicted by S. Symbolized by $\mathcal{W}$ is the class of analytic functions

$$w(\zeta)=\sum_{n=1}^{\infty}w_n\zeta^n, \quad \zeta\in U \tag{1.2}$$

such that $w(0)=0$ and $|w(\zeta)|<1$. These functions are known as Schwarz functions. If $f(\zeta)$ and $g(\zeta)$ are analytic functions in U, then $f(\zeta)$ is subordinate to $g(\zeta)$ (written as $f(\zeta)\prec g(\zeta)$) if there exists $w(\zeta)\in\mathcal{W}$ such that $f(\zeta)=g(w(\zeta))$, $\zeta\in U$.

Janowski [4] introduced the class $P(A,B)$, $-1 \leq B < A \leq 1$ of functions $p(\zeta)$ satisfying the subordination condition

$$p(\zeta) \prec \frac{1+A\zeta}{1+B\zeta},$$

or equivalently, satisfying the inequality

$$\left| p(\zeta) - \frac{1-ABr^2}{1-B^2r^2} \right| \leq \frac{(A-B)r}{1-B^2r^2}, \quad 0 < r < 1. \tag{1.3}$$

Also, $p \in P(A,B)$ if and only if

$$\left| \frac{p(\zeta)-1}{A-Bp(\zeta)} \right| < 1, \quad \zeta \in U. \tag{1.4}$$

As a special cases, $P(1,-1) \equiv P$ and $P(1-2\beta,-1) \equiv P(\beta)$ $(0 \leq \beta < 1)$ are the classes of functions of positive real part and those functions whose real parts are greater than β, respectively.

Definition 1. Let $q \in (0,1)$. Then the q-number $[n]_q$ is given as

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & n \in \mathbb{C}, \\ \sum\limits_{\iota=0}^{n-1} q^{\iota} = 1+q+q^2+\dots q^{n-1}, & n \in \mathbb{N}, \\ n, \quad \text{as} & q \to 1^-. \end{cases} \tag{1.5}$$

and the q-derivative of a complex valued function $f(\zeta)$ in U is given by

$$D_q f(\zeta) = \begin{cases} \frac{f(qz)-f(\zeta)}{(q-1)\zeta}, & \xi \neq 0 \\ f'(0), & \xi = 0, \\ f'(\zeta), \quad \text{as} & q \to 1^-. \end{cases} \tag{1.6}$$

From the above explanation, it is easy to see that for $f(\zeta)$ given by (1.1),

$$D_q f(\zeta) = 1 + \sum_{n=2}^{\infty} [n]_q a_n \zeta^n. \tag{1.7}$$

Let $f, g \in \mathcal{A}$, we have the following rules for q-difference operator D_q.

(i) $D_q(f(\zeta)g(\zeta)) = f(q\zeta)D_q g(\zeta) + g(\zeta)D_q f(\zeta)$;

(ii) $D_q(\sigma f(\zeta) \pm \delta g(\zeta)) = \sigma D_q f(\zeta) \pm \delta D_q g(\zeta)$, for $\sigma, \delta \in \mathbb{C} \setminus \{0\}$;

(iii)

$$D_q\left(\frac{f(\zeta)}{g(\zeta)}\right) = \frac{g(\zeta)D_q g(f) - f(\zeta)D_q g(\zeta)}{f(\zeta)g(q\zeta)}, \quad f(\zeta) \neq 0,\ g(q\zeta) \neq 0;$$

(iv) $D_q(\log f(\zeta)) = \log q^{\frac{1}{q-1}} \frac{D_q f(\zeta)}{f(\zeta)}$.

As a right inverse, Jackson [2] presented the q-integral of the analytic function $f(\zeta)$ as

$$\int f(\zeta) d_q\zeta = (1-q)\zeta \sum_{j=0}^{\infty} q^j f(q^j\zeta).$$

For example, $f(\zeta) = \zeta^n$ has q-anti derivative as

$$\int \zeta^n d_q\zeta = \frac{\zeta^{n+1}}{[n+1]_q}, n \neq -1.$$

Using the concept of q-calculus, the q-Caratheodory class of functions $P(q)$ was formulated [1,10] as follow: Let $p(\zeta)$ be analytic in U with $p(0) = 1$. Then $p \in P(q)$ if and only if

$$p(\zeta) \prec \frac{1+\zeta}{1-q\zeta}, \quad \zeta \in U, q \in (0,1).$$

or

$$\left| p(\zeta) - \frac{1+qr^2}{1-q^2r^2} \right| \leq \frac{(1+q)r}{1-q^2r^2}. \tag{1.8}$$

. If $f \in \mathcal{A}$ and $p(\zeta) = \frac{\zeta D_q f(\zeta)}{f(\zeta)}$ in (1.8), then $f \in ST_q$. Also, with $p(\zeta) = \frac{D_q(\zeta D_q f(\zeta))}{D_q f(\zeta)}$, then $f \in CV_q$. These are the classes of q- starlike and covex functions respectively (see [3,10]). We say a function $f \in \mathcal{A}$ is said to be starlike of order $\beta (0 \leq \beta < 1)$ if and only if

$$\operatorname{Re} \frac{\zeta f'(\zeta)}{f(\zeta)} > \beta, \quad \zeta \in U.$$

The class of this function is denoted by $S^*(\beta)$ and was introduced by Robertson [11]. For $\zeta f'(\zeta) \in S^*(\beta)$, $f(\zeta)$ is called a convex function of order β and the class containing such functions is symbolized with $C(\beta)$. These concept may be generalized by replacing the ordinary derivative with q-derivative, and the classes are tagged as $S_q^*(\beta)$ and $C_q(\beta)$, respectively.

Definition 2. [8] Let $p(\zeta) = 1 + \sum_{n=1}^{\infty} c_n\zeta^n$. Then $p \in P(\mathbb{L}_s)$ if and only if

$$p(\zeta) \prec (1+s\zeta)^2, \quad 0 < s \leq \frac{1}{\sqrt{2}}, \zeta \in U,$$

or equivalently, if $p(\zeta)$ satisfies the inequality

$$|p(\zeta) - 1| < 1 - (1-s)^2.$$

Presented in [8], was the inclusion relation

$$\{w \in \mathbb{C} \colon |w-1| < 1-(1-s)^2\} \subset \mathbb{L}_s(U) \subset \{w \in \mathbb{C} \colon |w-1| < (1+s)^2 - 1\}. \tag{1.9}$$

Definition 3. [8] Let $f \in \mathcal{A}$. Then $f \in ST_{\mathcal{L}}(s)$ if and only if

$$\frac{\zeta f'(\zeta)}{f(\zeta)} \in P(\mathbb{L}_s), \quad 0 < s \leq \frac{1}{\sqrt{2}}.$$

Also, $f \in CV_{\mathcal{L}}(s)$ if and only if

$$\zeta f'(\zeta) \in ST_{\mathcal{L}}(s), \quad 0 < s \leq \frac{1}{\sqrt{2}}.$$

Inspired with these definitions and the notion of q-calculus, we introduce the following novel classes of functions.

Definition 4. Let $p(\zeta) = 1 + \sum_{n=1}^{\infty} c_n \zeta^n$. Then $p \in P(\mathbb{L}_s(q))$ if and only if

$$p(\zeta) \prec \left(\frac{2(1+s\zeta)}{2+s(1-q)\zeta}\right)^2 := \mathbb{L}_{q,s}(\zeta), \quad 0 < q < 1,\, 0 < s \leq \frac{1}{\sqrt{2}},\, \zeta \in U.$$

Definition 5. Let $f \in \mathcal{A}$. Then $f \in ST_{\mathcal{L}_q}(s)$ if and only if

$$\frac{\xi D_q f(\zeta)}{f(\zeta)} \in P(\mathbb{L}_s(q)), \quad 0 < q < 1, 0 < s \leq \frac{1}{\sqrt{2}}.$$

Also, $f \in CV_{\mathcal{L}_q}(s)$ if and only if

$$\zeta D_q f(\zeta) \in ST_{\mathcal{L}_q}(s), \quad 0 < q < 1, 0 < s \leq \frac{1}{\sqrt{2}}.$$

In particular, as $q \to 1^-$, we are back to Definitions 2 and 3.

2. A Preliminary Lemma

Lemma 1. *[25] Let $w \in \mathcal{W}$. If $|w(\zeta)|$ attains its maximum value on the circle $|\zeta| = r$ at a point $\zeta_0 \in U$, then we have $\zeta_0 D_q w(\zeta_0) = kw(\zeta_0)$, for some $k \geq 1$.*

The main results of this manuscript are presented in subsequent sections with the condition that analytic functions $p(\zeta)$ in U satisfies $p(0) = 1$.

3. Properties Associated with the Class $P(\mathbb{L}_s(q))$

Theorem 1. *Let $p \in P(\mathbb{L}_s(q))$. Then*

$$4\left(\frac{1-sr}{2-s(1-q)r}\right)^2 \leq |p(\zeta)| \leq 4\left(\frac{1+sr}{2+s(1-q)r}\right)^2, \quad z = re^{i\theta}, 0 < r < 1.$$

Proof. Assume $A = s$ and $B = \frac{s(1-q)}{2}$. Since $p \in P(\mathbb{L}_s(q))$, then

$$p(\zeta) \prec \left(\frac{1+A\zeta}{1+B\zeta}\right)^2.$$

Taking the principal values of the square root and applying (1.3), we obtain the result. □

Remark 1. As $q \to 1^-$, in the above theorem, we obtain the bounds for $p(\zeta) \in P(\mathbb{L}_s)$ in [8]. Also, we can infer from the theorem that the smallest disc with center $(0,1)$ that contains $\mathbb{L}_{q,s}(U)$ and the largest disc with center $(0,1)$ that contained in $\mathbb{L}_{q,s}(U)$ are related with the following relation

$$\left\{ w \in \mathbb{C} \colon |w(\zeta) - 1| < 1 - 4\left(\frac{1-s}{2-s(1-q)}\right)^2 \right\} \subset \mathbb{L}_{q,s}(U) \tag{3.1}$$

and

$$\mathbb{L}_{q,s}(U) \subset \left\{ w \in \mathbb{C} \colon |w(\zeta) - 1| < 4\left(\frac{1+s}{2+s(1-q)}\right)^2 - 1 \right\}. \tag{3.2}$$

Lastly, $f \in \mathcal{A}$ belongs to $ST_{L_q}(s)$ if and only

$$\left|\frac{\zeta D_q f(\zeta)}{f(\zeta)}\right| < 1 - 4\left(\frac{1-s}{2-s(1-q)}\right).$$

Theorem 2. *If $p(\zeta)$ is analytic in U and satisfies the condition*

$$\left|\frac{\zeta D_q p(\zeta)}{p(\zeta)}\right| < \frac{2(1+q)s}{(1+s)(2+s(1-q))}, \tag{3.3}$$

then for $0 < s \leq \frac{1}{\sqrt{2}}$, $p \in P(\mathbb{L}_s(q))$.

Proof. Consider the function

$$p(\zeta) = \left(\frac{2(1+sw(\zeta))}{2+s(1-q)w(\zeta)}\right)^2,$$

where $w(\zeta)$ is analytic in U with $w(0) = 0$. To determine our result, it is enough to show that $|w(\zeta)| < 1$ in U. By q-logarithmic differentiation, it is easy to see that

$$\frac{\zeta D_q p(\zeta)}{p(\zeta)} = \frac{2(1+q)s\zeta D_q w(\zeta)}{(1+sw(\zeta))(2+s(1-q)w(\zeta))}.$$

If there exists $\zeta_0 \in U$ such that $\max_{|\xi|\leq|\zeta_0|} = |w(\zeta_0)| = 1$, then by Lemma 1, $\zeta_0 D_q w(\zeta_0) = kw(\zeta_0)$ for some $k \geq 1$. Setting $w(\zeta_0) = e^{i\theta}$ $(-\pi \leq \theta \leq \pi)$, we obtain

$$\begin{aligned} \left|\frac{\zeta_0 D_q p(\zeta_0)}{p(\zeta_0)}\right| &= 2s(1+q)\left|\frac{\zeta_0 D_q w(\zeta_0)}{(1+sw(\zeta_0))(2+s(1-q)w(\zeta_0))}\right| \geq \frac{2ks(1+q)}{(1+s)(2+s(1-q))} \\ &\geq \frac{2s(1+q)}{(1+s)(2+s(1-q))}, \end{aligned}$$

which contradicts (3.3) and so, there is no ζ_0 in U such that $|w(\zeta_0)| = 1$. Thus, $|w(\zeta)| < 1$ for all $\zeta \in U$. Hence, if (3.3) is satisfied, $p \in P(\mathbb{L}_s(q))$. $\square$

Let $f(\zeta)\in\mathcal{A}$ and setting $p(\zeta)=\frac{\zeta D_q f(\zeta)}{f(\zeta)}$ and $p(\zeta)=\frac{D_q(\zeta D_q f(\zeta))}{D_q f(\zeta)}$ in Theorem 2, respectively, we are left with the following corollary.

Corollary 1. *If $f\in\mathcal{A}$, then*

$$\left|1+\frac{q\zeta D_q^2 f(\zeta)}{D_q f(\zeta)}-\frac{\zeta D_q f(\zeta)}{f(\zeta)}\right|<\frac{2(1+q)s}{(1+s)(2+s(1-q))}\Longrightarrow f\in ST_{\mathcal{L}_q}(s)$$

and similarly

$$\left|1+\frac{q^2\zeta^2 D_q^3 f(\zeta)+\zeta D_q^2 f(\zeta)-D_q f(\zeta)}{q\zeta D_q^2 f(\zeta)+D_q f(\zeta)}-\frac{\zeta D_q^2 f(\zeta)}{D_q f(\zeta)}\right|<\frac{2(1+q)s}{(1+s)(2+s(1-q))}$$

implies $f\in CV_{\mathcal{L}_q}(s)$.

Theorem 2 reduces to the following result as $q\to 1^-$.

Corollary 2. *If $p(\zeta)$ is analytic in U and satisfies the condition*

$$\left|\frac{\xi p'(\zeta)}{p(\zeta)}\right|<\frac{2s}{(1+s)},\tag{3.4}$$

then for $0<s\le\frac{1}{\sqrt{2}}$, $p\in P(\mathbb{L}_s)$.

Theorem 3. *Let $0<s\le\frac{1}{\sqrt{2}}$. Then*

$$\max_{|\xi|=1}\mathrm{Re}\,(\mathbb{L}_{q,s}(\zeta))=4\left(\frac{1+s}{2+s(1-q)}\right),$$

$$\min_{|\xi|=1}\mathrm{Re}\,(\mathbb{L}_{q,s}(\zeta)):=\mu_q(s)=\begin{cases}4\left(\frac{1-s}{2-s(1-q)}\right), & 0<s\le s_0,\\ \frac{2\left[s^4(1-q)^2-2s^2(3+q)+4\right]}{(4-s^2(1-q)^2)^2}, & s_0\le s\le\frac{1}{\sqrt{2}},\end{cases}\tag{3.5}$$

where

$$s_0=\frac{\sqrt{3+q^2}-(1+q)}{1-q},\quad q\in(0,1).\tag{3.6}$$

Proof. We can write $\mathbb{L}_{q,s}(\zeta)=\left(\frac{1+A\zeta}{1+B\zeta}\right)^2$, where $A=s, B=\frac{s(1-q)}{2}$. Then

$$\mathrm{Re}\,(\mathbb{L}_{q,s}(\zeta))=\frac{(1+AB)^2+2AB+2(1+AB)(A+B)\cos\theta+(A^2+B^2)\cos 2\theta}{(1+2B\cos\theta+B^2)^2},$$

with $-\pi\le\theta\le\pi$. Let $\phi(\theta)=\mathrm{Re}\,(\mathbb{L}_{q,s}(\zeta))$. Then

$$\phi'(\theta)=\frac{2(A-B)\left[(2B^3-2A)\cos\theta+AB^3-3AB-1+3B^2\right]\sin\theta}{(1+2B\cos\theta+B^2)^3}.$$

Because $\phi(\theta)$ is an even function, we consider $\theta \in [0,\pi]$. Therefore, the maximum of $\phi(\theta)$ is attained at the stationary point, that is

$$2(A-B)\left[(2B^3-2A)\cos\theta + AB^3 - 3AB - 1 + 3B^2\right]\sin\theta = 0,$$

which are $\theta = 0, \pi$ and the solution of the equation

$$\cos\theta = \frac{3AB+1-AB^3-3B^2}{2B^3-2A}.$$

Similarly,

$$\phi''(\theta) = \frac{\gamma_1\cos^3\theta + \gamma_2\cos^2\theta + \gamma_3\cos\theta + C}{(1+2B\cos\theta+B^2)^4},$$

where

$$\begin{aligned}
\gamma_1 &= 2(A-B)\left(4AB-4B^4\right),\\
\gamma_2 &= 2(A-B)\left(4B^5-4AB^4-8B^3+8AB^2+4B-4A\right)\\
\gamma_3 &= 2(A-B)\left(AB^5+11B^4-2AB^3+2B^2-11AB-1\right)\\
C &= 2(A-B)\left(-2B^5+6AB^4+16B^3-16AB^2-6B+2A\right).
\end{aligned}$$

At $\theta = 0$,

$$\phi''(0) = -\frac{8s(1+q)\left[s^2(q-1)+2s(1+q)+2\right]}{(2+s(1-q))^4} < 0.$$

This implies that $\phi(\theta)$ attain maximum at $\theta = 0$. Similarly for $\theta = \pi$,

$$\phi''(\pi) = \frac{8s(1+q)\left[s^2(q-1)-2s(1+q)+2\right]}{(2-s(1-q))^4}.$$

Therefore, $\phi''(\pi) \le 0$ for $s_0 \le s \le \frac{1}{\sqrt{2}}$ and $\phi''(\pi) \ge 0$ for $0 < s \le s_0$, where s_0 is given by (3.6). These imply that $\phi(\theta)$ attains its maximum when $s_0 \le s \le \frac{1}{\sqrt{2}}$ and minimum when $0 < s \le s_0$. At

$$\begin{aligned}
\cos\theta &= \frac{3AB+1-AB^3-3B^2}{2B^3-2A},\\
\phi''(\theta) &= -\frac{4[s^2(q-1)+2s(1+q)+2][s^2(q-1)-2s(1+q)+2][s^2(q-1)^3+8]^2}{(2+s(1-q))^4\,(2-s(1-q))^4\,(1+q)},
\end{aligned}$$

which means that $\phi(\theta)$ attain maximum when $0 < s \le s_0$ and minimum when $s_0 \le s \le \frac{1}{\sqrt{2}}$.

Overall, if $s_0 \le s \le \frac{1}{\sqrt{2}}$, $\phi(\theta)$ attains maximum at $\theta = 0, \pi$ and minimum when at

$$\cos\theta = \frac{3AB+1-AB^3-3B^2}{2B^3-2A}$$

Also, for $0 < s \leq s_0$, $\phi(\theta)$ attains maximum at $\theta = 0$ and minimum at $\theta = \pi$. Hence, for $s_0 \leq s \leq \frac{1}{\sqrt{2}}$,

$$\max_{|\xi|=1} \operatorname{Re}\left(\mathbb{L}_{q,s}(\zeta)\right) = 4\left(\frac{1+s}{2+s(1-q)}\right),$$

and

$$\min_{|\xi|=1} \operatorname{Re}\left(\mathbb{L}_{q,s}(\zeta)\right) = \frac{2\left[s^4(1-q)^2 - 2s^2(3+q) + 4\right]}{(4-s^2(1-q)^2)^2}.$$

For $0 < s \leq s_0$,

$$\min_{|\xi|=1} \operatorname{Re}\left(\mathbb{L}_{q,s}(\zeta)\right) = 4\left(\frac{1-s}{2-s(1-q)}\right).$$

□

Remark 2. As $q \to 1^-$ in Theorem 3, we are led to the result of Kanas and Masih in [8, Theorem 1].

Theorem 4. *Let* $0 < s \leq \frac{1}{\sqrt{2}}$. *Then*

$$\frac{\mathbb{L}_{q,s}(\zeta) - 1}{s(1+q)} \in S_q^*(\beta_1),$$

where

$$\beta_1 = \frac{(q-1)s^2 - 2(1+q)s + 3 + q^2}{4 - s(3-q)}.$$

Proof. Let

$$g(\zeta) = \frac{\mathbb{L}_{q,s}(\zeta) - 1}{s(1+q)}.$$

Then by q-logarithmic differentiation, we have

$$\begin{aligned}
\operatorname{Re}\frac{\zeta D_q g(\zeta)}{g(\zeta)} &= \operatorname{Re}\frac{\zeta D_q\left(\mathbb{L}_{q,s}(\zeta)\right)}{\mathbb{L}_{q,s}(\zeta) - 1} \\
&= \operatorname{Re}\frac{1}{(1+Bq\zeta)^2} + \operatorname{Re}\frac{q[(A+B)\zeta + 2AB\zeta^2]}{(2+(A+B)\zeta)(1+Bq\zeta)^2}, \quad A = s, B = s(1-q)/2 \\
&= (1+q)\operatorname{Re}\frac{1}{(1+Bq\zeta)^2} + 2q\operatorname{Re}\frac{1 - AB\zeta^2}{(2+(A+B)\zeta)(1+Bq\zeta)^2} \\
&\geq \frac{1+q}{(1+Bq)^2} - 2q\left|\frac{1 - AB\zeta^2}{(2+(A+B)\zeta)(1+Bq\zeta)^2}\right| \\
&\geq \frac{1+q}{(1+Bq)^2} - \frac{2q(1+AB)}{(2-(A+B))(1-Bq)^2} \\
&= \frac{(q-1)s^2 - 2(1+q)s + 3 + q^2}{4 - s(3-q)}.
\end{aligned}$$

□

Corollary 3. *[8] Let $0 < s \leq \sqrt{2}$. Then as $q \to 1^-$,*

$$\frac{\mathbb{L}_{q,s}(\zeta)-1}{2s} \in S^*\left(\frac{2-2s}{2-s}\right). \tag{3.7}$$

Theorem 5. *Let $p \in P(q)$. Then $p \in P(\mathbb{L}_s(q))$ for z in the disc*

$$|\xi| < \frac{(1+q)(4-(3-q)s)s}{(1+2q)(2-s(1-q))^2-4q(1-s)^2}. \tag{3.8}$$

Proof. Since $p \in P(q)$, then

$$\left|p(\zeta)-\left(\frac{1+qr^2}{1-q^2r^2}\right)\right| < \frac{(1+q)r}{1-q^2r^2}.$$

Now,

$$\begin{aligned}|p(\zeta)-1| &\leq \left|p(\zeta)-\left(\frac{1+qr^2}{1-q^2r^2}\right)\right| + \frac{q(1+q)r^2}{1-q^2r^2}\\ &< \frac{(1+q)r}{1-q^2r^2}+\frac{q(1+q)r^2}{1-q^2r^2} = \frac{(1+q)r}{1-qr}.\end{aligned}$$

Therefore, $p \in \mathbb{L}_s(q)$ if

$$\frac{(1+q)r}{1-qr} < 1-\left(\frac{2(1-s)}{2+s(1-q)}\right)^2$$

and from this, we obtain the required result. □

Corollary 4. *Let $0 < s \leq \frac{1}{\sqrt{2}}$. Then*

$$C_q \subset CV_{\mathcal{L}_q}(s) \quad \text{and} \quad S_q^* \subset ST_{\mathcal{L}_q}(s),$$

for all z in the disc given by (3.8).

4. Subordination Results

In this section, we assume the parameter $\alpha \neq 0$.

Theorem 6. *Let $p(\zeta)$ be analytic in U and Suppose it satisfies the differential subordination*

$$1+\alpha\frac{\zeta D_q p(\zeta)}{p(\zeta)} \prec \mathbb{L}_{q,s}(\zeta).$$

If

$$|\alpha| \geq \frac{(1+s)(4-s(3-q))}{2(2-s(1-q))}, \tag{4.1}$$

then $p(\zeta) \prec \mathbb{L}_{q,s}(\zeta)$.

Proof. Suppose

$$p(\zeta)=\left(\frac{2(1+sw(\zeta))}{2+s(1-q)w(\zeta)}\right)^2, \tag{4.2}$$

where $w(\zeta)$ is analytic in U with $w(0)=0$. Let

$$H_1(\zeta)=1+\alpha\frac{\zeta D_q p(\zeta)}{p(\zeta)}.$$

Then

$$|H_1(\zeta)-1|=|\alpha|\left|\frac{\zeta D_q p(\zeta)}{p(\zeta)}\right|<1-\left(\frac{2(1+s)}{2+s(1-q)}\right)^2. \tag{4.3}$$

By q-logarithmic differential equation of (4.2), we have

$$\frac{\zeta D_q p(\zeta)}{p(\zeta)}=\frac{2(1+q)s\zeta D_q w(\zeta)}{(1+sw(\zeta))(2+s(1-q)w(\zeta))}.$$

Therefore,

$$|H_1(\zeta)-1|=\frac{2|\alpha|(1+q)s|\zeta D_q w(\zeta)|}{|1+sw(\zeta)|\,|2+s(1-q)w(\zeta)|}.$$

To show that $p(\zeta)\prec\mathbb{L}_{q,s}(\zeta)$, it is enough to prove that $|w(\zeta)|<1$ for all $\zeta\in U$. Assume there exists $\zeta_0\in U$ such that $\max\limits_{|\xi|\le|\zeta_0|}|w(\zeta)|=|w(\zeta_0)|=1$, then by Lemma 1, $\zeta_0 D_q w(\zeta_0)=kw(\zeta_0)$, for some $k\ge 1$. Thus, for $w(\zeta_0)=e^{1\theta}$, $-\pi\le\theta\le\pi$,

$$\begin{aligned}|H_1(\zeta_0)-1|&=\frac{2|\alpha|(1+q)s|\zeta_0 D_q w(\zeta_0)|}{|1+sw(\zeta_0)|\,|2+s(1-q)w(\zeta_0)|}\\&=\frac{2|\alpha|(1+q)sk}{|1+sw(\zeta_0)|\,|2+s(1-q)w(\zeta_0)|}\ge\frac{2|\alpha|(1+q)sk}{(1+s)(2+s(1-q))|}.\end{aligned}$$

In view of (4.1), we have a contradiction. Hence, there exists no $\zeta_0\in U$ such that $|w(\zeta_0)|=1$. Thus, $|w(\zeta)|<1$ for all $\zeta\in U$. This completes the proof. $\square$

Let $f(\zeta)\in\mathcal{A}$ and setting $p(\zeta)=\frac{\zeta D_q f(\zeta)}{f(\zeta)}$ and $p(\zeta)=\frac{D_q(\zeta D_q f(\zeta))}{D_q f(\zeta)}$ in Theorem 6, respectively, we are led to the following result.

Corollary 5. *Let $f\in\mathcal{A}$ and suppose*

$$|\alpha|\ge\frac{(1+s)(4-s(3-q))}{2(2-s(1-q))},$$

then

$$1+\alpha\left(\frac{q\zeta D_q^2 f(\zeta)}{D_q f(\zeta)}+1-\frac{\zeta D_q f(\zeta)}{f(\zeta)}\right)\prec\mathbb{L}_{q,s}(\zeta)\Longrightarrow f\in ST_{\mathcal{L}_q}(s),$$

and similarly

$$1+\alpha\left(\frac{q^2\zeta^2D_q^3 f(\zeta)+\zeta D_q^2 f(\zeta)-D_q f(\zeta)}{q\zeta D_q^2 f(\zeta)+D_q f(\zeta)}+1-\frac{\zeta D_q^2 f(\zeta)}{D_q f(\zeta)}\right)\prec\mathbb{L}_{q,s}(\zeta)\Longrightarrow f\in CV_{\mathcal{L}_q}(s).$$

Theorem 7. *Assume that*

$$|\alpha| \geq \frac{2(4+s(q-3))(1+s)^3}{(2+s(1-q))(2-s(1-q))^2}, \tag{4.4}$$

and $p(\zeta)$ is an analytic function in U obeying the subordination condition

$$1+\alpha\frac{\zeta D_q p(\zeta)}{p^2(\zeta)} \prec \mathbb{L}_{q,s}(\zeta).$$

Then $p(\zeta) \prec \mathbb{L}_{q,s}(\zeta)$.

Proof. Consider

$$p(\zeta) = \left(\frac{2(1+sw(\zeta))}{(2+s(1-q)w(\zeta))}\right)^2, \tag{4.5}$$

where $w(\zeta)$ is analytic in U with $w(0)=0$. Then

$$H_2(\zeta) := 1+\alpha\frac{\zeta D_q p(\zeta)}{p^2(\zeta)} = 1+\frac{\alpha s(1+q)(2+s(1-q)w(\zeta))\zeta D_q w(\zeta)}{2(1+sw(\zeta))^3}.$$

To show that $p(\zeta) \prec \mathbb{L}_{q,s}(\zeta)$, it suffices to establish that $|w(\zeta)|<1$ for all $\zeta \in U$. If otherwise, there is a $\zeta_0 \in U$ such that $\max\limits_{|\xi|\leq|\zeta_0}| = |w(\zeta_0)=1$. By Lemma 1, there is a $k \geq 1$ such that $\zeta_0 D_q w(\zeta_0) = kw(\zeta_0)$. Let $w(\zeta_0)=e^{i\theta}$, $-\pi \leq \theta \leq \pi$, then

$$|H_2(\zeta_0)-1| = \frac{|\alpha|(1+q)s|2+s(1-q)e^{i\theta}|k}{2|1+se^{i\theta}|^3} = \frac{s|\alpha|(1+q)}{2}\left[\frac{4+(4s\cos\theta+s^2)(1-q)}{(1+2s\cos\theta+s^2)^3}\right]^{\frac{1}{2}}$$
$$= \frac{s|\alpha|(1+q)}{2}\varphi(\theta),$$

where

$$\varphi(\theta) = \sqrt{\frac{4+(4s\cos\theta+s^2)(1-q)}{(1+2s\cos\theta+s^2)^3}}.$$

Since $\varphi(\theta)$ is an even function, it is enough to analyze $\varphi(\theta)$ for $\theta \in [0,\pi]$. Therefore,

$$\varphi'(\theta) = \frac{\left[(3q^2-4q+1)s^2+8(1-q)s+2(q+s)\right]s\sin\theta}{(2s\cos\theta+1+s^3)^3\sqrt{4+4(1-q)s\cos\theta+(1-q)^2s^2}}.$$

It is simple to see that $\varphi'(\theta)=0$ when $\theta=0,\pi$. Similarly, we find that $\varphi''(0)=g(s)>0$ and $\varphi''(\pi)=g(-s)<0$, where

$$g(s) = \frac{s\left[10+2q+8(1-q)s+(1-q)(1-3q)s^2\right]}{(2-s(1-q))(1+s)^5}.$$

Hence, $\varphi(\theta) > \varphi(0)$. Therefore,

$$|H_2(\zeta_0-1| \geq \frac{|\alpha|(1+q)}{2}\varphi(0)$$

$$= \frac{|\alpha|(1+q)(2+s(1-q))}{2(1+s)^3}.$$

Utilizing (4.4), we arrive at a contradiction. Thus, there exists no $\zeta_0 \in U$ such that $|w(\zeta_0)| = 1$. Consequently, $|w(\zeta)| < 1$ for all $\zeta \in U$. Hence, $p(\zeta) \prec \mathbb{L}_{q,s}(\zeta)$.

Let $f(\zeta) \in \mathcal{A}$ and putting $p(\zeta) = \frac{\zeta D_q f(\zeta)}{f(\zeta)}$ and $p(\zeta) = \frac{D_q(\zeta D_q f(\zeta))}{D_q f(\zeta)}$ in Theorem 7, respectively, we arrive at the following corollary. □

Corollary 6. *Let $f \in \mathcal{A}$ and suppose*

$$|\alpha| \geq \frac{2(4+s(q-3))(1+s)^3}{(2+s(1-q))(2-s(1-q))^2},$$

then

$$1+\alpha\frac{f(\zeta)}{\zeta D_q f(\zeta)}\left(\frac{q\zeta D_q^2 f(\zeta)}{D_q f(\zeta)}+1-\frac{\zeta D_q f(\zeta)}{f(\zeta)}\right) \prec \mathbb{L}_{q,s}(\zeta) \Longrightarrow f \in ST_{\mathcal{L}_q}(s)$$

and similarly

$$1+\alpha\frac{D_q f(\zeta)}{D_q(\zeta D_q f(\zeta))}\left(\frac{q^2\zeta^2 D_q^3 f(\zeta)+\zeta D_q^2 f(\zeta)-D_q f(\zeta)}{q\zeta D_q^2 f(\zeta)+D_q f(\zeta)}+1-\frac{\zeta D_q^2 f(\zeta)}{D_q f(\zeta)}\right) \prec \mathbb{L}_{q,s}(\zeta)$$

implies $f \in CV_{\mathcal{L}_q}(s)$.

Theorem 8. *If*

$$|\alpha| \geq \frac{(A-B)(1+s)(2+s(1-q))}{2s(1+q)(1-|B|)}, \quad -1 < B < A \leq 1, \tag{4.6}$$

and $p(\zeta)$ is analytic in U satisfying

$$1+\alpha\frac{\zeta D_q p(\zeta)}{p(\zeta)} \prec \frac{1+A\zeta}{1+B\zeta}$$

then $p(\zeta) \prec \mathbb{L}_{q,s}(\zeta)$.

Proof. Following the proof of Theorem 6, we have

$$H_3(\zeta); = 1+\alpha\frac{\zeta D_q p(\zeta)}{p(\zeta)} = 1+\alpha\frac{2(1+q)s\zeta D_q w(\zeta)}{(1+sw(\zeta))(2+s(1-q)w(\zeta))}.$$

Hence,

$$\left|\frac{H_3(\zeta)-1}{A-BH_3(\zeta)}\right| < 1.$$

But

$$\left|\frac{H_3(\zeta)-1}{A-BH_3(\zeta)}\right| = \left|\frac{2\alpha(1+q)s\zeta D_q w(\zeta)}{(A-B)(1+sw(\zeta))(2+s(1-q)w(\zeta))-2B\alpha(1+q)s\zeta D_q w(\zeta)}\right|.$$

To establish our result, we need to show that $|w(\zeta)| < 1$ in U. Suppose there is $\zeta_0 \in U$ such that $\max_{|\xi|\leq|\zeta_0} |w(\zeta)| = |w_{\zeta_0}| = 1$. Using Lemma 1, $\zeta_0 D_q w(\zeta_0) = kw(\zeta_0)$, we have for $w(\zeta_0) = e^{i\theta}$, $\theta \in [-\pi, \pi]$,

$$\begin{aligned}\left|\frac{H_3(\zeta_0)-1}{A-BH_3(\zeta_0)}\right| &= \left|\frac{2\alpha(1+q)ske^{i\theta}}{(A-B)(1+se^{i\theta})(2+s(1-q)e^{i\theta})-2B\alpha(1+q)ske^{i\theta}}\right| \\ &\geq \frac{2\alpha(1+q)sk}{(A-B)(1+s)(2+s(1-q))+2|B||\alpha|(1+q)sk} \\ &:= \varphi(k),\end{aligned}$$

where $\varphi(k)$ is an increase function of k on $[1, \infty)$. Thus

$$\left|\frac{H_3(\zeta_0)-1}{A-BH_3(\zeta_0)}\right| \geq \varphi(1) = \frac{2|\alpha|(1+q)s}{(A-B)(1+s)(2+s(1-q))+2|\alpha||B|(1+q)s}.$$

Applying the condition (4.6) leads to a contradiction. Hence, we have the desire result. ☐

Let $f(\zeta) \in \mathcal{A}$ and setting $p(\zeta) = \frac{\zeta D_q f(\zeta)}{f(\zeta)}$ and $p(\zeta) = \frac{D_q(\zeta D_q f(\zeta))}{D_q f(\zeta)}$ in Theorem 8, respectively, we have the following result.

Corollary 7. *Let $f \in \mathcal{A}$ and suppose*

$$|\alpha| \geq \frac{(A-B)(1+s)(2-s(1-q))}{2s(1+q)(1-|B|)}, \quad -1 < B < A \leq 1.$$

then

$$1+\alpha\left(\frac{q\zeta D_q^2 f(\zeta)}{D_q f(\zeta)}+1-\frac{\zeta D_q f(\zeta)}{f(\zeta)}\right) \prec \mathbb{L}_{q,s}(\zeta) \Longrightarrow f \in ST_{\mathcal{L}_q}(s)$$

and similarly

$$1+\alpha\left(\frac{q^2\zeta^2 D_q^3 f(\zeta)+\zeta D_q^2 f(\zeta)-D_q f(\zeta)}{q\zeta D_q^2 f(\zeta)+D_q f(\zeta)}+1-\frac{\zeta D_q^2 f(\zeta)}{D_q f(\zeta)}\right) \prec \mathbb{L}_{q,s}(\zeta) \Longrightarrow f \in CV_{\mathcal{L}_q}(s).$$

Theorem 9. *If*

$$|\alpha| > \frac{2(A-B)(1+s)^3}{(1+q)s(2-s(1-q)(1-|B|))} \tag{4.7}$$

and

$$1+\alpha\frac{\zeta D_q p(\zeta)}{p^2(\zeta)} \prec \frac{1+A\zeta}{1+B\zeta}, \quad -B < A \leq 1,$$

then $p(\zeta) \prec \mathbb{L}_{q,s}(\zeta)$.

Proof. The proof follows the same techniques of Theorem 7. ☐

5. CONCLUSION

In this work, we had successfully studied subfamilies of Ma and Mind classes of functions that were endowed with q- calculus. At first, we introduced a q-Limaçon function and investigated some of its geometric properties. Furthermore, the classes of q-Limaçon starlike and convex functions were initiated. The radius of inclusion relationship between these classes and that of q starlike and convex functions were established. Few differential subordination results associated with these new families were proved. Also, we presented a few consequences of our findings.

Moreover, to have more new theorems under present examinations, new generalization and applications can be explored with some positive and novel outcomes in various fields of science, especially, in applied mathematics. These new surveys will be presented in future research work being processed by the authors of the present paper.

However, the purported trivial (p,q)-calculus extension was clearly demonstrated to be relatively insignificant and inconsequential variation of the classical q-calculus, the extra parameter p being redundant or superfluous (see, for details, [17, p. 340] and [18, pp. 1511-1512]). This observation by Srivastava (see [17] and [18]) will indeed apply also to any future attempt to produce the rather straightforward (p,q)-variants of the results which we have presented in this paper.

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