Miskolc Mathematical Notes
Vol. 24 (2023), No. 1, pp. 31–46

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.3839

NONLINEAR COUPLED LIOUVILLE–CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH A NEW CLASS OF NONLOCAL BOUNDARY CONDITIONS

BASHIR AHMAD, AHMED ALSAEDI, FAWZIAH M. ALOTAIBI,
AND MADEAHA ALGHANMI

Received 14 May, 2021

Abstract. In this paper, we study a coupled system of nonlinear Liouville–Caputo fractional differential equations equipped with a new set of nonlocal boundary conditions involving an arbitrary strip together with two sets of nonlocal multi-points on either part of the strip on the given domain. We emphasize that the boundary conditions considered in this study are formulated with respect to the sum and difference of the unknown functions. We apply the well-known tools of the fixed point theory to derive the main results. Examples are presented for the illustration of the obtained results.

2010 *Mathematics Subject Classification:* 34A08; 34B10; 34B15

Keywords: Coupled fractional differential equations, nonlocal multipoint boundary conditions, existence, fixed point

1. INTRODUCTION

Fractional calculus has evolved as an important area of research in view of its diverse applications in a variety of applied fields. Examples include immunology [8], chaotic systems [21], ecology [12], virology [7], economic model [20], neural networks [6], etc. A salient feature distinguishing fractional-order differential and integral operators from the classical ones is their nonlocal nature, which can trace the past history of the phenomena and processes under investigation. In the recent years, many researchers contributed to the development of fractional calculus, for example, see [3], [4], [5], [9], [11], [13], [22] and the references cited therein. One can also find a substantial material dealing with coupled systems of fractional differential equations in the recent literature, for instance see [1], [2], [10], [14] and the references cited therein.

In survey-cum-expository review articles [15–18], Srivastava described some recent developments on the subject of fractional calculus and its applications. In the survey [15], some recent developments involving different types of the Mittag-Leffler type functions associated with generalized Riemann-Liouville and other related fractional derivative operators are reviewed. The article [16] contains a brief elementary and introductory overview of the theory and applications of the integral and derivative operators of fractional calculus. In [17], an overview of the fractional-calculus operators based on the general Fox-Wright function and its specialized forms is presented, while [18] deals with some variations of the operators of fractional calculus, related special functions and integral transformations.

In this paper, we consider a Liouville-Caputo type coupled system of nonlinear fractional differential equations supplemented with a new set of boundary conditions given by

$$\begin{cases} {}^{LC}D^{\alpha}\nu(t) = \varphi(t,\nu(t),\omega(t)), \quad t \in [0,T], \\ {}^{LC}D^{\beta}\omega(t) = \psi(t,\nu(t),\omega(t)), \quad t \in [0,T], \\ (\nu+\omega)(0) = -(\nu+\omega)(T), \\ \displaystyle\int_{\eta}^{\zeta}(\nu-\omega)(s)ds - \sum_{i=1}^{m} a_i(\nu-\omega)(\sigma_i) - \sum_{j=1}^{n} b_j(\nu-\omega)(\delta_j) = A, \end{cases} \tag{1.1}$$

where ${}^{LC}D^{\alpha}_{0+}$ and ${}^{LC}D^{\beta}_{0+}$ denote the Liouville-Caputo fractional derivative operators of order $\alpha,\beta \in (0,1]$, $0 < \sigma_i < \eta < \zeta < \delta_j < T$, $i = 1,\ldots,m$, $j = 1,\ldots,n$, and a_i, b_j, A are nonnegative constants, $\varphi,\psi : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions. The first condition in (1.1) is anti-periodic one with respect to the sum of unknown functions ν and ω, while the second condition describes that the contribution of the difference of the unknown functions ν and ω on an arbitrary strip (η,ξ) within the given domain $[0,T]$ differs from the sum of such contributions due to arbitrary positions at $\sigma_i, i = 1,\ldots,m$ and $\delta_j, j = 1,\ldots,n$ (off the strip (η,ξ)) by a constant.

The present work is motivated by a recent article [2] in which the authors studied the system in (1.1) with boundary conditions of the form:

$$(\nu+\omega)(0) = -(\nu+\omega)(T), \quad \int_{\eta}^{\zeta}(\nu-\omega)(s)ds = A.$$

Here our objective is to enhance and generalize the study established in [2] by introducing more general nonlocal boundary conditions describing the changes within the given domain.

In the next section, we provide the related definitions of fractional integral and derivatives, and prove an auxiliary lemma. Our main results, presented in Section 3, rely on Schaefer like fixed point theorem and contraction mapping principle.

2. Preliminaries

Let us commence this section with some definitions.

Definition 1. ([13]). For a function $\upsilon \in L_1[a,b], -\infty < a < b < +\infty$, we define the Riemann–Liouville fractional integral $I_a^{\rho}\upsilon$ of order $\rho > 0$ existing almost everywhere on $[a,b]$ as

$$I_a^{\rho}\upsilon(t) = \frac{1}{\Gamma(\rho)}\int_a^t (t-\tau)^{\rho-1}\upsilon(\tau)d\tau,$$

where Γ denotes the Euler gamma function.

Definition 2. [13]. The Riemann–Liouville fractional derivative $D_a^{\rho}\upsilon$ of order $\rho \in (p-1,p], p \in N$ existing almost everywhere on $[a,b]$ is defined as

$$D_a^{\rho}\upsilon(t) = \frac{d^p}{dt^p}I_a^{p-\rho}\upsilon(t) = \frac{1}{\Gamma(p-\rho)}\frac{d^p}{dt^p}\int_a^t (t-\tau)^{p-1-\rho}\upsilon(\tau)d\tau,$$

where $\upsilon, \upsilon^{(p)} \in L_1[a,b]$.

The Liouville-Caputo fractional derivative ${}^{LC}D_a^{\rho}\upsilon$ of order $\rho \in (p-1,p], p \in N$ is defined as

$${}^{LC}D_a^{\rho}\upsilon(t) = D_a^{\rho}\left[\upsilon(t) - \upsilon(a) - \upsilon'(a)\frac{(t-a)}{1!} - \ldots - \upsilon^{(p-1)}(a)\frac{(t-a)^{p-1}}{(p-1)!}\right].$$

Remark 1. [13]. The Liouville-Caputo fractional derivative ${}^{LC}D_a^{\rho}\upsilon$ of order $\rho \in (p-1,p], p \in N$ for a function $\upsilon \in AC^p[a,b]$, existing almost everywhere on $[a,b]$, is defined as

$${}^{LC}D_a^{\rho}\upsilon(t) = I_a^{p-\rho}\upsilon^{(p)}(t) = \frac{1}{\Gamma(p-\rho)}\int_a^t (t-\tau)^{p-1-\rho}\upsilon^{(p)}(\tau)d\tau.$$

In the follow lemma, we solve a linear variant of the problem (1.1).

Lemma 1. *Let $F,G \in C[0,T]$. Then the solution of the following linear coupled boundary value problem:*

$$\begin{cases} {}^{LC}D^{\alpha}\nu(t) = F(t), \; 0<\alpha<1, \; t \in [0,T], \\ {}^{LC}D^{\beta}\omega(t) = G(t), \; 0<\beta<1, \; t \in [0,T], \\ (\nu+\omega)(0) = -(\nu+\omega)(T), \\ \int_{\eta}^{\zeta}(\nu-\omega)(s)ds - \sum_{i=1}^{m} a_i(\nu-\omega)(\sigma_i) - \sum_{j=1}^{n} b_j(\nu-\omega)(\delta_j) = A, \end{cases} \tag{2.1}$$

is given by

$$\nu(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds \tag{2.2}$$
$$+\frac{1}{2}\Bigg\{\frac{A}{\Lambda} - \frac{1}{2}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds + \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}G(s)ds\Big)$$

$$
\begin{aligned}
&-\frac{1}{\Lambda}\int_{\eta}^{\zeta}\Big(\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)}F(x)dx-\int_0^s \frac{(s-x)^{\beta-1}}{\Gamma(\beta)}G(x)dx\Big)ds\\
&+\frac{\sum_{i=1}^m a_i}{\Lambda}\Big(\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\alpha-1}}{\Gamma(\alpha)}F(x)dx-\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\beta-1}}{\Gamma(\beta)}G(x)dx\Big)\\
&+\frac{\sum_{j=1}^n b_j}{\Lambda}\Big(\int_0^{\delta_j} \frac{(\delta_j-x)^{\alpha-1}}{\Gamma(\alpha)}F(x)dx-\int_0^{\delta_j} \frac{(\delta_j-x)^{\beta-1}}{\Gamma(\beta)}G(x)dx\Big)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\omega(t)=&\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}G(s)ds \qquad (2.3)\\
&+\frac{1}{2}\Bigg\{\frac{-A}{\Lambda}-\frac{1}{2}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds+\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}G(s)ds\Big)\\
&+\frac{1}{\Lambda}\int_{\eta}^{\xi}\Big(\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)}F(x)dx-\int_0^s \frac{(s-x)^{\beta-1}}{\Gamma(\beta)}G(x)dx\Big)ds\\
&-\frac{\sum_{i=1}^m a_i}{\Lambda}\Big(\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\alpha-1}}{\Gamma(\alpha)}F(x)dx-\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\beta-1}}{\Gamma(\beta)}G(x)dx\Big)\\
&-\frac{\sum_{j=1}^n b_j}{\Lambda}\Big(\int_0^{\delta_j} \frac{(\delta_j-x)^{\alpha-1}}{\Gamma(\alpha)}F(x)dx-\int_0^{\delta_j} \frac{(\delta_j-x)^{\beta-1}}{\Gamma(\beta)}G(x)dx\Big)\Bigg\},
\end{aligned}
$$

where

$$\Lambda:=\zeta-\eta-\sum_{i=1}^m a_i-\sum_{j=1}^n b_j\neq 0, \tag{2.4}$$

Proof. Applying the operators I^{α} and I^{β} on both sides of fractional differential equations in (2.1), respectively (for details, see [13]), we obtain

$$\nu(t)=\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds+c_1, \tag{2.5}$$

$$\omega(t)=\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}h_2(s)ds+c_2, \tag{2.6}$$

where $c_1,c_2\in\mathbb{R}$. Using (2.5) and (2.6) in the boundary conditions of the problem (2.1), we get

$$c_1+c_2=-\frac{1}{2}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds+\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}G(s)ds\Big), \tag{2.7}$$

$$
\begin{aligned}
c_1-c_2=\frac{1}{\Lambda}\Bigg\{&A-\int_{\eta}^{\xi}\Big(\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)}F(x)dx-\int_0^s \frac{(s-x)^{\beta-1}}{\Gamma(\beta)}G(x)dx\Big)ds\\
&+\sum_{i=1}^m a_i\Big(\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\alpha-1}}{\Gamma(\alpha)}F(x)dx-\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\beta-1}}{\Gamma(\beta)}G(x)dx\Big)
\end{aligned}
$$

$$+\sum_{j=1}^{n} b_j\Big(\int_0^{\delta_j}\frac{(\delta_j-x)^{\alpha-1}}{\Gamma(\alpha)}F(x)dx-\int_0^{\delta_j}\frac{(\delta_j-x)^{\beta-1}}{\Gamma(\beta)}G(x)dx\Big)\Bigg\}. \tag{2.8}$$

Solving (2.7) and (2.8) for c_1 and c_2 yields

$$\begin{aligned}
c_1 =&\frac{1}{2}\Bigg\{\frac{A}{\Lambda}-\frac{1}{2}\Big(\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds+\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}G(s)ds\Big)\\
&-\frac{1}{\Lambda}\int_\eta^\zeta\Big(\int_0^s\frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)}F(x)dx-\int_0^s\frac{(s-x)^{\beta-1}}{\Gamma(\beta)}G(x)dx\Big)ds\\
&+\frac{\sum_{i=1}^m a_i}{\Lambda}\Big(\int_0^{\sigma_i}\frac{(\sigma_i-x)^{\alpha-1}}{\Gamma(\alpha)}F(x)dx-\int_0^{\sigma_i}\frac{(\sigma_i-x)^{\beta-1}}{\Gamma(\beta)}G(x)dx\Big)\\
&+\frac{\sum_{j=1}^n b_j}{\Lambda}\Big(\int_0^{\delta_j}\frac{(\delta_j-x)^{\alpha-1}}{\Gamma(\alpha)}F(x)dx-\int_0^{\delta_j}\frac{(\delta_j-x)^{\beta-1}}{\Gamma(\beta)}G(x)dx\Big)\Bigg\},\\
c_2 =&\frac{1}{2}\Bigg\{\frac{-A}{\Lambda}-\frac{1}{2}\Big(\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds+\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}G(s)ds\Big)\\
&+\frac{1}{\Lambda}\int_\eta^\xi\Big(\int_0^s\frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)}F(x)dx-\int_0^s\frac{(s-x)^{\beta-1}}{\Gamma(\beta)}G(x)dx\Big)ds\\
&-\frac{\sum_{i=1}^m a_i}{\Lambda}\Big(\int_0^{\sigma_i}\frac{(\sigma_i-x)^{\alpha-1}}{\Gamma(\alpha)}F(x)dx-\int_0^{\sigma_i}\frac{(\sigma_i-x)^{\beta-1}}{\Gamma(\beta)}G(x)dx\Big)\\
&-\frac{\sum_{j=1}^n b_j}{\Lambda}\Big(\int_0^{\delta_j}\frac{(\delta_j-x)^{\alpha-1}}{\Gamma(\alpha)}F(x)dx-\int_0^{\delta_j}\frac{(\delta_j-x)^{\beta-1}}{\Gamma(\beta)}G(x)dx\Big)\Bigg\}.
\end{aligned}$$

Substituting the values of c_1 and c_2 in (2.5) and (2.6) yields the solution (2.2) and (2.3). By direct computation, one can prove the converse of this lemma. The proof is finished. □

3. Main results

Let $X = C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})$ be the Banach space equipped with the norm $\|(\nu,\omega)\| = \|\nu\|+\|\omega\| = \sup_{t\in[0,T]}|\nu(t)|+\sup_{t\in[0,T]}|\omega(t)|$, for $(\nu,\omega)\in X$. In order to transform the problem (1.1) into a fixed point problem, we define an operator $\Pi : X\to X$ via Lemma 1 as

$$\Pi(\nu,\omega)(t) := (\Pi_1(\nu,\omega)(t),\Pi_2(\nu,\omega)(t)), \tag{3.1}$$

$$\begin{aligned}
\Pi_1(\nu,\omega)(t) =& \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi(s,\nu(s),\omega(s))ds \\
&+\frac{1}{2}\Bigg\{\frac{A}{\Lambda}-\frac{1}{2}\Big(\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s,\nu(s),\omega(s))ds
\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&+\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}\psi(s,\nu(s),\omega(s))ds\Big) \\
&-\frac{1}{\Lambda}\int_\eta^\xi\Big(\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)}\varphi(x,\nu(x),\omega(x))dx \\
&-\int_0^s \frac{(s-x)^{\beta-1}}{\Gamma(\beta)}\psi(x,\nu(x),\omega(x))dx\Big)ds \\
&+\frac{\sum_{i=1}^m a_i}{\Lambda}\Big(\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\alpha-1}}{\Gamma(\alpha)}\varphi(x,\nu(x),\omega(x))dx \\
&-\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\beta-1}}{\Gamma(\beta)}\psi(x,u(x),v(x))dx\Big) \\
&+\frac{\sum_{j=1}^n b_j}{\Lambda}\Big(\int_0^{\delta_j} \frac{(\delta_j-x)^{\alpha-1}}{\Gamma(\alpha)}\varphi(x,\nu(x),\omega(x))dx \\
&-\int_0^{\delta_j} \frac{(\delta_j-x)^{\beta-1}}{\Gamma(\beta)}\psi(x,\nu(x),\omega(x))dx\Big)\Bigg\}
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi_2(\nu,\omega)(t) = &\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\psi(s,\nu(s),\omega(s))ds \qquad (3.3)\\
&+\frac{1}{2}\Bigg\{\frac{-A}{\Lambda}-\frac{1}{2}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s,\nu(s),\omega(s))ds \\
&+\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}\psi(s,\nu(s),\omega(s))ds\Big) \\
&+\frac{1}{\Lambda}\int_\eta^\xi\Big(\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)}\varphi(x,\nu(x),\omega(x))dx \\
&-\int_0^s \frac{(s-x)^{\beta-1}}{\Gamma(\beta)}\psi(x,\nu(x),\omega(x))dx\Big)ds \\
&-\frac{\sum_{i=1}^m a_i}{\Lambda}\Big(\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\alpha-1}}{\Gamma(\alpha)}\varphi(x,\nu(x),\omega(x))dx \\
&-\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\beta-1}}{\Gamma(\beta)}\psi(x,\nu(x),\omega(x))dx\Big) \\
&-\frac{\sum_{j=1}^n b_j}{\Lambda}\Big(\int_0^{\delta_j} \frac{(\delta_j-x)^{\alpha-1}}{\Gamma(\alpha)}\varphi(x,\nu(x),\omega(x))dx \\
&-\int_0^{\delta_j} \frac{(\delta_j-x)^{\beta-1}}{\Gamma(\beta)}\psi(x,\nu(x),\omega(x))dx\Big)\Bigg\}.
\end{aligned}
$$

In the sequel, we need the following assumptions.

(H_1) There exist continuous functions $\mu_i, \kappa_i \in C([0,T],\mathbb{R}^+), i = 1,2,3$, such that, $\forall(t,\nu,\omega) \in [0,T]\times\mathbb{R}^2$,

$$|\varphi(t,\nu,\omega)| \le \mu_1(t) + \mu_2(t)|\nu| + \mu_3(t)|\omega|;$$
$$|\psi(t,\nu,\omega)| \le \kappa_1(t) + \kappa_2(t)|\nu| + \kappa_3(t)|\omega|.$$

(H_2) There exist $C_i, K_i > 0$, $i = 1,2$, such that, $\forall t \in [0,T], \nu_i, \omega_i \in \mathbb{R}, i = 1,2$,

$$|\varphi(t,\nu_1,\omega_1) - \varphi(t,\nu_2,\omega_2)| \le C_1\|\nu_1-\nu_2\| + C_2\|\omega_1-\omega_2\|,$$
$$|\psi(t,\nu_1,\omega_1) - \psi(t,\nu_2,\omega_2)| \le K_1\|\nu_1-\nu_2\| + K_2\|\omega_1-\omega_2\|.$$

For the sake of computational convenience, we set the notation:

$$\Delta_1 = \frac{T^\alpha}{4\Gamma(\alpha+1)} + \frac{1}{2|\Lambda|}\Big[\frac{\zeta^{\alpha+1}-\eta^{\alpha+1}}{\Gamma(\alpha+2)} + \sum_{i=1}^m a_i \frac{\sigma_i^\alpha}{\Gamma(\alpha+1)} + \sum_{j=1}^n b_j \frac{\delta_j^\alpha}{\Gamma(\alpha+1)}\Big], \tag{3.4}$$

$$\Delta_2 = \frac{T^\beta}{4\Gamma(\beta+1)} + \frac{1}{2|\Lambda|}\Big[\frac{\zeta^{\beta+1}-\eta^{\beta+1}}{\Gamma(\beta+2)} + \sum_{i=1}^m a_i \frac{\sigma_i^\beta}{\Gamma(\beta+1)} + \sum_{j=1}^n b_j \frac{\delta_j^\beta}{\Gamma(\beta+1)}\Big], \tag{3.5}$$

and

$$M_0 = \min\Bigg\{1 - \Bigg[\|\mu_2\|\Big(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)}\Big) + \|\kappa_2\|\Big(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)}\Big)\Bigg], 1 - \Bigg[\|\mu_3\|\Big(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)}\Big) + \|\kappa_3\|\Big(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)}\Big)\Bigg]\Bigg\}.$$

Our first existence result for the problem (1.1) is based on the following fixed point theorem [19].

Lemma 2. *Let $\mathcal{E}$ be a Banach space and $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ be a completely continuous operator. If the set $\mathcal{S} = \{x \in \mathcal{E} | x = \mu\mathcal{F}x, 0 < \mu < 1\}$ is bounded, then the operator $\mathcal{F}$ has a fixed point in $\mathcal{E}$.*

Theorem 1. *Suppose that the assumption (H_1) holds. Then there exists at least one solution for the problem (1.1) on $[0,T]$ if*

$$\|\mu_2\|\Big(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)}\Big) + \|\kappa_2\|\Big(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)}\Big) < 1, \tag{3.6}$$

$$\|\mu_3\|\Big(2\Delta_1+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big)+\|\kappa_3\|\Big(2\Delta_2+\frac{T^{\beta}}{\Gamma(\beta+1)}\Big)<1,$$

where $\Delta_i\,(i=1,2)$ *are given by* (3.4)-(3.5).

Proof. Observe that continuity of functions φ and ψ implies that the operator $\Pi: X\to X$ is continuous. Let $\Omega_{\bar r}\subset X$ be bounded. Then we can find positive constants L_φ and L_ψ such that

$$|\varphi(t,\nu(t),\omega(t))|\le L_\varphi,\ |\psi(t,\nu(t),\omega(t))|\le L_\psi,\ \forall(\nu,\omega)\in\Omega_{\bar r}.$$

So, for any $(\nu,\omega)\in\Omega_{\bar r}$, $t\in[0,T]$, we get

$$|\Pi_1(\nu,\omega)(t)|\le L_\varphi\Big(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\Delta_1\Big)+L_\psi\Delta_2+\frac{A}{2|\Lambda|},$$
$$|\Pi_2(\nu,\omega)(t)|\le L_\varphi\Delta_1+L_\psi\Big(\frac{T^{\beta}}{\Gamma(\beta+1)}+\Delta_2\Big)+\frac{A}{2|\Lambda|}.$$

Thus

$$\begin{aligned}\|\Pi(\nu,\omega)\|&=\|\Pi_1(\nu,\omega)\|+\|\Pi_2(\nu,\omega)\|\\&\le L_\varphi\Big(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+2\Delta_1\Big)+L_\psi\Big(\frac{T^{\beta}}{\Gamma(\beta+1)}+2\Delta_2\Big)+\frac{A}{|\Lambda|},\end{aligned}$$

which implies that the operator Π is uniformly bounded.

Next it will be shown that Π maps bounded sets into equicontinuous sets of X. For $t_1,t_2\in[0,T]$, $t_1<t_2$, and $(\nu,\omega)\in\Omega_{\bar r}$, we have

$$\begin{aligned}&|\Pi_1(\nu,\omega)(t_2)-\Pi_1(\nu,\omega)(t_1)|\\&\le\Bigg|\frac{1}{\Gamma(\alpha)}\Big(\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]\varphi(s,\nu(s),\omega(s))ds\\&\qquad+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\varphi(s,\nu(s),\omega(s))ds\Big)\Bigg|\\&\le L_\varphi\Big(\frac{2(t_2-t_1)^{\alpha}+t_2^{\alpha}-t_1^{\alpha}}{\Gamma(\alpha+1)}\Big)\to0\text{ when }t_1\longrightarrow t_2,\end{aligned}$$

independently of $(u,v)\in\Omega_{\bar r}$. Likewise

$$|\Pi_2(\nu,\omega)(t_2)-\Pi_2(\nu,\omega)(t_1)|\le L_\psi\Big(\frac{2(t_2-t_1)^{\beta}+t_2^{\beta}-t_1^{\beta}}{\Gamma(\beta+1)}\Big)\to0$$

when $t_1\longrightarrow t_2$, independently of $(\nu,\omega)\in\Omega_{\bar r}$. Thus it follows by the Arzelá-Ascoli theorem that the operator $\Pi: X\longrightarrow X$ is completely continuous.

Next,we consider the set $\mathcal{U} = \{(\nu,\omega) \in X | u = \lambda\Pi(\nu,\omega), 0 < \lambda < 1\}$ and prove that it is bounded. Let $(\nu,\omega) \in \mathcal{U}$, then $(\nu,\omega) = \lambda\Pi(\nu,\omega)$, $0 < \lambda < 1$. For any $t \in [0,T]$, we have

$$\nu(t) = \lambda\Pi_1(\nu,\omega)(t), \;\; \omega(t) = \lambda\Pi_2(\nu,\omega)(t).$$

As in the previous step, using $\Delta_i\,(i = 1,2)$ given by (3.4)-(3.5), we find that

$$\begin{aligned}
|\nu(t)| = \lambda|\Pi_1(\nu,\omega)(t)| \leq & \Big(\|\mu_1\| + \|\mu_2\|\|\nu\| + \|\mu_3\|\|\omega\|\Big)\Big(\frac{T^\alpha}{\Gamma(\alpha+1)} + \Delta_1\Big) \\
& + \Big(\|\kappa_1\| + \|\kappa_2\|\|\nu\| + \|\kappa_3\|\|\omega\|\Big)\Delta_2 + \frac{A}{2|\Lambda|}, \\
|\omega(t)| = \lambda|\Pi_2(\nu,\omega)(t)| \leq & \Big(\|\mu_1\| + \|\mu_2\|\|\nu\| + \|\mu_3\|\|\omega\|\Big)\Delta_1 \\
& + \Big(\|\kappa_1\| + \|\kappa_2\|\|\nu\| + \|\kappa_3\|\|\omega\|\Big)\Big(\frac{T^\beta}{\Gamma(\beta+1)} + \Delta_2\Big) + \frac{A}{2|\Lambda|}.
\end{aligned}$$

In consequence, we get

$$\begin{aligned}
\|\nu\| + \|\omega\| \leq & \|\mu_1\|\Big(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)}\Big) + \|\kappa_1\|\Big(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)}\Big) + \frac{A}{|\Lambda|} \\
& + \Big[\|\mu_2\|\Big(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)}\Big) + \|\kappa_2\|\Big(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)}\Big)\Big]\|\nu\| \\
& + \Big[\|\mu_3\|\Big(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)}\Big) + \|\kappa_3\|\Big(2\Delta_2 + \frac{T^\alpha}{\Gamma(\alpha+1)}\Big)\Big]\|\omega\|.
\end{aligned}$$

By the condition (3.6), it follows that

$$\|(\nu,\omega)\| \leq \frac{\|\mu_1\|\Big(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)}\Big) + \|\kappa_1\|\Big(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)}\Big) + \frac{A}{|\Lambda|}}{M_0},$$

which implies that the set $\mathcal{U}$ is bounded. In consequence, we deduce that the operator Π has at least one fixed point by the conclusion of Lemma 2, which is a solution of the problem (1.1). □

The statement of Theorem 1 reduces to the the following special form by fixing $\mu_2(t) = \mu_3(t) \equiv 0$ and $\kappa_2(t) = \kappa_3(t) \equiv 0$ in it.

Corollary 1. *Let $\varphi,\psi : J \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions such that $|\varphi(t,\nu,\omega)| \leq \mu_1(t), |\psi(t,\nu,\omega)| \leq \kappa_1(t), \forall (t,\nu,\omega) \in [0,T] \times \mathbb{R}^2$, where $\mu_1, \kappa_1 \in C([0,T],\mathbb{R}^+)$. Then the problem (1.1) has at least one solution on $[0,T]$.*

Corollary 2. *If $\mu_i(t) = \lambda_i$, $\kappa_i(t) = \varepsilon_i, i = 1,2,3,$ then the condition (H_1) becomes:*
(H_1') *there exist real constants $\lambda_i, \varepsilon_i > 0$, $(i = 1,2)$ such that*

$$|\varphi(t,\nu,\omega)| \leq \lambda_1 + \lambda_2|\nu| + \lambda_3|\omega| \quad \forall (t,\nu,\omega) \in [0,T] \times \mathbb{R}^2;$$
$$|\psi(t,\nu,\omega)| \leq \varepsilon_1 + \varepsilon_2|\nu| + \varepsilon_3|\omega| \quad \forall (t,\nu,\omega) \in [0,T] \times \mathbb{R}^2;$$

and (3.6) *takes the form:*

$$\lambda_2\Big(2\Delta_1+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big)+\varepsilon_2\Big(2\Delta_2+\frac{T^{\beta}}{\Gamma(\beta+1)}\Big)<1,$$
$$\lambda_3\Big(2\Delta_1+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big)+\varepsilon_3\Big(2\Delta_2+\frac{T^{\beta}}{\Gamma(\beta+1)}\Big)<1.$$

In the following result, we prove the uniqueness of solutions to the problem (1.1) via Banach contraction mapping principle.

Theorem 2. *If the assumption* (H_2) *holds, then the problem (1.1) has a unique solution on* $[0,T]$ *provided that*

$$\mathcal{C}\Big(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+2\Delta_1\Big)+\mathcal{K}\Big(\frac{T^{\beta}}{\Gamma(\beta+1)}+2\Delta_2\Big)<1, \tag{3.7}$$

where $\mathcal{C}=\max\{C_1,C_2\}$, $\mathcal{K}=\max\{K_1,K_2\}$ *and* Δ_i, $i=1,2$ *are defined by (3.4)-(3.5).*

Proof. Fix

$$r>\frac{M_1\Big(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+2\Delta_1\Big)+M_2\Big(\frac{T^{\beta}}{\Gamma(\beta+1)}+2\Delta_2\Big)+\frac{A}{|\Lambda|}}{1-\Big(\mathcal{C}\Big(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+2\Delta_1\Big)+\mathcal{K}\Big(\frac{T^{\beta}}{\Gamma(\beta+1)}+2\Delta_2\Big)\Big)},$$

where $M_1=\sup_{t\in[0,T]}|\varphi(t,0,0)|$, and $M_2=\sup_{t\in[0,T]}|\psi(t,0,0)|$. Then we show that $\Pi B_r\subset B_r$, where $\Pi:X\to X$ is defined by (3.1) and $B_r=\{(\nu,\omega)\in X:\|(\nu,\omega)\|\le r\}$. By the assumption (H_1), for $(\nu,\omega)\in B_r$, $t\in[0,T]$, we have

$$|\varphi(t,\nu(t),\omega(t))|\le|\varphi(t,\nu(t),\omega(t))-\varphi(t,0,0)|+|\varphi(t,0,0)|\le\mathcal{C}(\|\nu\|+\|\omega\|)+M_1$$

and

$$|\psi(t,\nu(t),\omega(t))|\le\mathcal{K}(\|\nu\|+\|\omega\|)+M_2.$$

In consequence, for $(\nu,\omega)\in B_r$, we obtain

$$\begin{aligned}|\Pi_1(\nu,\omega)(t)|\le{}&\frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big(\mathcal{C}(\|\nu\|+\|\omega\|)+M_1\Big)\\&+\frac{1}{2}\Bigg[\frac{A}{|\Lambda|}+\frac{1}{2}\Big(\frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big(\mathcal{C}(\|\nu\|+\|\omega\|)+M_1\Big)\\&\quad+\frac{T^{\beta}}{\Gamma(\beta+1)}\Big(\mathcal{K}(\|\nu\|+\|\omega\|)+M_2\Big)\Big)\\&\quad+\frac{1}{|\Lambda|}\Big(\frac{\zeta^{\alpha+1}-\eta^{\alpha+1}}{\Gamma(\alpha+2)}\Big(\mathcal{C}(\|\nu\|+\|\omega\|)+M_1\Big)\\&\quad+\frac{\zeta^{\beta+1}-\eta^{\beta+1}}{\Gamma(\beta+2)}\Big(K(\|\nu\|+\|\omega\|)+M_2\Big)\Big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sum_{i=1}^{m} a_i}{|\Lambda|}\Big(\frac{\sigma_i^{\alpha}}{\Gamma(\alpha+1)}\Big(\mathcal{C}(\|\nu\|+\|\omega\|)+M_1\Big)\\
&+\frac{\sigma_i^{\beta}}{\Gamma(\beta+1)}\Big(\mathcal{K}(\|\nu\|+\|\omega\|)+M_2\Big)\Big)\\
&+\frac{\sum_{j=1}^{n} b_j}{|\Lambda|}\Big(\frac{\delta_j^{\alpha}}{\Gamma(\alpha+1)}\Big(\mathcal{C}(\|\nu\|+\|\omega\|)+M_1\Big)\\
&+\frac{\delta_j^{\beta}}{\Gamma(\beta+1)}\Big(\mathcal{K}(\|\nu\|+\|\omega\|)+M_2\Big)\Big)\Big],
\end{aligned}
$$

which, on taking the norm for $t\in[0,T]$, leads to

$$
\begin{aligned}
\|\Pi_1(\nu,\omega)\| \le{}& \left(\mathcal{C}\left(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\Delta_1\right)+\mathcal{K}\Delta_2\right)(\|\nu\|+\|\omega\|)\\
&+M_1\Big(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\Delta_1\Big)+M_2\Delta_2+\frac{A}{2|\Lambda|}.
\end{aligned}
$$

In the same way, for $(\nu,\omega)\in B_r$, one can obtain

$$
\begin{aligned}
\|\Pi_2(\nu,\omega)\| \le{}& \Big(\mathcal{C}\Delta_1+\mathcal{K}\Big(\frac{T^{\beta}}{\Gamma(\beta+1)}+\Delta_2\Big)\Big)(\|\nu\|+\|\omega\|)\\
&+M_1\Delta_1+M_2\Big(\frac{T^{\beta}}{\Gamma(\beta+1)}+\Delta_2\Big)+\frac{A}{2|\Lambda|}.
\end{aligned}
$$

Therefore, for any $(\nu,\omega)\in B_r$, we have

$$
\begin{aligned}
\|\Pi(\nu,\omega))\| &= \|\Pi_1(\nu,\omega)\|+\|\Pi_2(\nu,\omega)\|\\
&\le \Big(\mathcal{C}\Big(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+2\Delta_1\Big)+\mathcal{K}\Big(\frac{T^{\beta}}{\Gamma(\beta+1)}+2\Delta_2\Big)\Big)(\|\nu\|+\|\omega\|)\\
&\quad +M_1\Big(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+2\Delta_1\Big)+M_2\Big(\frac{T^{\beta}}{\Gamma(\beta+1)}+2\Delta_2\Big)+\frac{A}{|\Lambda|}<r,
\end{aligned}
$$

which shows that Π maps B_r into itself.

Now show that the operator Π is a contraction. For $(\nu_1,\omega_1),(\nu_2,\omega_2)\in X$, $t\in[0,T]$, we have

$$
\begin{aligned}
&|\Pi_1(\nu_1,\omega_1)(t)-\Pi_1(\nu_2,\omega_2)(t)|\\
&\quad\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big|\varphi(s,\nu_1(s),\omega_1(s))-\varphi(s,\nu_2(s),\omega_2(s))\big|ds\\
&\qquad+\frac{1}{2}\Bigg\{\frac{1}{2}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\big|\varphi(s,\nu_1(s),\omega_1(s))-\varphi(s,\nu_2(s),\omega_2(s))\big|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}\big|\psi(s,\nu_1(s),\omega_1(s))-\psi(s,\nu_2(s),\omega_2(s))\big|ds\Big)\\
&+\frac{1}{|\Lambda|}\int_\eta^\xi\Big(\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)}\big|\varphi(x,\nu_1(x),\omega_1(s))-\varphi(s,\nu_2(x),\omega_2(x))\big|dx\\
&+\int_0^s \frac{(s-x)^{\beta-1}}{\Gamma(\beta)}\big|\psi(x,\nu_1(x),\omega_1(x))-\psi(x,\nu_2(x),\omega_2(x))\big|dx\Big)ds\\
&+\frac{\sum_{i=1}^m a_i}{|\Lambda|}\Big(\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\alpha-1}}{\Gamma(\alpha)}\big|\varphi(x,\nu_1(x),\omega_1(s))-\varphi(s,\nu_2(x),\omega_2(x))\big|dx\\
&+\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\beta-1}}{\Gamma(\beta)}\big|\psi(x,\nu_1(x),\omega_1(x))-\psi(x,\nu_2(x),\omega_2(x))\big|dx\Big)\\
&+\frac{\sum_{j=1}^n b_j}{|\Lambda|}\Big(\int_0^{\delta_j} \frac{(\delta_j-x)^{\alpha-1}}{\Gamma(\alpha)}\big|\varphi(x,\nu_1(x),\omega_1(s))-\varphi(s,\nu_2(x),\omega_2(x))\big|dx\\
&+\int_0^{\delta_j} \frac{(\delta_j-x)^{\beta-1}}{\Gamma(\beta)}\big|\psi(x,\nu_1(x),\omega_1(x))-\psi(x,\nu_2(x),\omega_2(x))\big|dx\Big)\Bigg\}\\
\le&\left\{\mathcal{C}\Big(\frac{T^\alpha}{\Gamma(\alpha+1)}+\Delta_1\Big)+\mathcal{K}\Delta_2\right\}(\|\nu\|+\|\omega\|),
\end{aligned}
$$

and

$$
\begin{aligned}
&|\Pi_2(\nu_1,\omega_1)(t)-\Pi_2(\nu_2,\omega_2)(t)|\\
&\le\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big|\psi(s,\nu_1(s),\omega_1(s))-\psi(s,\nu_2(s),\omega_2(s))\big|ds\\
&+\frac{1}{2}\Bigg\{\frac{1}{2}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\big|\varphi(s,\nu_1(s),\omega_1(s))-\varphi(s,\nu_2(s),\omega_2(s))\big|ds\\
&+\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}\big|\psi(s,\nu_1(s),\omega_1(s))-\psi(s,\nu_2(s),\omega_2(s))\big|ds\Big)\\
&+\frac{1}{|\Lambda|}\int_\eta^\xi\Big(\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)}\big|\varphi(s,\nu_1(s),\omega_1(s))-\varphi(s,\nu_2(s),\omega_2(s))\big|dx\\
&+\int_0^s \frac{(s-x)^{\beta-1}}{\Gamma(\beta)}\big|\psi(s,\nu_1(s),\omega_1(s))-\psi(s,\nu_2(s),\omega_2(s))\big|dx\Big)ds\\
&+\frac{\sum_{i=1}^m a_i}{|\Lambda|}\Big(\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\alpha-1}}{\Gamma(\alpha)}\big|\varphi(x,\nu_1(x),\omega_1(x))-\varphi(s,\nu_2(x),\omega_2(x))\big|dx\\
&+\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\beta-1}}{\Gamma(\beta)}\big|\psi(x,\nu_1(x),\omega_1(x))-\psi(s,\nu_2(x),\omega_2(x))\big|dx\Big)
\end{aligned}
$$

$$+\frac{\sum_{j=1}^{n} b_j}{|\Lambda|}\Big(\int_0^{\delta_j}\frac{(\delta_j-x)^{\alpha-1}}{\Gamma(\alpha)}\big|\varphi(x,\nu_1(x),\omega_1(x))-\varphi(s,\nu_2(x),\omega_2(x))\big|dx$$
$$+\int_0^{\delta_j}\frac{(\delta_j-x)^{\beta-1}}{\Gamma(\beta)}\big|\psi(x,\nu_1(x),\omega_1(x))-\psi(s,\nu_2(x),\omega_2(x))\big|dx\Big)\Bigg\}$$
$$\leq\Bigg\{\mathcal{C}\Delta_1+\mathcal{K}\Big(\frac{T^{\beta}}{\Gamma(\beta+1)}+\Delta_2\Big)\Bigg\}(\|\nu\|+\|\omega\|).$$

In view of the foregoing inequalities, it follows that

$$\|\Pi(\nu_1,\omega_1)-\Pi(\nu_2,\omega_2)\|=\|\Pi_1(\nu_1,\omega_1)-\Pi_1(\nu_2,\omega_2)\|+\|\Pi_2(\nu_1,\omega_1)-\Pi_2(\nu_2,\omega_2)\|$$
$$\leq\Bigg\{\mathcal{C}\Big(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+2\Delta_1\Big)+\mathcal{K}\Big(\frac{T^{\beta}}{\Gamma(\beta+1)}+2\Delta_2\Big)\Bigg\}\|(\nu_1-\nu_2,\omega_1-\omega_2)\|,$$

which, by (3.7), implies that Π is a contraction mapping. Hence the conclusion of contraction mapping principle applies and Π has a unique fixed point, which is a unique solution of the problem (1.1). This finishes the proof. □

Example 1. Consider the following problem

$$\begin{cases} {}^{LC}D^{1/2}\nu(t)=\varphi(t,\nu(t),\omega(t)), \ \ t\in[0,2],\\ {}^{LC}D^{2/3}\omega(t)=\psi(t,\nu(t),\omega(t)), \ \ t\in[0,2],\\ (\nu+\omega)(0)=-(\nu+\omega)(2),\\ \int_{3/4}^{3/2}(\nu-\omega)(s)ds-(\nu-\omega)(1/4)-1/5(\nu-\omega)(1/3)-4(\nu-\omega)(7/4)=1, \end{cases} \tag{3.8}$$

where $\alpha=1/2$, $\beta=2/3$, $\eta=3/4$, $\zeta=3/2$, $a_1=1$, $a_2=1/5$, $b_1=4$, $\sigma_1=1/4,\sigma_2=1/3$, $\delta_1=7/4, A=1$, $T=2$, and $\varphi(t,\nu,\omega)$ and $\psi(t,\nu,\omega)$ will be fixed later.

Using the given values, it is found that

$$\Lambda=-4.45,\Delta_1=0.7477415389,\Delta_2=0.7168620998,$$

where Λ, Δ_1 and Δ_2 are respectively given by (2.4), (3.4) and (3.5). In order to illustrate Theorem 1, we consider

$$\varphi(t,\nu,\omega)=\frac{\cos t}{(2+t)^4}\Big(\sin\nu+\frac{\omega}{2}+e^{-t}\Big) \tag{3.9}$$

and

$$\psi(t,\nu,\omega)=\frac{e^{-t}}{2\sqrt{36+t}}\Big(|\nu|\Big(\frac{|\nu|}{2+|\nu|}\Big)+t\sin\omega+1\Big).$$

Clearly φ and ψ are continuous and satisfy the condition (H_1) with

$$\mu_1(t)=\frac{e^{-t}\cos t}{(2+t)^4},\mu_2(t)=\frac{\cos t}{(2+t)^4},\mu_3=\frac{\cos t}{2(2+t)^4},$$

$$\kappa_1 = \kappa_2 = \frac{e^{-t}}{2\sqrt{36+t}}, \kappa_3 = \frac{te^{-t}}{2\sqrt{36+t}}.$$

Also

$$\|\mu_2\|\left(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)}\right) + \|\kappa_2\|\left(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)}\right) \approx 0.4592148800$$

and

$$\|\mu_3\|\left(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)}\right) + \|\kappa_3\|\left(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)}\right) \approx 0.3626132487 < 1.$$

Clearly the assumptions of Theorem 1 are satisfied. Therefore, the problem (3.8) with $\varphi(t,\nu,\omega)$ and $\psi(t,\nu,\omega)$ given by (3.9) has at least one solution on $[0,2]$.

Now we explain Theorem 2 by taking

$$\varphi(t,\nu,\omega) = \frac{\tan^{-1}\nu + \omega}{\sqrt{900+t^2}} \quad \text{and} \quad \psi(t,\nu,\omega) = \frac{1}{25(1+t^2)}\left(\frac{|\nu|}{2+|\nu|} + \sin\omega\right). \tag{3.10}$$

Observe that φ and ψ are continuous and satisfy the condition (H_2) with $C_1 = C_2 = 1/30 = \mathcal{C}$ and $K_1 = 1/50$, $K_2 = 1/25$ and so, $\mathcal{K} = 1/25$. Also

$$\mathcal{C}\left(\frac{T^\alpha}{\Gamma(\alpha+1)} + 2\Delta_1\right) + \mathcal{K}\left(\frac{T^\beta}{\Gamma(\beta+1)} + 2\Delta_2\right) \approx 0.2307273163 < 1.$$

Clearly the hypothesis of Theorem 2 is satisfied and hence its conclusion applies to the problem (3.8) with $\varphi(t,\nu,\omega)$ and $\psi(t,\nu,\omega)$ given by (3.10).

ACKNOWLEDGMENTS

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (KEP-PhD-80-130-42). The authors, therefore, acknowledge with thanks DSR technical and financial support. The authors also thank the reviewer for his/her positive remarks on their work.

REFERENCES

[1] B. Ahmad, N. Alghamdi, A. Alsaedi, and s. K. Ntouyas, "A system of coupled multi-term fractional differential equations with three-point boundary conditions." *Fract. Calc. Appl. Anal.*, vol. 22, no. 3, pp. 601–616, 2019, doi: 10.1515/fca-2019-0034.

[2] B. Ahmad, M. Alghanmi, A. Alsaedi, and J. J. Nieto, "Existence and uniqueness results for a nonlinear coupled system involving Caputo fractional derivatives with a new kind of coupled boundary conditions." *Appl. Math. Lett.*, vol. 116, no. 107018, pp. 1–10, 2021, doi: 10.1016/j.aml.2021.107018.

[3] B. Ahmad, Y. Alruwaily, A. Alsaedi, and S. K. Ntouyas, "Riemann-Stieltjes Integral boundary value problems involving mixed Riemann-Liouville and Caputo fractional derivatives." *J. Nonlinear Funct. Anal.*, vol. 2021, no. 11, pp. 1–19, 2021, doi: 10.23952/jnfa.2021.11.

[4] B. Ahmad, A. Alsaedi, S. K. Ntouyas, and J. Tariboon, *Hadamard-type fractional differential equations, inclusions and inequalities*. Cham: Springer, 2017. doi: 10.1007/978-3-319-52141-1.

[5] B. Ahmad, G. Garout, S. K. Ntouyas, and A. Alsaedi, "Caputo fractional differential inclusions of arbitrary order with nonlocal integro-multipoint boundary conditions." *Miskolc Mathematical Notes*, vol. 20, no. 2, pp. 683–699, 2019, doi: 10.18514/MMN.2019.2241.

[6] M. Ali, G. Narayanan, V. Shekher, A. Alsaedi, and B. Ahmad, "Global Mittag-Leffler stability analysis of impulsive fractional-order complex-valued BAM neural networks with time varying delays." *Commun. Nonlinear Sci. Numer. Simul.*, vol. 83, no. 105088, pp. 1–22, 2020, doi: 10.1016/j.cnsns.2019.105088.

[7] A. Carvalho and C. M.A. Pinto, "A delay fractional order model for the co-infection of malaria and HIV/AIDS." *Int. J. Dyn. Control*, vol. 5, no. 1, pp. 168–186, 2017, doi: 10.1007/s40435-016-0224-3.

[8] Y. Ding, Z. Wang, and H. Ye, "Optimal control of a fractional-order HIV-immune system with memory." *IEEE Trans. Contr. Sys. Techn.*, vol. 20, no. 3, pp. 763–769, 2012, doi: 10.1109/TCST.2011.2153203.

[9] J. Henderson and N. Kosmatov, "Eigenvalue comparison for fractional boundary value problems with the Caputo derivative." *Fract. Calc. Appl. Anal.*, vol. 17, no. 3, pp. 872–880, 2014, doi: 10.2478/s13540-014-0202-4.

[10] J. Henderson and R. Luca, "Nonexistence of positive solutions for a system of coupled fractional boundary value problems." *Bound. Value Probl.*, vol. 2015, no. 138, pp. 1–12, 2015, doi: 10.1186/s13661-015-0403-8.

[11] G. Iskenderoglu and D. Kaya, "Symmetry analysis of initial and boundary value problems for fractional differential equations in Caputo sense." *Chaos Solitons Fractals*, vol. 134, no. 109684, pp. 1–7, 2020, doi: 10.1016/j.chaos.2020.109684.

[12] M. Javidi and B. Ahmad, "Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton–zooplankton system." *Ecol. Model.*, vol. 318, no. SI, pp. 8–18, 2015, doi: 10.1016/j.ecolmodel.2015.06.016.

[13] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, 204*. Amsterdam: Elsevier Science B.V., 2006.

[14] R. Luca, "Positive solutions for a system of fractional differential equations with p-Laplacian operator and multi-point boundary conditions." *Nonlinear Anal. Model. Control.*, vol. 23, no. 5, pp. 771—-801, 2018, doi: 10.15388/NA.2018.5.8.

[15] H. M. Srivastava, "Some families of Mittag-Leffler type functions and associated operators of fractional calculus." *TWMS J. Pure Appl. Math.* , vol. 7, no. 2, pp. 123–145, 2016.

[16] H. M. Srivastava, "Fractional-order derivatives and integrals: Introductory overview and recent developments." *Kyungpook Math. J.* , vol. 60, no. 1, pp. 73–116, 2020, doi: 10.5666/KMJ.2020.60.1.73.

[17] H. M. Srivastava, "An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions." *J. Adv. Engrg. Comput.* , vol. 5, no. 3, pp. 135–166, 2021, doi: 10.25073/jaec.202153.340.

[18] H. M. Srivastava, "Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations." *J. Nonlinear Convex Anal.* , vol. 22, no. 8, pp. 1501–1520, 2021.

[19] D. R. Smart, *Fixed point theorems. Cambridge Tracts in Mathematics, No. 66.* London-New York: Cambridge University Press, 1974.

[20] V. V. Tarasova and V. E. Tarasov, "Logistic map with memory from economic model." *Chaos Solitons Fractals*, vol. 95, pp. 84–91, 2017, doi: 10.1016/j.chaos.2016.12.012.

[21] Y. Xu and W. X. Li, "Finite-time synchronization of fractional-order complex-valued coupled systems." *Phys. A*, vol. 549, no. 123903, pp. 1–11, 2020, doi: 10.1016/j.physa.2019.123903.

[22] C. Zhai and L. Xu, "Properties of positive solutions to a class of four-point boundary value problem of Caputo fractional differential equations with a parameter." *Commun. Nonlinear Sci. Numer. Simul.*, vol. 19, no. 8, pp. 2820–2827, 2014, doi: 10.1016/j.cnsns.2014.01.003.

Authors' addresses

Bashir Ahmad
(**Corresponding author**) King Abdulaziz University, Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: bashirahmad_qau@yahoo.com

Ahmed Alsaedi
King Abdulaziz University, Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: aalsaedi@hotmail.com

Fawziah M. Alotaibi
King Abdulaziz University, Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: foozalotaibi_22@hotmail.com

Madeaha Alghanmi
Department of Mathematics, College of Science & Arts, King Abdulaziz University, Rabigh 21911, Saudi Arabia
E-mail address: madeaha@hotmail.com