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APPROXIMATION BY APOSTOL-GENOCCHI SUMMATION-INTEGRAL TYPE OPERATORS

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Abstract. In the present paper, we proposed a new sequence of summation-integral type operators involving Apostol-Genocchi polynomials. We study some approximation results of the proposed operators using first and second-order modulus of continuity, the global rate of convergence using Voronovskaja-type asymptotic theorem, Lipschitz-type space, and Ditzian-Totik modulus of smoothness. Lastly, we study the weighted approximation.

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1. Introduction

The Appell polynomial $A_n(x)$ is defined by

$$e^{xt}f(t)=\sum_{n=0}^{\infty}A_n(x)\frac{t^n}{n!},$$

where $f(t)$ be a formal power series in t. This polynomial found stunning applications in several field of mathematics, see [7, 21, 22]. One special case known as Genocchi polynomials $G_n(x)$ are defined by

$$\frac{2te^{xt}}{e^t+1}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},\ |t|<\pi.$$

In this case the Genocchi numbers G_n have many applications in number theory, special functions, combinatorics and numerical analysis, where

$$\frac{2t}{e^t+1}=\sum_{n=0}^{\infty}G_n\frac{t^n}{n!},\ |t|<\pi.$$

The Apostol-Genocchi polynomials $G_k^{\alpha}(x;\beta)$, $\beta\in\mathbb{C}$, of order α (non negative integers) are defined and found some results in [7].

Recently, Prakash et al. [15] proposed a sequence of operators involving Apostol-Genocchi polynomials given in (1.1). They studied some approximation results using second-order modulus of continuity, Voronovskaja-type theorem and weighted approximation theorem for these operators. They also proposed the Kantorovich form of (1.1), and discussed some approximation results. Many researchers have worked on new modifications of various operators and deliberate the approximation properties ([1, 3, 9, 10, 16, 17, 19, 22–24]). The operators discussed in [15] are given as follows. For $f \in C[0,\infty)$,

$$\mathcal{M}_n^{\alpha,\beta}(f;x) = e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha}\sum_{k=0}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}f\left(\frac{k}{n}\right), \tag{1.1}$$

where $G_k^{\alpha}(x;\beta)$ are the generalized Apostol-Genocchi polynomials, with the generating function of the form

$$\left(\frac{2t}{1+\beta e^t}\right)^{\alpha} e^{xt} = \sum_{k=0}^{\infty} G_k^{\alpha}(x;\beta)\frac{t^k}{k!}, \qquad (\beta \in \mathbb{C}, \alpha \in \mathbb{N}\cup 0, |t| < \pi).$$

For more information about the Apostol-Genocchi polynomials and their properties, readers should refer articles [6, 8, 11, 12, 14, 18, 20].

In 2007, Srivastava et al. proposed a family of summation-integral type operators and estimated rate of convergence and function having derivative of bounded variation [20]. Here we refer to some more articles related to summation-integral type operators for the readers ([4, 5, 13]). Motivated by the above work, for non negative integer α and $f \in L_1[0,\infty)$ we propose the integral-type generalization of the operators (1.1) as follows:

$$T_{n,\rho}^{\alpha,\beta}(f,x) = e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha}\left(G_0^{\alpha}(nx;\beta)f(0) + \sum_{k=1}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}\int_0^{\infty}\Theta_{n,k}^{\rho}(t,c)f(t)dt\right), \tag{1.2}$$

where $\rho > 0$, and

$$\Theta_{n,k}^{\rho}(t,c) = \begin{cases} \dfrac{n\rho}{\Gamma(k\rho)}e^{-n\rho t}(n\rho t)^{k\rho-1}, & c = 0, \\ \dfrac{\Gamma\left(\frac{n\rho}{c}+k\rho\right)}{\Gamma(k\rho)\Gamma\left(\frac{n\rho}{c}\right)}\dfrac{c^{k\rho}t^{k\rho-1}}{(1+ct)^{\frac{n\rho}{c}+k\rho}}, & c = 1,2,3,\ldots \end{cases}$$

It can be easily observed by simple computation that

$$\int_0^{\infty}\Theta_{n,k}^{\rho}(t,c)t^r dt = \begin{cases} \frac{\Gamma(k\rho+r)}{\Gamma(k\rho)}\dfrac{1}{\prod_{i=1}^{r}(n\rho - ic)}, & r \neq 0 \\ 1, & r = 0. \end{cases} \tag{1.3}$$

2. LEMMAS

In this section we deal with some useful lemmas and results.

Lemma 1 ([15]). *For $\mathcal{M}_n^{\alpha,\beta}(t^i;x)$, $i = \overline{0,4}$, we have $\mathcal{M}_n^{\alpha,\beta}(1;x) = 1$;*
$\mathcal{M}_n^{\alpha,\beta}(t;x) = x + \dfrac{\alpha}{n(1+e\beta)}$;

$$\mathcal{M}_n^{\alpha,\beta}(t^2;x) = x^2 + \frac{(1+2\alpha+e\beta)}{n(1+e\beta)}x + \frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{n^2(1+e\beta)^2};$$

$$\mathcal{M}_n^{\alpha,\beta}(t^3;x) = x^3 + \frac{(3+3\alpha+3e\beta)}{n(1+e\beta)}x^2 + \frac{\left(3\alpha^2+3\alpha+e^2\beta^2-3\alpha e^2\beta^2-3\alpha e\beta+2e\beta+1\right)}{n^2(1+e\beta)^2}x$$
$$+\frac{\left(\alpha^3-6\alpha^2 e\beta-3\alpha^2e^2\beta^2-5\alpha e\beta-4\alpha e^2\beta^2-\alpha e^3\beta^3\right)}{n^3(1+e\beta)^3};$$

$$\mathcal{M}_n^{\alpha,\beta}(t^4;x) = x^4 + \frac{(3+2\alpha+3e\beta)}{n(1+e\beta)}x^3 + \frac{\left(6\alpha^2+25e^2\beta^2-50e\beta-6\alpha e^2\beta^2+12\alpha+25\right)}{n^2(1+e\beta)^2}x^2$$
$$+\frac{x}{n^3(1+e\beta)^3}(4\alpha^3+6\alpha^2+42e\beta+48e\alpha\beta-18e\alpha^2\beta+42e^2\beta^2+6e^2\alpha\beta^2$$
$$-12e^2\alpha^2\beta^2+14e^3\beta^3-10e^3\alpha\beta^2+40\alpha+14)$$
$$+\frac{1}{n^4(1+e\beta)^4}(16\alpha^4-1056e\alpha\beta+256e\alpha^2\beta-192e\alpha^3\beta-1888e^2\alpha\beta^2$$
$$+224e^2\alpha^2\beta^2-96e^2\alpha^3\beta^2-1312e^3\alpha\beta^3+128e^3\alpha^2\beta^3-304e^4\alpha\beta^4$$
$$+48e^4\alpha^2\beta^4+288\alpha^2-80\alpha).$$

Lemma 2. *The moments of the proposed operator* $T_{n,\rho}^{\alpha,\beta}\left(t^i,x\right)$, $i=\overline{0,3}$, *we have*

$$T_{n,\rho}^{\alpha,\beta}(1,x) = 1;$$
$$T_{n,\rho}^{\alpha,\beta}(t,x) = \frac{n\rho}{(n\rho-c)}x + \frac{\alpha\rho}{(n\rho-c)(1+e\beta)};$$
$$T_{n,\rho}^{\alpha,\beta}\left(t^2,x\right) = \frac{n^2\rho^2}{(n\rho-c)(n\rho-2c)}x^2 + \frac{n\rho^2+2n\alpha\rho^2+n\rho^2e\beta}{(n\rho-c)(n\rho-2c)(1+e\beta)}x$$
$$+\frac{\alpha^2\rho^2-2\alpha\rho^2e\beta-\alpha\rho^2e^2\beta^2}{(n\rho-c)(n\rho-2c)(1+e\beta)^2};$$
$$T_{n,\rho}^{\alpha,\beta}\left(t^3,x\right) = \frac{n^3\rho^3}{(n\rho-c)(n\rho-2c)(n\rho-3c)}x^3$$
$$+\frac{3n^2\rho^2+3n^2\rho^3+3\alpha n^2\rho^3+3n^2\rho^2e\beta+3n^2\rho^3e\beta}{(n\rho-c)(n\rho-2c)(n\rho-3c)(1+e\beta)}x^2$$
$$+\frac{x}{(n\rho-c)(n\rho-2c)(n\rho-3c)(1+e\beta)^2}\left[n\alpha^2\rho^3+9n\alpha\rho^2\right.$$
$$\left.+n\rho^2e^2\beta^2(3-2n\rho)+3n\alpha\rho^2e\beta+n\rho(1+e\beta)^2+8n\rho^2e\beta+3n\rho^2+n\rho\right]$$
$$+\frac{1}{(n\rho-c)(n\rho-2c)(n\rho-3c)(1+e\beta)^3}\left[\alpha^3\rho^3+3\alpha^2\rho^2e\beta(1-2\rho-\rho e\beta)\right.$$
$$+3\alpha^2\rho^2-5\alpha\rho^3e\beta-\alpha\rho^2e^2\beta^2(4\rho+\rho e\beta+9e\beta+3)-6\alpha\rho^2e\beta+2\alpha\rho$$
$$\left.+2\alpha\rho e^2\beta^2+4\alpha\rho e\beta\right].$$

The proof the above lemma follows from (1.2), (1.3) and Lemma 1.

Remark 1. From Lemma 2 and simple estimation, we have

$$T_{n,\rho}^{\alpha,\beta}(t-x,x)=\frac{cx}{(n\rho-c)}+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)};$$

$$T_{n,\rho}^{\alpha,\beta}\left((t-x)^2,x\right)=\frac{n\rho c+2c^2}{(n\rho-c)(n\rho-2c)}x^2+\frac{n\rho^2+n\rho^2e\beta+4c\alpha\rho}{(n\rho-c)(n\rho-2c)(1+e\beta)}x$$
$$+\frac{\alpha^2\rho^2-2\alpha\rho^2e\beta-\alpha\rho^2e^2\beta^2}{(n\rho-c)(n\rho-2c)(1+e\beta)^2}.$$

Remark 2. For the central moments $T_{n,\rho}^{\alpha,\beta}((t-x)^m;x)$ for $m=1,2$, we have

$$\lim_{n\to\infty} nT_{n,\rho}^{\alpha,\beta}((t-x);x)=\frac{xc}{\rho}+\frac{\alpha}{(1+e\beta)};$$

$$\lim_{n\to\infty} nT_{n,\rho}^{\alpha,\beta}((t-x)^2;x)=\frac{x(cx+\rho)}{\rho}.$$

For $n\in\mathbb{N}$, $T_{n,\rho}^{\alpha,\beta}((t-x)^2;x)\leq\frac{2\phi^2(x)}{n\rho}$, where $\phi(x)=\sqrt{x(cx+\rho)}$.

Let $C_B[0,\infty)$ be the set of all continuous and bounded functions on $[0,\infty)$ endowed with the norm $\|f\|=\sup\{|f(x)|:x\in[0,\infty)\}$.

Lemma 3. *Let $f\in C_B[0,\infty)$, $0\leq x<\infty$ and $n\in\mathbb{N}$, then*

$$|T_{n,\rho}^{\alpha,\beta}(f;x)|\leq\|f\|,$$

where $\|.\|$ is the uniform norm on $[0,\infty)$.

Proof. We have $T_{n,\rho}^{\alpha,\beta}(e_0;x)=1$, so $|T_{n,\rho}^{\alpha,\beta}(f;x)|\leq T_{n,\rho}^{\alpha,\beta}(e_0;x)\|f\|=\|f\|$. □

3. Direct Results

In this section we establish the uniform convergence of the operators (1.2) using the Bohman-Korovkin theorem, the rate of convergence with the aid of different kind of modulus of smoothness and for functions in Lipschitz-type spaces, and a Voronovskaja-type asymptotic theorem.

Theorem 1. *Let $f\in C[0,\infty)$ and an adequately large n, then the sequence $\{T_{n,\rho}^{\alpha,\beta}(f,.)\}$ converges uniformly to f in $[a,b]$, where $0\leq a<b<\infty$.*

Proof. From Lemma 2 we have $T_{n,\rho}^{\alpha,\beta}(1,x)=1$ for every $n\in\mathbf{N}$,

$$T_{n,\rho}^{\alpha,\beta}(t,x)=\frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}$$

tends to x and

$$T_{n,\rho}^{\alpha,\beta}\left(t^2,x\right) = \frac{n^2\rho^2}{(n\rho-c)(n\rho-2c)}x^2 + \frac{n\rho^2+2n\alpha\rho^2+n\rho^2 e\beta}{(n\rho-c)(n\rho-2c)(1+e\beta)}x + \frac{\alpha^2\rho^2-2\alpha\rho^2 e\beta-\alpha\rho^2 e^2\beta^2}{(n\rho-c)(n\rho-2c)(1+e\beta)^2}$$

tends to x^2 as $n\to\infty$, similarly $T_{n,\rho}^{\alpha,\beta}\left(t^3,x\right)$ tends to x^3 uniformly on every compact subset of $[0,\infty)$. Hence, by Bohman-Korovkin theorem the required result holds. □

Consider the K-functional

$$K_2(f,\delta) = \inf_{j\in W^2}\{\|f-j\|+\delta\|j''\|\}, \tag{3.1}$$

where $\delta>0$ and $W^2=\{j\in C_B[0,\infty): j',j''\in C_B[0,\infty)\}$. For a constant $C>0$ such that

$$K_2(f,\delta)\le C\,\omega_2(f,\sqrt{\delta}), \tag{3.2}$$

where $f\in C_B[0,\infty)$ and

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\le\sqrt{\delta}}\ \sup_{x\in[0,\infty)} |f(x+2h)-2f(x+h)+f(x)|$$

is the second order modulus of smoothness.

Theorem 2. *Let $f\in C_B[0,\infty)$ and $x\in[0,\infty)$. Then we have*

$$\left|T_{n,\rho}^{\alpha,\beta}(f,x)-f(x)\right| \le C\omega_2\left(f,\sqrt{\delta_n}\right)+\omega\left(f,|\alpha_2|\right),$$

where C is a positive constant, $\delta_n=\frac{1}{2}[\alpha_1+\alpha_2^2]$, $\alpha_1=T_{n,\rho}^{\alpha,\beta}\left((t-x)^2,x\right)$ and

$$\alpha_2=\left(\frac{cx}{n\rho-c}+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}\right).$$

Proof. Define the auxiliary operators $L_{n,\rho}^{\alpha,\beta}: C_B[0,\infty)\to C_B[0,\infty)$ as follows:

$$L_{n,\rho}^{\alpha,\beta}(f,x) = T_{n,\rho}^{\alpha,\beta}(f,x) - f\left(\frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}\right)+f(x). \tag{3.3}$$

These operators are linear and $L_{n,\rho}^{\alpha,\beta}(t-x,x)=0$.

Let $j\in W^2$ and $x,t\in[0,\infty)$. By Taylor's series expansion

$$j(t)=j(x)+(t-x)j'(x)+\int_x^t (t-u)j''(u)du.$$

Applying the operator $L_{n,\rho}^{\alpha,\beta}$ on above, we obtain

$$L_{n,\rho}^{\alpha,\beta}(j;x) = j(x)+j'(x)L_{n,\rho}^{\alpha,\beta}((t-x);x)+L_{n,\rho}^{\alpha,\beta}\left(\int_x^t (t-u)j''(u)du;x\right)$$

implying that

$$\begin{aligned}\left|L_{n,\rho}^{\alpha,\beta}(j,x)-j(x)\right| &\leq L_{n,\rho}^{\alpha,\beta}\left(\left|\int_x^t (t-u)j''(u)du\right|,x\right)\\ &\leq T_{n,\rho}^{\alpha,\beta}\left(\left|\int_x^t (t-u)j''(u)du\right|,x\right)\\ &\quad+\left|\int_x^{\frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}}\left(\frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}-u\right)j''(u)du\right|\\ &\leq \frac{1}{2}T_{n,\rho}^{\alpha,\beta}\left((t-x)^2,x\right)||j''||\\ &\quad+\left|\int_x^{\frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}}\left(\frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}-u\right)du\right|||j''||\end{aligned}$$

$$\begin{aligned}\left|L_{n,\rho}^{\alpha,\beta}(j,x)-j(x)\right| &\leq \frac{1}{2}\left[T_{n,\rho}^{\alpha,\beta}\left((t-x)^2,x\right)+\left(\frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}-x\right)^2\right]||j''||\\ &\leq \frac{1}{2}\left[T_{n,\rho}^{\alpha,\beta}\left((t-x)^2,x\right)+\left(\frac{c}{n\rho-c}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}\right)^2\right]||j''||\\ &\leq \frac{1}{2}\left[\alpha_1+\alpha_2^2\right]||j''||=\delta_n||j''||. \end{aligned} \tag{3.4}$$

Since

$$\begin{aligned}\left|T_{n,\rho}^{\alpha,\beta}(f,x)\right| &\leq e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha}\left(G_0^{\alpha}(nx;\beta)|f(0)|+\left|\sum_{k=1}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}\int_0^{\infty}\Theta_{n,k}^{\rho}(t,c)f(t)dt\right|\right)\\ &\leq ||f||.\end{aligned}$$

Now by (3.3), we have

$$||L_{n,\rho}^{\alpha,\beta}(f,.)|| \leq ||T_{n,\rho}^{\alpha,\beta}(f,.)||+2||f|| \leq 3||f||,\ f\in C_B[0,\infty). \tag{3.5}$$

Using (3.3), (3.4) and (3.5), we have

$$\begin{aligned}\left|T_{n,\rho}^{\alpha,\beta}(f,x)-f(x)\right| &\leq \left|L_{n,\rho}^{\alpha,\beta}(f-j,x)-(f-j)(x)\right|+\left|L_{n,\rho}^{\alpha,\beta}(j,x)-j(x)\right|\\ &\quad+\left|f\left(\frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}\right)-f(x)\right|\\ &\leq 4||f-j||+\delta_n||j''||+\left|f(x)-f\left(\frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}\right)\right|\\ &\leq 4||f-j||+\delta_n||j''||+\omega(f,\alpha_2).\end{aligned}$$

Taking infimum over all $j\in W^2$, and using (3.2), we get the required result. □

Our next result is the Voronovskaja-type asymptotic formula.

Theorem 3. *Let f be a bounded and integrable function on $[0,\infty)$ such that $f''(x)$ exists at $0 \le x < \infty$, then*

$$\lim_{n\to\infty} n\left[T_{n,\rho}^{\alpha,\beta}(f,x) - f(x)\right] = \left[\frac{cx}{\rho} + \frac{\alpha}{(1+e\beta)}\right] f'(x) + \left[\frac{c}{\rho}x^2 + x\right]\frac{f''(x)}{2!}.$$

Proof. Using the well known Taylor's series expansion

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + \sigma(t,x)(t-x)^2,$$

where $\sigma(t,x) \to 0$ as $t \to x$ and the function ρ is bounded on $[0,\infty)$. Now,

$$\begin{aligned}
& n\left[T_{n,\rho}^{\alpha,\beta}(f,x) - f(x)\right] \\
&= n\left[T_{n,\rho}^{\alpha,\beta}(t-x,x) f'(x) + \frac{T_{n,\rho}^{\alpha,\beta}\left((t-x)^2,x\right)}{2!} f''(x) + T_{n,\rho}^{\alpha,\beta}\left(\sigma(t,x)(t-x)^2,x\right)\right] \\
&= n\left[\left(\frac{cx}{(n\rho-c)} + \frac{\alpha\rho}{(n\rho-c)(1+e\beta)}\right) f'(x) + \frac{n\rho c + 2c^2}{(n\rho-c)(n\rho-2c)}\frac{x^2}{2!}f''(x)\right. \\
&\quad \left. + \frac{n\rho^2 + n\rho^2 e\beta + 4c\alpha\rho}{(n\rho-c)(n\rho-2c)(1+e\beta)}\frac{x}{2!}f''(x) + \frac{\alpha^2\rho^2 - 2\alpha\rho^2 e\beta - \alpha\rho^2 e^2\beta^2}{2!(n\rho-c)(n\rho-2c)(1+e\beta)^2} f''(x)\right] \\
&\quad + \hbar(n,x),
\end{aligned}$$

where

$$\begin{aligned}
\hbar(n,x) &= e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha} \\
&\qquad \cdot \left(G_0^{\alpha}(nx;\beta)\,\sigma(0,x)(-x)^2 + \sum_{k=1}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}\int_0^{\infty}\Theta_{n,k}^{\rho}(t,c)\,\sigma(t,x)(t-x)^2 dt\right).
\end{aligned}$$

Now it is sufficient to show that $\hbar(n,x) \to 0$ as large n. Since $\sigma(t,x) \to 0$ as $t \to x$, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\sigma(t,x)| < \varepsilon$, whenever $-\delta < t - x < \delta$,

$$\begin{aligned}
|\hbar(n,x)| &\le e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha}\sum_{k=1}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!} \\
&\qquad \left[\int_{|t-x|<\delta}\Theta_{n,k}^{\rho}(t,c)\,|\sigma(t,x)|(t-x)^2 dt + \int_{|t-x|\ge\delta}\Theta_{n,k}^{\rho}(t,c)\,|\sigma(t,x)|(t-x)^2 dt\right] \\
&\qquad + e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha} G_0^{\alpha}(nx;\beta)\,\sigma(0,x)(-x)^2.
\end{aligned}$$

By using Remark 1, we have $I_1 = \sigma O(1)$ and for $m \ge 2$, we obtain

$$\begin{aligned}
I_2 &\le Ce^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha}\left(G_0^{\alpha}(nx;\beta)\,\sigma(0,x)(-x)^2 + \sum_{k=1}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}\int_0^{\infty}\Theta_{n,k}^{\rho}(t,c)\frac{(t-x)^{2m}}{\delta^{2m-2}}dt\right) \\
&= O(n^{-m+1}),
\end{aligned}$$

where $C = \sup_{t\in[x,\infty)} |\sigma(t,x)|$. Due to arbitrariness of $\varepsilon > 0$, $\hbar(n,x) \to 0$ for sufficiently large n, the proof is completed. □

In the following theorem, we succeed the rate of convergence of the proposed operators (1.2) for functions in Lipschitz-type spaces.

Suppose that $x \in (0,\infty)$, $t \in [0,\infty)$, and consider the Lipschitz-type space defined as

$$Lip_M^*(\sigma) = \left\{ f \in C[0,\infty) : |f(t) - f(x) \leq M\frac{|t-x|^{\sigma}}{(t+x)^{\frac{\sigma}{2}}} \right\},$$

where $M > 0$ is a constant and $0 < \sigma \leq 1$.

Theorem 4. *Let $f \in Lip_M^*(\sigma)$ and $0 < \sigma \leq 1$. Then for each $x \in (0,\infty)$, we have*

$$|T_{n,\rho}^{\alpha,\beta}(f(t);x) - f(x)| \leq M\left(\frac{T_{n,\rho}^{\alpha,\beta}((t-x)^2;x)}{x}\right)^{\frac{\sigma}{2}}.$$

Proof. Let $f \in Lip_M^*(\sigma)$ and $x \in (0,\infty)$, $t \in [0,\infty)$, we have

$$\begin{aligned} |T_{n,\rho}^{\alpha,\beta}(f(t);x) - f(x)| &\leq T_{n,\rho}^{\alpha,\beta}(|f(t) - f(x)|;x) \\ &\leq M\cdot T_{n,\rho}^{\alpha,\beta}\left(\frac{|t-x|^{\sigma}}{(t+x)^{\frac{\sigma}{2}}};x\right) \\ &\leq \frac{M}{x^{\frac{\sigma}{2}}} T_{n,\rho}^{\alpha,\beta}(|t-x|^{\sigma};x). \end{aligned} \tag{3.6}$$

Taking $p = \frac{2}{\sigma}$ and $q = \frac{2}{2-\sigma}$ and applying Hölder's inequality, we obtain

$$\begin{aligned} T_{n,\rho}^{\alpha,\beta}(|t-x|^{\sigma};x) &\leq \left\{T_{n,\rho}^{\alpha,\beta}(|t-x|^2;x)\right\}^{\frac{\sigma}{2}} \cdot \left\{T_{n,\rho}^{\alpha,\beta}(1^{\frac{2}{2-\sigma}};x)\right\}^{\frac{2-\sigma}{2}} \\ &\leq \left\{T_{n,\rho}^{\alpha,\beta}(|t-x|^2;x)\right\}^{\frac{\sigma}{2}}. \end{aligned} \tag{3.7}$$

Using (3.6) and (3.7), we get the required result

$$|T_{n,\rho}^{\alpha,\beta}(f(t);x) - f(x)| \leq M\left(\frac{T_{n,\rho}^{\alpha,\beta}((t-x)^2;x)}{x}\right)^{\frac{\sigma}{2}}.$$

□

Here, we estimate the convergence using the first order Ditzian-Totik modulus of smoothness $\omega_{\phi^{\gamma}}(f;t)$ and The Peetre $K-$functional $K_{\phi^{\gamma}}(f;t)$, $0 \leq \gamma \leq 1$. The Ditzian-Totik modulus of smoothness is defined as

$$\omega_{\phi^{\gamma}}(f;t) = \sup_{0<h\leq t}\left\{\left|f\left(x+\frac{h\phi^{\gamma}(x)}{2}\right) - f\left(x-\frac{h\phi^{\gamma}(x)}{2}\right)\right|, x \pm \frac{h\phi^{\gamma}(x)}{2} \in [0,\infty)\right\},$$

where $\phi^2(x) = x(cx+\rho)$ and $f \in C[0,\infty)$.

The Peetre $K-$functional is defined as

$$K_{\phi^\gamma}(f;t) = \inf_{j\in W_\gamma}\{\|f-j\| + t\|\phi^\gamma j'\|\}\,(t>0),$$

where $W_\gamma[0,\infty) = \{j : j \in AC_{loc}[0,\infty),\, \|\phi^\gamma j'\| < \infty\}$ and $AC_{loc}[0,\infty)$ is the class of the locally absolutely continuous functions. In [2], (Theorem 2.1.1), there is a relation between K-functional and the Ditzian-Totik first order modulus of smoothness

$$C^{-1}\omega_{\phi^\gamma}(f,t) \le K_{\phi^\gamma}(f;t) \le C\omega_{\phi^\gamma}(f;t), \tag{3.8}$$

where $C > 0$ is a constant.

Theorem 5. *For $f \in C_B[0,\infty)$ and $x \in (0,\infty)$, we have*

$$|T_{n,\rho}^{\alpha,\beta}(f(t);x) - f(x)| \le C\omega_{\phi^\gamma}\left(f;\frac{\phi^{1-\gamma}(x)}{\sqrt{n\rho}}\right),$$

for sufficiently large n and constant $C > 0$, which is independent of f and n.

Proof. For $j \in W_\gamma$, we get

$$j(t) = j(x) + \int_x^t j'(u)du. \tag{3.9}$$

Applying $T_{n,\rho}^{\alpha,\beta}$ in (3.8) and using Hölder's inequality, we obtain

$$\begin{aligned} |T_{n,\rho}^{\alpha,\beta}(j(t);x) - j(x)| &\le T_{n,\rho}^{\alpha,\beta}\left(\int_x^t |j'|du;x\right) \\ &\le \|\phi^\gamma j'\| T_{n,\rho}^{\alpha,\beta}\left(\left|\int_x^t \frac{du}{\phi^\gamma(u)}\right|;x\right) \\ &\le \|\phi^\gamma j'\| T_{n,\rho}^{\alpha,\beta}\left(|t-x|^{1-\gamma}\left|\int_x^t \frac{du}{\phi(u)}\right|^\gamma;x\right). \end{aligned} \tag{3.10}$$

Take $A = \left|\int_x^t \frac{du}{\phi(u)}\right|$, we find

$$\begin{aligned} A &\le \left|\int_x^t \frac{du}{\sqrt{u}}\right|\left|\left(\frac{1}{\sqrt{cx+\rho}} + \frac{1}{\sqrt{ct+\rho}}\right)\right| \\ &\le 2|\sqrt{t}-\sqrt{x}|\left(\frac{1}{\sqrt{cx+\rho}} + \frac{1}{\sqrt{ct+\rho}}\right) \\ &\le 2\frac{|t-x|}{\sqrt{x}+\sqrt{t}}\left(\frac{1}{\sqrt{cx+\rho}} + \frac{1}{\sqrt{ct+\rho}}\right) \\ &\le 2\frac{|t-x|}{\sqrt{x}}\left(\frac{1}{\sqrt{cx+\rho}} + \frac{1}{\sqrt{ct+\rho}}\right). \end{aligned} \tag{3.11}$$

Using the inequality $|a+b|^{\gamma} \leq |a|^{\gamma}+|b|^{\gamma}$, $0 \leq \gamma \leq 1$ and (3.11), we get

$$\| \int_x^t \frac{du}{\phi(u)} \|^{\gamma} \leq 2^{\beta} \frac{|t-x|^{\gamma}}{x^{\frac{\gamma}{2}}} \left(\frac{1}{(cx+\rho)^{\frac{\gamma}{2}}} + \frac{1}{(ct+\rho)^{\frac{\gamma}{2}}} \right). \tag{3.12}$$

From (3.10), (3.12) and using Cauchy inequality, we get

$$\begin{aligned} |T_{n,\rho}^{\alpha,\beta}(j(t);x)-j(x)| &\leq \frac{2^{\gamma}\|\phi^{\gamma}j'\|}{x^{\frac{\gamma}{2}}} T_{n,\rho}^{\alpha,\beta}\left(|t-x| \left(\frac{1}{(cx+\rho)^{\frac{\gamma}{2}}} + \frac{1}{(ct+\rho)^{\frac{\gamma}{2}}} \right);x \right) \\ &\leq \frac{2^{\gamma}\|\phi^{\gamma}j'\|}{x^{\frac{\gamma}{2}}} \left(\frac{1}{(ct+\rho)^{\frac{\gamma}{2}}} (T_{n,\rho}^{\alpha,\beta}((t-x)^2;x))^{\frac{1}{2}} \right. \\ &\quad \left. +(T_{n,\rho}^{\alpha,\beta}\left((t-x)^2;x\right))^{\frac{1}{2}} .(T_{n,\rho}^{\alpha,\beta}\left((ct+\rho)^{-\gamma};x\right))^{\frac{1}{2}} \right). \end{aligned}$$

If n is adequately large, then we get

$$\left(T_{n,\rho}^{\alpha,\beta}\left((t-x)^2;x\right) \right)^{\frac{1}{2}} \leq \sqrt{\frac{2}{n\rho}}\phi(x), \tag{3.13}$$

where $\phi(x) = \sqrt{x(cx+\rho)}$.

For each $x \in (0,\infty)$, $T_{n,\rho}^{\alpha,\beta}\left((ct+\rho)^{-\gamma};x\right) \to (cx+\rho)^{-\gamma}$ as $n \to \infty$. For $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$T_{n,\rho}^{\alpha,\beta}\left((ct+\rho)^{-\gamma};x\right) \leq (cx+\rho)^{-\gamma}+\varepsilon, \ \forall\, n \geq n_0 = n_0(c,x,\rho,\beta).$$

By choosing $\varepsilon = (cx+\rho)^{-\gamma}$, we obtain

$$T_{n,\rho}^{\alpha,\beta}\left((ct+\rho)^{-\gamma};x\right) \leq 2(cx+\rho)^{-\gamma}, \ \forall\, n \geq n_0. \tag{3.14}$$

From (3.10) to (3.14), we get

$$\begin{aligned} |T_{n,\rho}^{\alpha,\beta}(j(t);x)-j(x)| &\leq 2^{\gamma}\|\phi^{\gamma}j'\| \sqrt{\frac{2}{n\rho}}\phi(x) \left(\phi^{-\gamma}(x) + \sqrt{2}x^{\frac{-\gamma}{2}}(cx+\rho)^{\frac{-\gamma}{2}} \right) \\ &\leq 2^{\gamma+\frac{1}{2}}(1+\sqrt{2})\|\phi^{\gamma}j'\| \sqrt{\frac{2}{n\rho}}\phi^{1-\gamma}(x). \end{aligned} \tag{3.15}$$

We may write

$$\begin{aligned} |T_{n,\rho}^{\alpha,\beta}(f(t);x)-f(x)| &\leq |T_{n,\rho}^{\alpha,\beta}(f(t)-j(x);x)| + |T_{n,\rho}^{\alpha,\beta}(j(t);x)-j(x)| + |j(x)-f(x)| \\ &\leq 2\|f-j\| + |T_{n,\rho}^{\alpha,\beta}(j(t);x)-j(x)|. \end{aligned} \tag{3.16}$$

From (3.15) to (3.16) and for adequately large n, we get

$$|T_{n,\rho}^{\alpha,\beta}(f(t);x)-f(x)| \leq 2\|f-j\| + 2^{\gamma+\frac{1}{2}}(1+\sqrt{2})\sqrt{\frac{2}{n\rho}}\phi^{1-\gamma}\|\phi^{\beta}j'\|$$

$$\leq m_1 \left\{ \|f-j\| + \frac{\phi^{1-\gamma}(x)}{\sqrt{n\rho}} \|\phi^\gamma j'\| \right\}$$

$$\leq CK_{\phi^\gamma}\left(f, \frac{\phi^{1-\gamma}(x)}{\sqrt{n\rho}}\right), \tag{3.17}$$

where $m_1 = \max(2, 2^{\gamma+\frac{1}{2}}(1+\sqrt{2})\sqrt{2})$, from (3.8) and (3.17) we obtain the result. □

4. Weighted Approximation

In this section, we will examine the weighted estimation hypothesis. Let $B_{x^2}[0,\infty) = \{f : \text{for every}\, x \in [0,\infty), |f(x)| \leq \varphi_f(1+x^2), \varphi_f \text{ is a constant depending on } f\}$. Here $C^*_{x^2}[0,\infty)$, denotes the subspace of all continuous functions belonging to $B_{x^2}[0,\infty)$ and satisfying the condition $\lim\limits_{x\to\infty} \frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is given by $||f||_{x^2} = \sup\limits_{x\in[0,\infty)} \frac{f(x)}{1+x^2}$.

Theorem 6. *For each $f \in C^*_{x^2}[0,\infty)$ and $n > 2c$, we have*

$$\lim_{n\to\infty} ||T^{\alpha,\beta}_{n,\rho}(f) - f||_{x^2} = 0.$$

Proof. Using Lemma 2, we see that it is sufficient to verify the conditions

$$\lim_{n\to\infty} ||T^{\alpha,\beta}_{n,\rho}(t^r,x) - x^r||_{x^2} = 0, \; r = 0,1,2. \tag{4.1}$$

Since $T^{\alpha,\beta}_{n,\rho}(1,x) = 1$, for $r = 0$, (4.1) holds. For $n\rho > c$, we have

$$||T^{\alpha,\beta}_{n,\rho}(t,x) - x||_{x^2} = \sup_{x\in[0,\infty)} \frac{|T^{\alpha,\beta}_{n,\rho}(t,x) - x|}{1+x^2}$$
$$= \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\left|\frac{n\rho x}{(n\rho - c)} + \frac{\alpha\rho}{(n\rho-c)(1+e\beta)} - x\right|$$
$$\leq \left[\frac{n\rho}{(n\rho-c)} - 1\right] \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{\alpha\rho}{(n\rho-c)(1+e\beta)} \sup_{x\in[0,\infty)} \frac{1}{1+x^2},$$

condition (4.1) holds for $r = 1$ as $n \to \infty$. Again $n\rho > 2c$, we have

$$||T^{\alpha,\beta}_{n,\rho}(t^2,x) - x^2||_{x^2} = \sup_{x\in[0,\infty)} \frac{|T^{\alpha,\beta}_{n,\rho}(t^2,x) - x^2|}{1+x^2}$$
$$= \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\left|\frac{n^2\rho^2}{(n\rho-c)(n\rho-2c)}x^2\right.$$
$$+ \frac{n\rho^2(1+2\alpha+e\beta)}{(n\rho-c)(n\rho-2c)(1+e\beta)}x$$

$$+\frac{\rho^2(\alpha^2-2\alpha e\beta-\alpha e^2\beta^2)}{(n\rho-c)(n\rho-2c)(1+e\beta)^2}-x^2\Bigg|$$
$$\leq\left[\frac{n^2\rho^2}{(n\rho-c)(n\rho-2c)}-1\right]\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}$$
$$+\frac{n\rho^2(1+2\alpha+e\beta)}{(n\rho-c)(n\rho-2c)(1+e\beta)}\sup_{x\in[0,\infty)}\frac{x}{1+x^2}$$
$$+\frac{\rho^2(\alpha^2-2\alpha e\beta-\alpha e^2\beta^2)}{(n\rho-c)(n\rho-2c)(1+e\beta)^2}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}.$$

Condition (4.1) holds for $r=2$ as $n\to\infty$. This completes the proof of the theorem. □

Corollary 1. *For each $f\in C_{x^2}[0,\infty)$, and $\alpha>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|T_{n,\rho}^{\alpha,\beta}(f;x)-f(x)|}{(1+x^2)^{\alpha}}=0.$$

Proof. For any fixed $x_0>0$,

$$\sup_{x\in[0,\infty)}\frac{|T_{n,\rho}^{\alpha,\beta}(f;x)-f(x)|}{(1+x^2)^{1+\alpha}}\leq\sup_{x\leq x_0}\frac{|T_{n,\rho}^{\alpha,\beta}(f;x)-f(x)|}{(1+x^2)^{1+\alpha}}+\sup_{x\geq x_0}\frac{|T_{n,\rho}^{\alpha,\beta}(f;x)-f(x)|}{(1+x^2)^{1+\alpha}}$$
$$\leq\|T_{n,\rho}^{\alpha,\beta}(f;.)-f\|_{C[0,x_0]}+\|f\|_{x^2}\sup_{x\geq x_0}\frac{|T_{n,\rho}^{\alpha,\beta}(f;x)-f(x)|}{(1+x^2)^{1+\alpha}}$$
$$+\sup_{x\geq x_0}\frac{|f(x)|}{(1+x^2)^{1+\alpha}}.$$

From Theorem 2, in the above inequality first term tends to zero and by Lemma 2 for any fixed x_0 it can be easily seen that

$$\sup_{x\geq x_0}\frac{|T_{n,\rho}^{\alpha,\beta}(1+t^2;x)-f(x)|}{(1+x^2)^{1+\alpha}}\leq\frac{\varphi}{(1+x_0^2)^{\alpha}}.$$

Constant $\varphi>0$ is independent of x, and choose adequately large x_0 the right-hand side of the earlier inequality and last part can be made small, we get the required result. □

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