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SOME IDENTITIES FOR DERANGEMENT NUMBERS

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Abstract. In this paper, we give some identities involving derangement numbers. In addition, we derive nonlinear differential equation from the generating function of r-derangement numbers and obtain some identities including generalized harmonic numbers and derangement numbers by using this differential equation. For example, for any positive integers N, n and r,

$$\sum_{k=0}^{N}\sum_{i=0}^{N-k}(-1)^i\binom{r}{k}\binom{r+N-i-k}{N-i-k}\frac{d_{n-r+k}^{(r+N-i-k+1)}}{i!(n-r+k)!}=\frac{D_r(n+N)}{n!N!}.$$

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1. INTRODUCTION

Harmonic numbers and generalized harmonic numbers have been studied since the distant past and are involved in a wide range of diverse fields such as analysis, computer science and various in a wide range of diverse fields such as analysis, computer science and various branches of number theory [1–8, 16, 17, 23–25]. The harmonic numbers are defined by

$$H_0=0 \quad \text{and} \quad H_n=\sum_{i=1}^{n}\frac{1}{i} \quad \text{for } n\geq 1.$$

For any $\alpha\in\mathbb{R}^+$, the generalized harmonic numbers $H_n(\alpha)$ [8] are defined by

$$H_0(\alpha)=0 \quad \text{and} \quad H_n(\alpha)=\sum_{i=1}^{n}\frac{1}{i\alpha^i} \quad \text{for } n\geq 1.$$

For $\alpha=1$, the usual harmonic numbers are $H_n(1)=H_n$ and the generating function of $H_n(\alpha)$ is

$$\sum_{n=1}^{\infty}H_n(\alpha)x^n=-\frac{\ln\left(1-\frac{x}{\alpha}\right)}{1-x}.$$

For the generalized harmonic numbers $H_n(\alpha)$, Ömür et al. [16] defined the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$ as follows. For $r<0$ or $n\leq 0$,

$H_n^r(\alpha) = 0$ and for $n \geq 1$,

$$H_n^r(\alpha) = \sum_{i=1}^{n} H_i^{r-1}(\alpha), \quad r \geq 1,$$

where $H_n^0(\alpha) = \frac{1}{n\alpha^n}$. For $\alpha = 1$, $H_n^r(1) = H_n^r$ are the hyperharmonic numbers of order r. The generating function of these numbers is

$$\sum_{n=1}^{\infty} H_n^r(\alpha) x^n = -\frac{\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^r}. \tag{1.1}$$

The generalized harmonic numbers $H(n,r)$ of rank r [7, 23] are defined as for $n \geq 1$ and $r \geq 0$,

$$H(n,r) = \sum_{1 \leq n_0+n_1+\cdots+n_r \leq n} \frac{1}{n_0 n_1 \cdots n_r}.$$

It is clear that $H(n,0) = H_n$.

$H(n,r,\alpha)$ [6] are defined as for $n \geq 1$ and $r \geq 0$,

$$H(n,r,\alpha) = \sum_{1 \leq n_0+n_1+\cdots+n_r \leq n} \frac{1}{n_0 n_1 \cdots n_r \alpha^{n_0+n_1+\cdots+n_r}}$$

or equivalently as

$$H(n,r,\alpha) = \frac{(-1)^{r+1}}{n!} \left(\frac{d^n}{dx^n} \frac{\left[\ln\left(1-\frac{x}{\alpha}\right)\right]^{r+1}}{1-x} \right) \Bigg|_{x=0}.$$

For $\alpha = 1$, $H(n,r,1) = H(n,r)$.

The generating function of these numbers is given by

$$\sum_{n=0}^{\infty} H(n,r,\alpha) x^n = \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x}. \tag{1.2}$$

The Cauchy numbers of order r, C_n^r, are defined by the generating functions to be

$$\sum_{n=0}^{\infty} C_n^r \frac{x^n}{n!} = \left(\frac{x}{\ln(1+x)} \right)^r. \tag{1.3}$$

For $r = 1$, $C_n^1 = C_n$ are called Cauchy numbers.

The Daehee numbers of order r, D_n^r, are defined by the generating functions to be

$$\sum_{n=0}^{\infty} D_n^r \frac{x^n}{n!} = \left(\frac{\ln(1+x)}{x} \right)^r. \tag{1.4}$$

For $r = 1$, $D_n^1 = D_n$ are called Daehee numbers.

The Stirling numbers of the first kind $S_1(n,k)$ are defined by

$$x^{\underline{n}} = \sum_{k=0}^{n} S_1(n,k) x^k,$$

and the Stirling numbers of the second kind $S_2(n,k)$ are defined by

$$x^n = \sum_{k=0}^{n} S_2(n,k)x^{\underline{k}},$$

where $x^{\underline{n}}$ stands for the falling factorial defined by $x^{\underline{0}} = 1$ and $x^{\underline{n}} = x(x-1)\cdots\cdot(x-n+1)$.

The generating function of the Stirling numbers of the first kind $S_1(n,k)$ is given by

$$\sum_{n=0}^{\infty} S_1(n,k)\frac{x^n}{n!} = \frac{(\ln(1+x))^k}{k!}, \quad k \geq 0, \tag{1.5}$$

and the generating function of the Stirling numbers of the second kind $S_2(n,k)$ is given by

$$\sum_{n=k}^{\infty} S_2(n,k)\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}, \quad k \geq 0.$$

The generalized geometric series are given by for positive integers a and b,

$$\sum_{n=b}^{\infty} \binom{n+a-b}{a} x^n = \frac{x^b}{(1-x)^{a+1}}. \tag{1.6}$$

The exponential generating function is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x. \tag{1.7}$$

The generating function of the r-derangement numbers $D_r(n)$ is given by

$$\sum_{n=r}^{\infty} D_r(n)\frac{x^n}{n!} = \frac{x^r e^{-x}}{(1-x)^{r+1}}. \tag{1.8}$$

For $r \in \mathbb{N}$, the derangement numbers $d_n^{(r)}$ of order r $(n \geq 0)$ [13] are defined by the generating functions to be

$$\sum_{n=0}^{\infty} d_n^{(r)}\frac{x^n}{n!} = \left(\frac{1}{1-x}\right)^r e^{-x}. \tag{1.9}$$

For $r = 1$, $d_n^{(1)} = D_0(n) = d_n$ are called derangement numbers. The derangement numbers d_n are given by the closed form formula

$$d_n = n!\sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

It is known that for an exponential series $f(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$,

$$\frac{d^m}{dx^m} f(x) = \sum_{n=0}^{\infty} f_{n+m}\frac{x^n}{n!}, \tag{1.10}$$

and for an ordinary series $f(x) = \sum_{n\geq 0} f_n x^n$,

$$\frac{d^m}{dx^m} f(x) = \sum_{n=0}^{\infty} m! \binom{m+n}{n} f_{n+m} x^n. \tag{1.11}$$

There have been many studies involving derangement numbers [4–6, 14, 20, 21, 24].

Duran et al. [6] obtained sums involving generalized harmonic numbers of rank r, derangement and special numbers. For example, for any positive integers n and r,

$$\sum_{i=0}^{n} (-1)^{i-r} \binom{n}{i} r! \alpha^{-i} S_1(i,r) d_{n-i} = n! \sum_{i=0}^{n} \frac{(-1)^{n-i}}{(n-i)!} H(i, r-1, \alpha).$$

Recently, the combinatorial properties of special numbers and polynomials have been studied using differential equations associated with the generation functions [9–12, 15, 18, 19, 22].

Kwon et al. [15] showed that for positive integer N and nonnegative integer n,

$$D_{n+N-1} = \frac{(-1)^N (N-1)!}{n+N} \sum_{k=0}^{n} (-1)^k \binom{n}{k} N^k D_{n-k}^k.$$

Ömür et al. [18] gave that for any positive integers N, n, r and m,

$$S_1(n+N, r+1) \binom{n+N+m}{m} \binom{r+m+1}{m}^{-1} = \sum_{i=0}^{n+N} \binom{n+N+m}{i} C_i^m S_1(n+N+m-i, r+m+1).$$

Rim et al. [22] obtained some identities involving hyperharmonic numbers and Daehee number as follows: for any positive integer N and nonnegative integer n,

$$n! (n+N)^{\overline{N}} \sum_{k=0}^{n+N} (-1)^{n+N-k} \binom{r}{n+N-k} H_k^r = (N-1)! \sum_{k=0}^{n} (-1)^k \binom{n}{k} N^{n-k} D_k^{n-k}.$$

In this paper, we give some identities involving derangement numbers. In addition, we derive nonlinear differential equation from the generating function of r-derangement numbers and using this differential equation, obtain some identities including generalized harmonic numbers and derangement numbers. For example, for any positive integers N, n and r,

$$\sum_{k=0}^{N} \sum_{i=0}^{N-k} (-1)^i \binom{r}{k} \binom{r+N-i-k}{N-i-k} \frac{d_{n-r+k}^{(r+N-i-k+1)}}{i!(n-r+k)!} = \frac{D_r(n+N)}{n!N!}.$$

2. Some identities with derangement numbers

In this section, we will give some identities involving Daehee, Cauchy, derangement, r-derangement and harmonic numbers.

Theorem 1. *For any positive integers n, r and α, we have*

$$C_n^r = \sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}C_k^r C_{j-k}D_{n-j}, \tag{2.1}$$

$$= \sum_{i=0}^{n}\binom{i+r}{r}^{-1} S_1(n,i)\, S_2(i+r,r) \tag{2.2}$$

$$= -\frac{n!}{r+1}\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{i=0}^{n-k}\frac{(-1)^{k+j}\alpha^k}{(n-k)!}\binom{i+r+1}{i}^{-1}\binom{r}{j} \tag{2.3}$$

$$\times S_1(n-k,i)S_2(i+r+1,r)H_{k-j}^r(\alpha).$$

Proof. We will give proof of (2.1). By (1.3) and (1.4), we write

$$\sum_{n=0}^{\infty} C_n^r \frac{x^n}{n!} = \left(\frac{x}{\ln(1+x)}\right)^r = \frac{x}{\ln(1+x)}\left(\frac{x}{\ln(1+x)}\right)^r \frac{\ln(1+x)}{x}$$

$$= \sum_{n=0}^{\infty} C_n^r \frac{x^n}{n!}\sum_{k=0}^{\infty} C_k \frac{x^k}{k!}\sum_{j=0}^{\infty} D_j \frac{x^j}{j!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}C_k^r C_{n-k}\frac{x^n}{n!}\sum_{j=0}^{\infty} D_j \frac{x^j}{j!}$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}C_k^r C_{j-k}D_{n-j}\frac{x^n}{n!}.$$

Thus, comparing the coefficients on both sides, the first result is obtained. The proofs of (2.2) and (2.3) are similar to the proof of (2.1). Thus the proof is complete. □

Theorem 2. *For any positive integers n, r and α, we have*

$$(-1)^{n+r}H(n,r-1,\alpha) = \sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^i \frac{D_{j-r}^r}{\alpha^j (j-r)!}\frac{S_1(n-j,i)}{(n-j)!} \tag{2.4}$$

$$= \frac{r!}{\alpha^r}\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{l=0}^{n}\sum_{j=0}^{l}(-1)^k \frac{D_j^r}{\alpha^j j!}\frac{S_1(l-j,i)}{(l-j)!}\frac{S_1(n-l,k)}{(n-l)!}S_2(i,r).$$

Proof. We will give proof of (2.4). From (1.2), (1.4) and (1.7), we get

$$\sum_{n=0}^{\infty}(-1)^n H(n,r-1,\alpha)x^n = \frac{\left(-\ln\left(1+\frac{x}{\alpha}\right)\right)^r}{1+x} = \left(-\frac{x}{\alpha}\right)^r \frac{\left(\ln\left(1+\frac{x}{\alpha}\right)\right)^r}{\left(\frac{x}{\alpha}\right)^r} e^{-\ln(1+x)}$$

$$=\left(-\frac{x}{\alpha}\right)^{r}\sum_{n=0}^{\infty}D_{n}^{r}\frac{x^{n}}{\alpha^{n}n!}\sum_{i=0}^{\infty}(-1)^{i}\frac{(\ln(1+x))^{i}}{i!}.$$

By (1.5), we have

$$\begin{aligned}(-1)^{r}\sum_{n=0}^{\infty}(-1)^{n}H(n,r-1,\alpha)x^{n}&=\sum_{n=0}^{\infty}D_{n}^{r}\frac{x^{n+r}}{\alpha^{n+r}n!}\sum_{i=0}^{\infty}(-1)^{i}\sum_{n=i}^{\infty}S_{1}(n,i)\frac{x^{n}}{n!}\\&=\sum_{n=0}^{\infty}D_{n-r}^{r}\frac{x^{n}}{\alpha^{n}(n-r)!}\sum_{n=0}^{\infty}\sum_{i=0}^{n}(-1)^{i}S_{1}(n,i)\frac{x^{n}}{n!}\\&=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^{i}\frac{D_{n-j-r}^{r}}{\alpha^{n-j}(n-j-r)!}\frac{S_{1}(j,i)}{j!}x^{n}\\&=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i}\frac{D_{n-j-r}^{r}}{\alpha^{n-j}(n-j-r)!}\frac{S_{1}(j,i)}{j!}x^{n}\\&=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i}\frac{D_{j-r}^{r}}{\alpha^{j}(j-r)!}\frac{S_{1}(n-j,i)}{(n-j)!}x^{n},\end{aligned}$$

as claimed. The proof of other equality is similar to proof of (2.4). □

Now, for any positive integer r, we take the function $F_r(x)=\dfrac{e^{-x}x^{r}}{(1-x)^{r+1}}$. By taking the derivative of this function with respect to x, we write

$$\frac{d}{dx}F_r(x)=\frac{r+x}{1-x}F_r(x)+\frac{r}{1-x}F_{r-1}(x),$$

and

$$\begin{aligned}\frac{d^{2}}{dx^{2}}F_r(x)=&\frac{2}{(1-x)^{2}}\left(\binom{r+2}{2}F_r(x)+2\binom{r+1}{2}F_{r-1}(x)+\binom{r}{2}F_{r-2}(x)\right)\\&-\frac{2}{1-x}\left((r+1)F_r(x)+rF_{r-1}(x)\right)+F_r(x).\end{aligned}$$

By repeating this process, we easily have

$$\frac{d^{N}}{dx^{N}}F_r(x)=N!\sum_{k=0}^{N}\sum_{i=0}^{N-k}\frac{(-1)^{i}}{i!}\binom{r}{k}\binom{r+N-i-k}{r}\frac{F_{r-k}(x)}{(1-x)^{N-i}}. \tag{2.5}$$

Theorem 3. *For any positive integers N, n and r, we have*

$$\begin{aligned}\frac{D_r(n+N)}{n!N!}&=\sum_{k=0}^{r}\sum_{i=0}^{N-k}\sum_{j=0}^{n}(-1)^{i}\binom{r}{k}\binom{r+N-i-k}{r}\binom{n-j+N-i-1}{n-j}\frac{D_{r-k}(j)}{i!j!}\\&=\sum_{k=0}^{N}\sum_{i=0}^{N-k}(-1)^{i}\binom{r}{k}\binom{r+N-i-k}{r}\frac{d_{n-r+k}^{(r+N-i-k+1)}}{i!(n-r+k)!}.\end{aligned}$$

Proof. From (1.6), (1.8) and (2.5), we have

$$\frac{d^N}{dx^N}\left(\frac{x^r e^{-x}}{(1-x)^{r+1}}\right)$$
$$= N!\sum_{k=0}^{N}\sum_{i=0}^{N-k}\frac{(-1)^i}{i!}\binom{r}{k}\binom{r+N-i-k}{r}\sum_{n=0}^{\infty}D_{r-k}(n)\frac{x^n}{n!}\sum_{n=0}^{\infty}\binom{n+N-i-1}{n}x^n.$$

By product of the generating functions, we write

$$\frac{d^N}{dx^N}\left(\frac{x^r e^{-x}}{(1-x)^{r+1}}\right)$$
$$= \sum_{k=0}^{N}\sum_{i=0}^{N-k}(-1)^i\frac{N!}{i!}\binom{r}{k}\binom{r+N-i-k}{r}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{D_{r-k}(j)}{j!}\binom{n-j+N-i-1}{n-j}x^n$$
$$= N!\sum_{n=0}^{\infty}\sum_{k=0}^{N}\sum_{i=0}^{N-k}\sum_{j=0}^{n}\frac{(-1)^i}{i!j!}\binom{r}{k}\binom{r+N-i-k}{r}\binom{n-j+N-i-1}{n-j}D_{r-k}(j)x^n.$$

Also from (1.10)

$$\frac{d^N}{dx^N}\left(\frac{x^r e^{-x}}{(1-x)^{r+1}}\right) = \sum_{n=0}^{\infty}\frac{d^N}{dx^N}\left(D_r(n)\frac{x^n}{n!}\right) = \sum_{n=0}^{\infty}D_r(n+N)\frac{x^n}{n!}. \tag{2.6}$$

Comparing the coefficients of x^n in the first and last series, we obtain the result. Similarly, (1.9) and (2.5) yield that

$$\frac{d^N}{dx^N}\left(\frac{x^r e^{-x}}{(1-x)^{r+1}}\right)$$
$$= \sum_{k=0}^{N}\sum_{i=0}^{N-k}(-1)^i\frac{N!}{i!}\binom{r}{k}\binom{r+N-i-k}{r}\frac{e^{-x}}{(1-x)^{r+N-i-k+1}}x^{r-k}$$
$$= \sum_{k=0}^{N}\sum_{i=0}^{N-k}(-1)^i\frac{N!}{i!}\binom{r}{k}\binom{r+N-i-k}{r}\sum_{n=0}^{\infty}d_n^{(r+N-i-k+1)}\frac{x^{n+r-k}}{n!}$$
$$= \sum_{k=0}^{N}\sum_{i=0}^{N-k}(-1)^i\frac{N!}{i!}\binom{r}{k}\binom{r+N-i-k}{r}\sum_{n=0}^{\infty}d_{n-r+k}^{(r+N-i-k+1)}\frac{x^n}{(n-r+k)!}$$
$$= N!\sum_{n=0}^{\infty}\sum_{k=0}^{N}\binom{r}{k}\sum_{i=0}^{N-k}\frac{(-1)^i}{i!}\binom{r+N-i-k}{r}\frac{d_{n-r+k}^{(r+N-i-k+1)}}{(n-r+k)!}x^n. \tag{2.7}$$

Thus, with the help of (2.6) and (2.7), comparing the coefficients of x^n on both sides, the other result is proof. □

Theorem 4. *For any positive integers N, n and r, we have*

$$\begin{aligned}
&\frac{1}{N!}\sum_{i=0}^{n} D_r(i+N)S_1(n,i)\\
&=\sum_{k=0}^{N}\sum_{i=0}^{N-k}\sum_{l=0}^{n}\sum_{j=0}^{l}(-1)^i\frac{l!}{i!j!}\binom{r}{k}\binom{r+N-i-k}{r}\binom{l-j+N-i-1}{l-j}\\
&\quad\times D_{r-k}(j)S_1(n,l)\\
&=\sum_{k=0}^{N}\sum_{i=0}^{N-k}\sum_{l=0}^{n}\sum_{j=0}^{l}(-1)^i\frac{(r-k)!}{i!}\binom{n}{l}\binom{r}{k}\binom{r+N-i-k}{r}\\
&\quad\times S_1(l,j)S_1(n-l,r-k)\,d_j^{(r+N-i-k+1)}\\
&=n!\sum_{k=0}^{N}\sum_{i=0}^{N-k}\sum_{j=0}^{n}\sum_{l=0}^{j}\frac{(-1)^{n+i+j+r+k}}{i!}\frac{l!}{j!}\binom{r}{k}\binom{r+N-i-k}{r}\binom{l+r+N-i-k}{l}\\
&\quad\times H(n-j,r-k-1)S_1(j,l).
\end{aligned}\tag{2.8}$$

Proof. We will give proof of (2.8). From (1.6), (1.8) and (2.5) we have

$$\begin{aligned}
&\frac{d^N}{dx^N}\left(\frac{(\ln(1-x))^r e^{-\ln(1-x)}}{(1-\ln(1-x))^{r+1}}\right)\\
&=N!\sum_{k=0}^{N}\sum_{i=0}^{N-k}\frac{(-1)^i}{i!}\binom{r}{k}\binom{r+N-i-k}{r}\frac{F_{r-k}(\ln(1-x))}{(1-\ln(1-x))^{N-i}}\\
&=N!\sum_{k=0}^{N}\sum_{i=0}^{N-k}\frac{(-1)^i}{i!}\binom{r}{k}\binom{r+N-i-k}{r}\\
&\quad\times\sum_{n=0}^{\infty}D_{r-k}(n)\frac{(\ln(1-x))^n}{n!}\sum_{n=0}^{\infty}\binom{n+N-i-1}{n}(\ln(1-x))^n.
\end{aligned}$$

By product of the generating functions and (1.5), we write

$$\begin{aligned}
&\frac{d^N}{dx^N}\left(\frac{(\ln(1-x))^r e^{-\ln(1-x)}}{(1-\ln(1-x))^{r+1}}\right)\\
&=\sum_{k=0}^{N}\sum_{i=0}^{N-k}(-1)^i\frac{N!}{i!}\binom{r}{k}\binom{r+N-i-k}{r}\\
&\quad\times\sum_{l=0}^{\infty}\sum_{j=0}^{l}\frac{l!}{j!}\binom{l-j+N-i-1}{l-j}D_{r-k}(j)\frac{(\ln(1-x))^l}{l!}\\
&=\sum_{k=0}^{N}\sum_{i=0}^{N-k}(-1)^i\frac{N!}{i!}\binom{r}{k}\binom{r+N-i-k}{r}
\end{aligned}$$

$$\times \sum_{l=0}^{\infty}\sum_{j=0}^{l}\frac{l!}{j!}\binom{l-j+N-i-1}{l-j}D_{r-k}(j)\sum_{n=l}^{\infty}(-1)^n S_1(n,l)\frac{x^n}{n!}$$

$$=\sum_{k=0}^{N}\sum_{i=0}^{N-k}(-1)^i\frac{N!}{i!}\binom{r}{k}\binom{r+N-i-k}{r}$$

$$\times \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{j=0}^{l}(-1)^n\frac{l!}{j!n!}\binom{l-j+N-i-1}{l-j}D_{r-k}(j)S_1(n,l)x^n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{N}\sum_{i=0}^{N-k}\sum_{l=0}^{n}\sum_{j=0}^{l}(-1)^{n+i}\frac{N!l!}{i!j!n!}\binom{r}{k}\binom{r+N-i-k}{r}$$

$$\times\binom{l-j+N-i-1}{l-j}D_{r-k}(j)S_1(n,l)x^n. \tag{2.9}$$

By (2.9) and taking $x\longrightarrow \ln(1-x)$ in (2.6), comparing the coefficients on both sides, the desired result is obtained. The proofs of other equalities are similar to the proof of (2.8). □

Theorem 5. *For any positive integers N, n, r and α, we have*

$$\binom{n+N}{N}^{-1}\sum_{k=0}^{N}\sum_{i=0}^{N-k}\frac{(-1)^i}{i!}\binom{r}{k}\binom{r+N-i-k}{r}\frac{d_{n-r+k}^{(r+N-i-k+1)}}{(n-r+k)!}$$

$$=\sum_{k=0}^{n+N}\sum_{i=0}^{k}(-1)^{n+N-i-r+1}\frac{C_{k-i-r+1}}{(n+N-k)!(k-i-r+1)!}H_i^{r+1}$$

$$=\sum_{k=0}^{n+N}\sum_{i=0}^{k}(-1)^{k-i}\alpha^{r-k+i}\frac{C_{k-i}^{r}}{(k-i)!}\frac{d_{n+N-k}^{(r)}}{(n+N-k)!}H(i,r-1,\alpha) \tag{2.10}$$

$$=\sum_{i=0}^{n+N}\frac{(-1)^i}{i!}\binom{n+N-i}{r} \tag{2.11}$$

$$=\sum_{i=0}^{n+N}\binom{n+N-i}{r-1}\frac{d_{i-1}}{(i-1)!} \tag{2.12}$$

$$=\frac{d_{n+N-r}^{(r+1)}}{(n+N-r)!}. \tag{2.13}$$

Proof. By (1.1) and (1.3), we have

$$\frac{e^{-x}x^r}{(1-x)^{r+1}}=\frac{-\ln(1-x)}{(1-x)^{r+1}}\frac{-x}{\ln(1-x)}x^{r-1}e^{-x}$$

$$=\sum_{n=0}^{\infty}H_n^{r+1}x^n\sum_{i=0}^{\infty}(-1)^iC_i\frac{x^{i+r-1}}{i!}\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{k!}$$

$$=\sum_{n=0}^{\infty} H_n^{r+1} x^n \sum_{i=0}^{\infty} (-1)^{i-r+1} C_{i-r+1} \frac{x^i}{(i-r+1)!} \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$$
$$=\sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{i=0}^{k} (-1)^{n-i-r+1} H_i^{r+1} \frac{C_{k-i-r+1}}{(n-k)!(k-i-r+1)!} x^n,$$

and by (1.11)

$$\frac{d^N}{dx^N}\left(\frac{e^{-x}x^r}{(1-x)^{r+1}}\right)$$
$$=\sum_{n=N}^{\infty} \sum_{k=0}^{n} \sum_{i=0}^{k} (-1)^{n-i-r+1} \binom{n}{N} N! \frac{C_{k-i-r+1} H_i^{r+1}}{(k-i-r+1)!} \frac{x^{n-N}}{(n-k)!}$$
$$=\sum_{n=0}^{\infty} \sum_{k=0}^{n+N} \sum_{i=0}^{k} (-1)^{n+N-i-r+1} \binom{n+N}{N} N! \frac{C_{k-i-r+1} H_i^{r+1}}{(k-i-r+1)!} \frac{x^n}{(n+N-k)!}.$$

From here, by (2.7), comparing the coefficients on both sides, the desired result is obtained. Similarly, from the equalities

$$\frac{e^{-x}x^r}{(1-x)^{r+1}} = \frac{(-\ln(1-\frac{x}{\alpha}))^r}{(1-x)} \left(\frac{(-\frac{x}{\alpha})}{\ln(1-\frac{x}{\alpha})}\right)^r \alpha^r \left(\frac{1}{1-x}\right)^r e^{-x},$$
$$\frac{e^{-x}x^r}{(1-x)^{r+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \sum_{n=r}^{\infty} \binom{n}{r} x^n = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(-1)^i}{i!} \binom{n-i}{r} x^n,$$
$$\frac{e^{-x}x^r}{(1-x)^{r+1}} = \sum_{n=r-1}^{\infty} \binom{n}{r-1} x^n \sum_{n=0}^{\infty} d_n \frac{x^{n+1}}{n!} = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n-i}{r-1} d_{i-1} \frac{x^n}{(i-1)!},$$
$$\frac{e^{-x}x^r}{(1-x)^{r+1}} = \sum_{n=r}^{\infty} d_{n-r}^{(r+1)} \frac{x^n}{(n-r)!},$$

respectively, and (1.11), we have the proofs of (2.10)-(2.13). □

For example, by setting $n+N \longrightarrow n$ in (2.11)-(2.13), it is seen that for any positive integers n and r,

$$\sum_{i=0}^{n} \binom{n-i}{r} \frac{(-1)^i}{i!} = \sum_{i=0}^{n} \binom{n-i}{r-1} \frac{d_{i-1}}{(i-1)!} = \frac{d_{n-r}^{(r+1)}}{(n-r)!}. \tag{2.14}$$

Corollary 1. *For any positive integers n, r and α, we have*

$$\sum_{k=0}^{n} \sum_{i=0}^{k} \sum_{j=0}^{n-k+r-1} (-1)^{j+k-i} \alpha^{r-k+i} \binom{n-k-j+r-1}{r-1} \frac{C_{k-i}^r}{(k-i)!} \frac{H(i,r-1,\alpha)}{j!}$$
$$= \sum_{k=0}^{n} \sum_{i=0}^{k} (-1)^{n-i-r+1} \frac{C_{k-i-r+1}}{(k-i-r+1)!} \frac{H_i^{r+1}}{(n-k)!}.$$

Proof. By setting $n+N \longrightarrow n$ in Theorem 5 and (2.14), the proof is clearly given. □

Theorem 6. *Let n, r and m be positive integers such that $r > m$. We have*

$$\sum_{i=0}^{n} S_1(n,i)\, D_r(i) = \sum_{j=0}^{n}\sum_{i=0}^{j} (-1)^{j-i}\, i! \binom{j}{i}\binom{i}{r} S_1(n,j)$$
$$= \sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{i=0}^{j} (-1)^{j-i}(i+r)! \binom{j}{i}\binom{n}{k} S_1(k,j)\, S_1(n-k,r)$$
$$= (-1)^{r-m} m! n! \sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{i=0}^{n-k} (-1)^{j-k} \frac{i!}{j!\,(n-k)!}\binom{i+r}{i} H(k-j, r-m-1)$$
$$\times S_1(j,m)\, S_1(n-k,i),$$

and

$$\sum_{i=0}^{n} (-1)^n S_1(n,i) D_r(i) = r! \sum_{i=r}^{n} (-1)^n \binom{i}{r} d_{i-r}^{(r+1)} S_1(n,i)$$
$$= n! \sum_{k=0}^{n}\sum_{i=0}^{k} (-1)^{r+k} \frac{i!}{k!}\binom{i+r}{i} S_1(k,i) H(n-k, r-1).$$

Proof. Taking $\ln(1-x)$ instead of x and setting $n+N \longrightarrow n$ in Teorem 5, we can easily obtain these equalities. □

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