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ON JORDAN HOMO-DERIVATION OF TRIANGULAR ALGEBRAS

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Abstract. We define an $\mathbb{R}$-linear map ϕ from $\mathbb{A}$ to an $\mathbb{A}$-bimodule $\mathbb{M}$ is said to be Jordan homo-derivation if $\phi(x^2) = \phi(x)x + x\phi(x) + \phi(x)^2$ for each $x \in \mathbb{A}$. In this article, we proved that every Jordan homo-derivation to be homo-derivation on triangular algebras.

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1. INTRODUCTION

Let $\mathbb{R}$ be a commutative ring, and $\mathbb{A}$ be an algebra over $\mathbb{R}$. An $\mathbb{R}$-linear map ψ from $\mathbb{A}$ into an $\mathbb{A}$-bimodule $\mathbb{M}$ is said to be a derivation if $\psi(xy) = \psi(x)y + x\psi(y)$ hold $\forall$ $x, y \in \mathbb{A}$. It is called a Jordan derivation if $\psi(a^2) = \psi(a)a + a\psi(a)$ for all $a \in \mathbb{A}$. Any derivation is clearly a Jordan derivation. In general, the converse is not true (see[1]). Any Jordan derivation on a 2-torsion free prime ring is a derivation, according to a well-known Herstein [8] result. [4] contains a brief proof of this finding. Bresar [5], demonstrated in 1988 that Herstein's result holds in 2-torsion free semiprime rings. Jordan derivations of unital algebras with idempotents were investigated by Benkovic and Sirovnik [3] in 2012.

Let an $\mathbb{R}$-linear map ϕ from $\mathbb{A}$ to an $\mathbb{A}$-bimodule $\mathbb{M}$ is said to be a homo-derivation if $\phi(xy) = \phi(x)y + x\phi(y) + \phi(x)\phi(y)$ for any $x, y \in \mathbb{A}$, (see [7]). We call ϕ a Jordan homo-derivation if it is $\mathbb{R}$-linear and $\phi(x^2) = \phi(x)x + x\phi(x) + \phi(x)^2$ for each $x \in \mathbb{A}$, (see [9]). Obviously, every homo-derivation is a Jordan homo-derivation. But the converse, in general, is not true, for example see Example 1. If $\mathfrak{S} \subseteq \mathbb{R}$, then a mapping $\phi\colon \mathbb{R} \to \mathbb{R}$ preserves $\mathfrak{S}$ if $\phi(\mathfrak{S}) \subseteq \mathfrak{S}$. A mapping $\phi : \mathbb{R} \to \mathbb{R}$ is said to be zero-power valued on $\mathfrak{S}$ if ϕ preserves $\mathfrak{S}$ and if for each $a \in \mathfrak{S}$, there exists a positive integer $n(a) > 1$ such that $\phi^{n(a)} = 0$, [7].

The $\mathbb{R}$-algebra $\mathfrak{T} = \mathrm{Tri}(\mathbb{A}, \mathbb{M}, \mathbb{B})$ is an algebra of the form

$$\mathrm{Tri}(\mathbb{A}, \mathbb{M}, \mathbb{B}) = \left\{ \begin{pmatrix} a & m \\ & b \end{pmatrix} : \; a \in \mathbb{A}, \; b \in \mathbb{B}, \; m \in \mathbb{M} \right\}$$

under the usual matrix operations will be called a triangular algebra [2], where $\mathbb{A}$ and $\mathbb{B}$ are two algebras over the commutative ring $\mathbb{R}$, and $\mathbb{M}$ is a $(\mathbb{A},\mathbb{B})$-bimodule that is faithful as both a left $\mathbb{A}$-module and a right $\mathbb{B}$-module. Recently, some problems on triangular algebra have been determined by many authors. Any Jordan derivation from the algebra of all upper triangular matrices into its arbitrary bimodule, as demonstrated by Benkovic [1], is the sum of a derivation and an antiderivation on triangular matrices over commutative rings. Jordan derivations of triangular algebras are derivations, according to Zhang and Yu (2006) ([10]). It's normal to ask if a Jordan homo-derivation from a triangular algebra to itself is really a derivation.The primary goal of this article is to investigate this problem.

2. Jordan homo-derivations on triangular matrix rings

For the sake of notational clarity, let $\mathfrak{T} = \mathrm{Tri}(\mathbb{A},\mathbb{M},\mathbb{B})$ be a triangular algebra. Set

$$\mathfrak{T}_{11} = \left\{ \begin{pmatrix} a & 0 \\ & 0 \end{pmatrix} : a \in \mathbb{A} \right\}, \quad \mathfrak{T}_{12} = \left\{ \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} : m \in \mathbb{M} \right\}$$

and

$$\mathfrak{T}_{22} = \left\{ \begin{pmatrix} 0 & 0 \\ & b \end{pmatrix} : b \in \mathbb{B} \right\}.$$

Then we can write $\mathfrak{T} = \mathfrak{T}_{11} \oplus \mathfrak{T}_{12} \oplus \mathfrak{T}_{22}$. Henceforth the element A_{ij} belongs $\mathfrak{T}_{ij}$ and the corresponding elements are in $\mathbb{A},\mathbb{B}$ or $\mathbb{M}$. By a direct calculation $A_{ij}A_{kl} = 0$ if $j \neq k$ where $i, j, k \in \{1,2\}$.

Throughout our discussion, $\mathbb{A}$ and $\mathbb{B}$ will be two algebras over 2-torsion free ring $\mathbb{R}$ with the property:

(♣) Let $r \in \mathbb{A}$ (resp $\mathbb{B}$). If $prq + qrp = 0$ holds $\forall\, p, q \in \mathbb{A}$ (rep. $\mathbb{A}$), then $r = 0$.

Moreover, ϕ will be a Jordan homo-derivation from triangular algebra $\mathfrak{T}$ into itself and zero-power valued on $\mathfrak{T}$, and preserves on $\mathfrak{T}_{11}$ and $\mathfrak{T}_{22}$, where $\mathbb{M}$ is a faithful $(\mathbb{A},\mathbb{B})$-bimodule.

We begin with Cheng and Jing's [6, Lemma 2.1] result, which is essential for developing the proof of our main theorem.

Lemma 1. *Let r be in $\mathbb{A}$ (resp. $\mathbb{B}$).*

(i) If $prp = 0$ for any $p \in \mathbb{A}$ (resp. $\mathbb{B}$), then $r = 0$;
(ii) If $rp + pr = 0$ for any $p \in \mathbb{A}$ (resp. $\mathbb{B}$), then $r = 0$.

The main result of the paper is as follows:

Theorem 1. *Let $\mathbb{A}$ and $\mathbb{B}$ be two algebras over a 2-torsion free commutative ring $\mathbb{R}$ with the property (♣). Let $\mathbb{M}$ be a faithful ($\mathbb{A}$, $\mathbb{B}$)-bimodule and $\mathfrak{T}$ be the triangular algebra* $\mathrm{Tri}(\mathbb{A},\mathbb{M},\mathbb{B})$. *Let a mapping $\phi\colon \mathfrak{T} \to \mathfrak{T}$ be zero-power valued on $\mathfrak{T}$, and preserves on $\mathfrak{T}_{11}$ and $\mathfrak{T}_{22}$. Then every Jordan homo-derivation ϕ on $\mathfrak{T}$ into itself is a homo-derivation.*

In Example 1, we are putting $\mathbb{M} = \mathbb{Z}$ and $\mathbb{A} = \mathbb{B} = \mathbb{K}$ and $\mathbb{R} = \mathbb{Z}$. It is easy to see that $\mathfrak{T}$ does not satisfy the property (♣) , and ϕ is a Jordan homo-derivation, but it is not homo-derivation.

Example 1. Consider S be a ring such that $s^2 = 0$ for all $s \in S$, but the product of some elements of S is nonzero. Since $s^2 = 0$, so $(s+t)^2 = 0$ for all $s,t \in S$ this implies that $s \circ t = 0$ for all $s,t \in S$. Suppose $\mathbb{K} = \left\{ \begin{pmatrix} 0 & p & q \\ 0 & 0 & r \\ 0 & 0 & 0 \end{pmatrix} : p,q,r \in S \right\}$. Let $\phi \colon \mathbb{K} \to \mathbb{K}$ such that $\phi \begin{pmatrix} 0 & p & q \\ 0 & 0 & r \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & r & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Note that ϕ is a Jordan homo-derivation, but it is not homo-derivation.

We start with the following Lemma due to Herstein [8]

Lemma 2. *Let $\mathbb{A}$ be an algebra over a 2-torsion free ring commutative ring, and Let ϕ be a Jordan homo-derivation from $\mathbb{A}$ into its bimodule. Then*

(i) $\phi(A \circ B) = \phi(A) \circ B + A \circ \phi(B) + \phi(A) \circ \phi(B)$ *for all* $A, B \in \mathbb{A}$ *;*

(ii)

$$\begin{aligned}\phi(ABA) = {} & \phi(A)BA + A\phi(B)A + AB\phi(A) + \phi(A)\phi(B)A \\ & + A\phi(B)\phi(A) + \phi(A)B\phi(A) + \phi(A)\phi(B)\phi(A)\end{aligned}$$

for all $A, B \in \mathbb{A}$.

Proof.
(i) We compute $(A+B)^2$ in two ways, we get the required result.
(ii) Note that $A \circ (A \circ B) = A^2 \circ B + 2ABA$. We compute $\phi(A \circ (A \circ B))$ and $\phi(A^2 \circ B + 2ABA)$ by using Lemma 2 (i) and definition of Jordan homo-derivation.

$$\begin{aligned}\phi(A^2 \circ B + 2ABA) &= \phi(A^2 \circ B) + 2\phi(ABA) \\ &= \phi(A^2) \circ B + A^2 \circ \phi(B) + \phi(A^2) \circ \phi(B) \\ &\quad + 2\phi(ABA) \\ &= \phi(A)AB + A\phi(A)B + \phi(A)^2 B \\ &\quad + B\phi(A)A + BA\phi(A) + B\phi(A)^2 \\ &\quad + A^2\phi(B) + \phi(B)A^2 + \phi(A)A\phi(B) \\ &\quad + A\phi(A)\phi(B) + \phi(A)^2\phi(B) + \phi(B)\phi(A)A \\ &\quad + \phi(B)A\phi(A) + \phi(B)\phi(A)^2 + 2\phi(ABA).\end{aligned} \tag{2.1}$$

$$\phi(A \circ (A \circ B)) = \phi(A) \circ (A \circ B) + A \circ \phi(A \circ B)$$

$$
\begin{aligned}
&+\phi(A)\circ\phi(A\circ B)\\
&=\phi(A)AB+\phi(A)BA+AB\phi(A)\\
&\quad+BA\phi(A)+A\phi(A)B+AB\phi(A)+A^2\phi(B)\\
&\quad+A\phi(B)A+A\phi(A)\phi(B)+A\phi(B)\phi(A)\\
&\quad+\phi(A)BA+B\phi(A)A+A\phi(B)A\\
&\quad+\phi(B)A^2+\phi(A)\phi(B)A+\phi(B)\phi(A)A\\
&\quad+\phi(A)^2B+\phi(A)B\phi(A)+\phi(A)A\phi(B)\\
&\quad+\phi(A)\phi(B)A+\phi(A)^2\phi(B)+\phi(A)\phi(B)\phi(A)\\
&\quad+\phi(A)B\phi(A)+B\phi(A)^2+A\phi(B)\phi(A)\\
&\quad+\phi(B)A\phi(A)+\phi(A)\phi(B)\phi(A)+\phi(B)\phi(A)^2.
\end{aligned}
\tag{2.2}
$$

Since $\mathbb{R}$ is 2-torsion free and compare (2.1) and (2.2), we get (ii).

□

We note that condition (i) is analogous to the Jordan homo-derivation condition.

Lemma 3. $\phi(0)=0$.

Proof. Putting $B=0$ and $A=A_{11}$ in Lemma 2 (i), we have $\phi(0)=A_{11}\circ\phi(0)+\phi(A_{11})\circ\phi(0)$. Since there exists a positive integer $n(A_{11})>1$ such that $\phi^n(A_{11})=0$, taking A_{11} by $\phi^{n-1}(A_{11})$ in the last relation, we get $\phi(0)=\phi^{n-1}(A_{11})\circ\phi(0)$. Replacing A_{11} by $\phi(A_{11})$ in the last relation, we obtain $\phi(0)=0$. □

Lemma 4. *For arbitrary $A_{11}\in\mathfrak{T}_{11}$ and $A_{22}\in\mathfrak{T}_{22}$, we have*

(i) $\phi(A_{11}A_{22})=\phi(A_{11})A_{22}+A_{11}\phi(A_{22})+\phi(A_{11})\phi(A_{22})$;
(ii) $\phi(A_{22}A_{11})=\phi(A_{22})A_{11}+A_{22}\phi(A_{11})+\phi(A_{22})\phi(A_{11})$.

Proof.

(i) From conditions of Theorem 1, we have $\phi(A_{22})_{11}=0$ and $\phi(A_{11})_{22}=0$. Now, note that

$$
\begin{aligned}
&\phi(A_{22})A_{11}+A_{22}\phi(A_{11})+\phi(A_{22})\phi(A_{11})\\
&=[\phi(A_{22})_{11}+\phi(A_{22})_{12}+\phi(A_{22})_{22}]A_{11}\\
&\quad+A_{22}[\phi(A_{11})_{11}+\phi(A_{11})_{12}+\phi(A_{11})_{22}]\\
&\quad+[\phi(A_{22})_{11}+\phi(A_{22})_{12}+\phi(A_{22})_{22}][\phi(A_{11})_{11}+\phi(A_{11})_{12}+\phi(A_{11})_{22}]=0.
\end{aligned}
\tag{2.3}
$$

We compute

$$
\begin{aligned}
\phi(A_{11}A_{22})&=\phi(A_{11}A_{22}+A_{22}A_{11})\\
&=\phi(A_{11})A_{22}+A_{11}\phi(A_{22})+\phi(A_{11})\phi(A_{22})\\
&\quad+\phi(A_{22})A_{11}+A_{22}\phi(A_{11})+\phi(A_{22})\phi(A_{11})\\
&=\phi(A_{11})A_{22}+A_{11}\phi(A_{22})+\phi(A_{11})\phi(A_{22}),\text{by using}(2.3).
\end{aligned}
$$

(ii) Also, by using (2.3), we get (ii).

□

Lemma 5. *For any $A_{12} \in \mathfrak{T}_{12}$, the following are true.*
(i) $\phi(A_{12})_{11} = 0$;
(ii) $\phi(A_{12})_{22} = 0$.

Proof. (i) We compute

$$\begin{aligned}
0 &= \phi(\phi^{n-1}(A_{11})A_{12}A_{11}) \\
&= \phi^{n}(A_{11})A_{12}A_{11} + \phi^{n-1}(A_{11})\phi(A_{12})A_{11} + \phi^{n-1}(A_{11})A_{12}\phi(A_{11}) \\
&\quad + \phi^{n}(A_{11})\phi(A_{12})A_{11} + \phi^{n-1}(A_{11})\phi(A_{12})\phi(A_{11}) + \phi^{n}(A_{11})A_{12}\phi(A_{11}) \\
&\quad + \phi^{n}(A_{11})\phi(A_{12})\phi(A_{11}).
\end{aligned}$$

That is

$$(\phi^{n-1}(A_{11}) + \phi^{n}(A_{11}))[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})] = 0. \tag{2.4}$$

Since $\phi^{n}(A_{11}) = 0$, we have

$$\phi^{n-1}(A_{11})[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})] = 0.$$

Similarly, we can compute $\phi(\phi^{n-2}(A_{11})A_{12}A_{11}) = 0$, we get

$$(\phi^{n-2}(A_{11}) + \phi^{n-1}(A_{11}))[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})] = 0.$$

By using (2.4) in last relation, we have

$$\phi^{n-2}(A_{11})[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})] = 0.$$

Also, we can compute $\phi(\phi^{k}(A_{11})A_{12}A_{11}) = 0$ for $k = n-3, n-4, \ldots, 1$ and $\phi(A_{11}A_{12}A_{11}) = 0$ and so we get

$$A_{11}[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})] = 0.$$

Hence

$$A_{11}[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})]A_{11} = 0,$$

thus

$$A_{11}[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})]_{11}A_{11} = 0$$

and by using Lemma 1 (i)

$$[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})]_{11} = 0. \tag{2.5}$$

Also, we have

$$A_{11}[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})]_{12} = 0$$

and since $\mathbb{M}$ is a faithful left $\mathbb{A}$-module, we have

$$[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})]_{12} = 0. \tag{2.6}$$

Similarly, we can compute $\phi(\phi^k(A_{22})A_{12}A_{11}) = 0$ for $k = n-3, n-4, \ldots, 1$ and $\phi(A_{22}A_{12}A_{11}) = 0$ and so we get

$$A_{22}[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})] = 0.$$

Hence

$$A_{22}[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})]A_{22} = 0$$

thus

$$A_{22}[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})]_{22}A_{22} = 0$$

and by using Lemma 1 (i), we get

$$[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})]_{22} = 0. \tag{2.7}$$

From (2.5), (2.6) and (2.7), we have

$$[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})] = 0.$$

That is

$$\phi(A_{12})_{11}(A_{11} + \phi(A_{11})) = 0. \tag{2.8}$$

Replacing A_{11} by $\phi^{n-1}(A_{11})$, in (2.8), we get $\phi(A_{12})_{11}\phi^{n-1}(A_{11}) = 0$. Now, replacing A_{11} by $\phi^{n-2}(A_{11})$, in (2.8) and using last relation, we have

$$\phi(A_{12})_{11}\phi^{n-2}(A_{11}) = 0,$$

by continuing, we obtain

$$\phi(A_{12})_{11}A_{11} = 0 \quad \text{that is} \quad A_{11}\phi(A_{12})_{11}A_{11} = 0$$

and by Lemma 1 (i), we get $\phi(A_{12})_{11} = 0$. Similarly we can get (ii) by considering

$$\phi(A_{22}A_{12}\phi^k(A_{22})) = 0$$

for $k = n-1, n-2, \ldots, 1$ and $\phi(A_{22}A_{12}A_{22}) = 0$. □

Lemma 6. *For any $A_{11} \in \mathfrak{T}_{11}$ and $A_{12} \in \mathfrak{T}_{12}$,*

(i) $\phi(A_{12}A_{11}) = \phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})$;
(ii) $\phi(A_{11}A_{12}) = \phi(A_{11})A_{12} + A_{11}\phi(A_{12}) + \phi(A_{11})\phi(A_{12})$.

Proof.

(i) We compute

$$\begin{aligned} &\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11}) \\ &= \phi(A_{12})_{11}A_{11} + A_{12}\phi(A_{11})_{22} \\ &\quad + [\phi(A_{12})_{11} + \phi(A_{12})_{12} + \phi(A_{12})_{22}][\phi(A_{11})_{11} + \phi(A_{11})_{12} + \phi(A_{11})_{22}] \\ &= 0 = \phi(A_{12}A_{11}). \end{aligned} \tag{2.9}$$

(ii) We compute

$$\begin{aligned}\phi(A_{11}A_{12}) &= \phi(A_{11}A_{12}+A_{12}A_{11})\\ &= \phi(A_{11})A_{12}+A_{11}\phi(A_{12})+\phi(A_{11})\phi(A_{12})\\ &\quad +\phi(A_{12})A_{11}+A_{12}\phi(A_{11})+\phi(A_{12})\phi(A_{11})\\ &= \phi(A_{11})A_{12}+A_{11}\phi(A_{12})+\phi(A_{11})\phi(A_{12}) \text{ by using (2.9).}\end{aligned}$$

□

Using similar techniques as used in the proof of Lemma 6, we get the following.

Lemma 7. *For any $A_{12}\in\mathfrak{T}_{12}$ and $A_{22}\in\mathfrak{T}_{22}$,*

(i) $\phi(A_{12}A_{22}) = \phi(A_{12})A_{22}+A_{12}\phi(A_{22})+\phi(A_{12})\phi(A_{22})$;
(ii) $\phi(A_{22}A_{12}) = \phi(A_{22})A_{12}+A_{22}\phi(A_{12})+\phi(A_{22})\phi(A_{12})$.

Lemma 8. *ϕ is a homo-derivation on $\mathfrak{T}_{12}$.*

Proof. For any $A_{12},B_{12}\in\mathfrak{T}_{12}$, by Lemma 5

$$\begin{aligned}&\phi(A_{12})B_{12}+A_{12}\phi(B_{12})+\phi(A_{12})\phi(B_{12})\\ &= \phi(A_{12})_{11}B_{12}+A_{12}\phi(B_{12})_{22}\\ &\quad +[\phi(A_{12})_{11}+\phi(A_{12})_{12}+\phi(A_{12})_{22}][\phi(B_{12})_{11}+\phi(B_{12})_{12}+\phi(B_{12})_{22}]\\ &= 0=\phi(A_{12}B_{12}).\end{aligned}$$

□

Lemma 9. *ϕ is a homo-derivation on $\mathfrak{T}_{11}$.*

Proof. Let $A_{11},B_{11}\in\mathfrak{T}_{11}$ and $C_{12}\in\mathfrak{T}_{12}$ be arbitrary. On one side, we have

$$\begin{aligned}\phi(A_{11}B_{11}C_{12}) &= \phi(A_{11})B_{11}C_{12}+A_{11}\phi(B_{11})C_{12}+A_{11}B_{11}\phi(C_{12})\\ &\quad +\phi(A_{11})\phi(B_{11})C_{12}+A_{11}\phi(B_{11})\phi(C_{12})+\phi(A_{11})B_{11}\phi(C_{12})\\ &\quad +\phi(A_{11})\phi(B_{11})\phi(C_{12}).\end{aligned}$$

On the other side, we get

$$\phi(A_{11}B_{11}C_{12}) = \phi(A_{11}B_{11})C_{12}+A_{11}B_{11}\phi(C_{12})+\phi(A_{11}B_{11})\phi(C_{12}).$$

Then we can see that

$$[\phi(A_{11}B_{11})-\phi(A_{11})B_{11}-A_{11}\phi(B_{11})-\phi(A_{11})\phi(B_{11})](C_{12}+\phi(C_{12})) = 0. \quad (2.10)$$

Replacing C_{12} by $\phi^{n-1}(C_{12})$ in (2.10) and since $\phi^n(C_{12})=0$, we have

$$[\phi(A_{11}B_{11})-\phi(A_{11})B_{11}-A_{11}\phi(B_{11})-\phi(A_{11})\phi(B_{11})]\phi^{n-1}(C_{12}) = 0. \quad (2.11)$$

Replacing C_{12} by $\phi^{n-2}(C_{12})$ in (2.10) and using (2.11), we get

$$[\phi(A_{11}B_{11})-\phi(A_{11})B_{11}-A_{11}\phi(B_{11})-\phi(A_{11})\phi(B_{11})]\phi^{n-2}(C_{12}) = 0.$$

By continuing, we obtain

$$[\phi(A_{11}B_{11}) - \phi(A_{11})B_{11} - A_{11}\phi(B_{11}) - \phi(A_{11})\phi(B_{11})]C_{12} = 0.$$

This yields that

$$\phi(A_{11}B_{11}) = \phi(A_{11})B_{11} + A_{11}\phi(B_{11}) + \phi(A_{11})\phi(B_{11})$$

since $\mathbb{M}$ is a faithful left $\mathbb{A}$-module. □

Using the similar arguments as used in the Lemma 9, one can get the following.

Lemma 10. ϕ *is a homo-derivation on* $\mathfrak{T}_{22}$.

Proof of Theorem 1. The proof of is follows from Lemmas 4-10. □

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