Miskolc Mathematical Notes
Vol. 24 (2023), No. 1, pp. 165–178
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.3835

AVERAGE CONSENSUS STABILITY ANALYSIS FOR STOCHASTIC MULTI-AGENT SYSTEMS WITH COMPOUND NOISES UNDER MARKOVIAN SWITCHING TOPOLOGIES

XIAOJIN HUANG, YONGJIAN LIU, AND JIANHUA HUANG

Received 10 May, 2021

Abstract. In this paper, a new class of multi-agent systems with addictive and multiplicative measurement noises under Markovian switching topologies is introduced. Some sufficient conditions of average consensus to the systems are also established under consideration. First, based on the continuous-time Markov chain with finite modes, the time-varying topologies of the considering system is figured. Then, the mean square average consensus for the multi-agent systems with a time-varying gain under the time-varying topologies and the Markov chain is studied. Finally, the example and simulation results are given to illustrate the effectiveness of the obtained theoretical results.

2010 *Mathematics Subject Classification:* 93A16; 93E99; 93E15; 60H10

Keywords: multi-agent systems, stochastic systems, average consensus, Markov switchings, compound noises

1. INTRODUCTION

The mathematical models subjected to the phenomenon of uncertain environment for collective behavior have been widely introduced and analyzed (see e.g., [9,16,17], and its special cases, flocking[1, 2] and consensus [5–7, 10, 11, 15–18]). Noises are ubiquitous stochastic phenomena in the real world which are usually modeled as additive noise [5, 7, 16] or multiplicative noises [1, 6, 11]. Recetnly, Li et al. considered the addictive noise [7] and multiplicative noises [6] for continuous-time models under fixed topologies, respectively. Besides, random link failures often happen in the various systems which motives to study the dynamics with stochastic switching. The average consensus for a linear system with Markov switching topologies has been developed by Matei et al.[10]. After that it was extended to a more general linear system in [15]. Under the assumption that the systems have addictive noises, the sufficient conditions for average consensus were proposed by Zhang et al. [16]. Then,

The first author was supported in part by the Natural Science Foundation of Guangxi Province, Grant No.2018JJA110105, the Foundation of Yulin Normal University, Grant No.G2019ZK12, Middle-aged and Young Teachers' Basic Ability Promotion Project of Guangxi, Grant No.2019KY0587.

Li et al. proposed the weaker necessary and sufficient conditions to obtain weaker consensus results. Since the measurements of multiple sensors are often influenced by compound noises[3], Wang et al.[13, 14] gave measurement models with compound noises and constant control gains, and Zong et al.[17] studied a first-order consensus problem with addictive and multiplicative noises to seek consensus. If the topology is fixed, the necessary and sufficient conditions for mean square weak and strong consensus were developed with a time-varying gain by Zong et al.[17]. Furthermore, when time-varying topologies were considered in the systems, they also obtained sufficient conditions for weak consensus and average consensus. However, the time-varying topologies were not figured explicitly and no numerical simulations were presented. This is the main motivation of the paper.

To fill this gap, the goal of this paper is to establish the asymptotic consensus for a class of first-order continuous-time multi-agent systems with addictive and multiplicative noises under Markovian random graphs. Towards the asymptotic consensus, we transform the dynamics of agents into a stochastic error systems. In order to decrease the errors, we combine with stochastic analysis and the tools of symmetrized graph in the theory of stochastic Lyapunov to systemically form a good expression for the ease of analysis. To ensure the connectivity of the network to some extent, we assume that the graph resulting from the union of graphs which are switched by ergodic Markov chain is contained into a spanning tree. Under the assumption, we can show that the error systems in the mean square sense tend to zero as time goes on making sure that all agents may asymptotically agree on their states. Moreover, we prove that the state of each agent converges to the average of initial states under some convergence and robust conditions.

The paper is organized as follows. In Section 2, we recall some related concepts and basic results on graphs and matrices, and present the setup and formulation of the continuous-time systems under consideration. Section 3 is devoted to state our main convergence theorem for average consensus. In Section 4, the example and simulation results are given to illustrate the effectiveness of the obtained theoretical results. Section 5 gives a brief summary of the paper.

2. Preliminary and problem formulation

In this section, we will give some related preliminaries and formulate the considered problem in this work.

Notation 1. In this paper, unless otherwise specified, the following symbols will be used in next sections. For any positive integers N, I_N and $e_{N,i}$ represents N-by-N identity matrix and its ith column, $\mathbb{R}^N$ denotes the N dimensional Euclidean space, $\mathbb{R}^{m\times n}$ denotes the set of all $m\times n$ real matrices and 1_N denotes the N dimensional column vector with all ones. For any given vector x or matrix X, $\|x\|$ denotes the Euclid norm and $\|X\|_F$ denotes the Frobenius norm, $\mathbb{E}x$ denotes the mathematical expectation of x and Var$x$ denotes the variance of x. For any given matrices $A\in\mathbb{R}^{n\times n}$

and $B \in \mathbb{R}^{m\times m}$, A^T denotes the transpose of A, $A \otimes B$ denotes the Kronecker product. Set $A \oplus B = (I_m \otimes A) + (B \otimes I_n) \in \mathbb{R}^{mn\times mn}$. The Kronecker product has the properties: $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

2.1. *Preliminary*

The generator matrix of a continuous-time Markov chain $\{\theta(t)\}_{t\geq 0}$ is denoted by $\Gamma = (\gamma_{ij})_{s\times s}$, so that for a sufficiently small $h > 0$, the probability of mode k jumps to mode l is determined by the generator matrix

$$\mathbb{P}\{\theta(t+h) = l|\theta(t) = k\} = \begin{cases} \gamma_{kl} h + o(h), & k \neq l \\ 1 + \gamma_{kk} h + o(h), & k = l \end{cases},$$

where $o(h)$ is a high-order infinitesimal with respect to infinitesimal h, and $\gamma_{lk} \geq 0$ is the transition rate from state l to state k if $k \neq l$. Let $\lambda_{kk} = -\sum_{k\neq l} \lambda_{kl}$. From [9], for almost every sample path of $\theta(t)$ in any finite interval of $\mathbb{R}_+ := [0, +\infty)$, we know there are finite simple jumps.

Throughout this paper, we associate to a dynamic system with a communication graph within N agents. Let the set of nodes $\mathcal{V} = \{1, 2, \ldots, N\}$ be the set of agents with node i representing the ith agent. The interactive communication topology among the agents is modeled by a time-dependent and weighted digraph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$ with no self-loops and $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$. An ordered pair (j, i) belongs to the edge set $\mathcal{E}(t)$, which means that the ith agent can receive information from the jth agent directly at t. If $(k_1, k_2), (k_2, k_3), \ldots, (k_{m-1}, k_m) \in \mathcal{E}(t)$, then there is a directed path from node k_1 to k_m at t. The weighted adjacency matrix is defined by $\mathcal{A}(t) = [a_{ij}(t)] \in \mathbb{R}^{N\times N}$. For any $i, j \in \mathcal{V}, a_{ij}(t) \geq 0$, and $a_{ij}(t) > 0$, i.e.,$(j, i) \in \mathcal{E}$, the graph corresponding to a given adjacency matrix $\mathcal{A}(t)$ is denoted by $\mathcal{G}_{\mathcal{A}(t)}$. The Laplacian matrix is given by $\mathcal{L}(t) = \mathcal{D}(t) - \mathcal{A}(t)$, where $\mathcal{D}(t) = \mathrm{diag}(\sum_{j=1}^{N} a_{ij}(t), i = 1, \ldots, N)$. A graph $\mathcal{G}$ is called to be balanced digraph, if $\sum_{j=1}^{N} a_{ji}(t) = \sum_{j=1}^{N} a_{ij}(t),\ i = 1, 2, \ldots, N$. It is clear that an undirected graph is a balanced digraph. If there exists a directed path from some agent of the systems to the rest of agents, then we say that the digraph $\mathcal{G}$ contains a spanning tree.

Definition 1. Let $\mathbb{S} = \{1, 2, \ldots, s\}$ be the index set of modes corresponding to all possible network topologies, and let $\{A(i) \in \mathbb{R}^{N\times N}, i \in \mathbb{S}\}$ be a set of matrices with respect to a set of graphs $\{\mathcal{G}_{A(i)}, i = 1, \ldots, s\}$. We say that the graph $\mathcal{G}_{\mathcal{A}}$ corresponds to the set $\mathcal{A}$ if it is given by the union of graphs, i.e.,

$$\mathcal{G}_{\mathcal{A}} \triangleq \bigcup_{i\in\mathbb{S}} \mathcal{G}_{A(i)} = \left(\mathcal{V}, \bigcup_{i\in\mathbb{S}} \mathcal{E}_i, \mathcal{A}\right),$$

where $\mathcal{A} = \sum_{i=1}^{s} A(i)$.

Lemma 1. *[7] If the fixed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ is undirected, then the corresponding Laplacian matrix $\mathcal{L}$ is symmetric with N real eigenvalues, and the eigenvalues are in an ascending order:*

$$0 = v_1 \leq v_2 \leq \ldots \leq v_N,$$

and

$$\min_{x \neq 0, \mathbb{1}^T x = 0} \frac{x^T \mathcal{L} x}{\|x\|^2} = v_2.$$

We call that $v_2(\mathcal{L})$ is the algebraic connectivity of $\mathcal{G}$. Moreover, if $\mathcal{G}$ is connected, then $v_2 > 0$.

2.2. *Problem formulation*

We consider the consensus problem for the multi-agent systems with the following distributed control coordination:

$$\dot{x}_i(t) = c(t) \sum_{j=1}^{N} a_{ij}(t)(y_{ji}(t) - x_i(t)),\ t \geq 0,\ i = 1, 2, \ldots, N, \tag{2.1}$$

where $x_i(t) \in \mathbb{R}$, and $c(t) : [0, +\infty) \to [0, +\infty)$ is a time-varying gain. The communication information from jth agent to ith agent is measured by addictive and multiplicative noises as the form:

$$y_{ji}(t) = x_j(t) + \alpha_{ji}\varpi_{1ji}(t) + f_{ji}(x_j(t) - x_i(t))\varpi_{2ji}(t), \tag{2.2}$$

where $\alpha_{ji} \geq 0$ in the additive noisy intensity, $f_{ji}(\cdot) : \mathbb{R} \to \mathbb{R}_+$ is the multiplicative noisy intensity function, $\{\varpi_{nji}(t), i, j = 1, 2, \ldots, N\}$, $n = 1, 2$ are the noise processes. In this work, $\alpha_{ji}\varpi_{1ji}(t) + f_{ji}(x_j(t) - x_i(t))\varpi_{2ji}(t)$ is the noise term including two case:

- $\alpha_{ji}\varpi_{1ji}(t)$ are addictive and independent of the agents' states;
- $f_{ji}(x_j(t) - x_i(t))\varpi_{2ji}(t)$ are multiplicative depending on the relative states.

Remark 1. The measurements and information transmission are often disturbed by addictive and multiplicative noises [12]. Generally speaking, $x_j(t) - x_i(t)$ is an ideal measurement and cannot be obtained accurately, because noises appear everywhere. Li et al. considered the addictive noise [7] and multiplicative noises [6] for continuous-time models under fixed topologies, respectively. Zong et al. [17] studied the consensus problem for multi-agent systems with addictive noise and fixed topology under weaker sufficient conditions obtained by [7]. Li et al. [5] generalized the addictive results to a Markovian switching topologies case. Since the measurements to multiple sensors are often influenced by compound noises [3], Wang et al. [13, 14] proposed measurement models with compound noises and constant control gains, and Zong et al. [17] also considered a first-order consensus problem with addictive and multiplicative noises and designed a time-varying control gain to seek consensus.

Assumption 1. *The noise processes* $\{\varpi_{nji}(t), i, j = 1, 2, \dots, N\}$*,* $n = 1, 2$ *satisfying* $\int_0^t \varpi_{nji}(s)\mathrm{d}s = w_{nji}(t),\ t \geq 0$ *and* $\{w_{nji}(t), i, j = 1, 2, \dots, N\}$ *are scalar independent Brownian motions.*

From Assumption 1, (2.1) and (2.2), the dynamic behavior of the ith agent is given by

$$\mathrm{d}x_i(t) = c(t)\sum_{j=1}^{N} a_{ij}(t)\left(x_j(t) - x_i(t)\right)\mathrm{d}t + c(t)\sum_{j=1}^{N} a_{ij}(t)\alpha_{ji}\mathrm{d}w_{1ji}(t) + c(t)\sum_{j=1}^{N} a_{ij}(t) f_{ji}\left(x_j(t) - x_i(t)\right)\mathrm{d}w_{2ji}(t), \tag{2.3}$$

where $f_{ji}(\cdot) : \mathbb{R} \to \mathbb{R}$ is the noise intensity function.

Let $\eta_i(t)^T$ be the first row of $\mathcal{A}(t)$, $\Sigma_i(t) = \eta_i^T(t)\mathrm{diag}(\alpha_{1i}, \alpha_{2i}, \dots, \alpha_{Ni})$, $\alpha^{(t)} = \mathrm{diag}(\Sigma_1(t), \dots, \Sigma_N(t))_{N\times N^2}$ and $\mathrm{d}W_1 = (\mathrm{d}w_{111} \dots \mathrm{d}w_{1N1} \dots\dots \mathrm{d}w_{11N} \dots \mathrm{d}w_{1NN})^T$. Then, (2.3) can be rewritten as

$$\mathrm{d}x(t) = -c(t)\mathcal{L}(t)x(t)\mathrm{d}t + c(t)\alpha^{(t)}\mathrm{d}W_1(t) + c(t)\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(t)e_{N,i} f_{ji}\left(x_j(t) - x_i(t)\right)\mathrm{d}w_{2ji}(t), \tag{2.4}$$

where $x(t) = (x_1(t), \dots, x_N(t))^T$.

Next, we give the definition of consensus as follows.

Definition 2. The agents reach asymptotically unbiased mean square average consensus(AUMSAC), if there is a random vector $x^* \in \mathbb{R}$ such that $\mathbb{E}[x^*] = \frac{1}{N}\sum_{j=1}^{N} x_j(0)$, $\mathbb{E}[x^*]^2 < \infty$, and $\lim\limits_{t\to\infty} \mathbb{E}\|x(t) - x^* 1_N\|^2 = 0$.

Remark 2. Definition 2 follows from [18], and we use the mean square consensus to figure such asymptotical behavior.

3. CONSENSUS ANALYSIS

In the section, we are going to give several sufficient conditions for asymptotically unbiased mean square average-consensus to the multi-agent systems. To this end, we make the following assumptions.

Assumption 2. *All digraphs* $\mathcal{G}_{\mathcal{A}(k)}, k \in \mathbb{S}$ *are assumed to be balanced and the union graph of all available digraphs contains a spanning tree.*

Assumption 3. *The homogeneous, finite-state Markov chain* $\{\theta(t)\}_{t\geq 0}$ *is right continuous-time, and ergodic with generator matrix* $\Gamma = (\gamma_{ij})_{s\times s}$ *satisfies* $\sum_{j\neq i}^{s} \gamma_{ij} = \sum_{j\neq i}^{s} \gamma_{ji}$ *and initial probability distribution has* $\{q_i, i \in \mathbb{S}\}, \mathbb{S} = \{1, 2, \dots, s\}$*.*

Assumption 4. $c(t) \geq 0,\ \int_0^\infty c(t) = \infty,\ \int_0^\infty c^2(t) < \infty.$

Assumption 5. *The initial data $x(0)$ satisfies $\mathbb{E}\|x(0)\|^2 < \infty$. Besides, $x(0)$, $\{\theta(t)\}_{t\geq 0}$, $\{\varpi_{nji}(t), i, j = 1, 2, \ldots, N\}$ are independent.*

Remark 3. The Assumptions above are sufficient for seeking consensus of the considered systems in this work.

(1) Assumption 2 is often used to analysis the dynamics for systems with switching topologies (see [5]).
(2) Assumption 3 makes the generator matrix Γ to be doubly stochastic and to correspond to a balanced digraph[10]. Besides $Q := \frac{\Gamma^T+\Gamma}{2}$ is irreducible and negative semidefinite.
(3) The two integrals in Assumption 4 are convergence condition and robust condition, respectively (see [7]). And it is easy to find that $c(t) \to 0,\ t \to \infty$.

Theorem 1. *Suppose that Assumptions 1,2,3,4 hold. If $f_{ji}(x) = \beta_{ji}x, 0 \leq \beta_{ji} \leq \kappa_\beta, i, j = 1, 2, \ldots, N$ and $a_{ij}(k) = 1$ or 0, $i, j = 1, \ldots, N$, $k \in \mathbb{S}$, then the multi-agent systems achieve unbiased mean square average consensus under the distributed control coordination (2.1) with measurements (2.2), that is,*

$$\lim_{t\to\infty} \|x(t) - x^* 1_N\|^2 = 0,$$

where

$$x^* = \frac{1}{N}1_N^T x(0) + \frac{1}{N}\int_0^\infty c(t)1_N^T \alpha^{(\theta(t))} dW_1(t) + \frac{1}{N}\sum_{i,j=1}^N \int_0^\infty c(t)\beta_{ji}1_N^T S_{ij}^{(\theta(t))} x(t) dw_{2ji}(t).$$

Proof. Since $f_{ji}(x) = \beta_{ji}x, i, j = 1, 2, \ldots, N$, let $S_{ij}^{(t)} = (s_{ij}(t))_{N\times N}$ be an N-th-order square matrix with $s_{ij}(t) = a_{ij}(t), s_{ii}(t) = -a_{ij}(t)$ and the rest elements being zero, $i, j = 1, 2, \ldots, N$. Then, (2.4) can be rewritten as

$$\mathrm{d}x(t) = -c(t)\mathcal{L}(t)x(t)\mathrm{d}t + c(t)\alpha^{(t)}\mathrm{d}W_1(t) + c(t)\sum_{i=1}^N\sum_{j=1}^N \beta_{ji}S_{ij}^{(t)}x(t)\mathrm{d}w_{2ji}(t). \tag{3.1}$$

Let $\delta(t) = (I_N - J_N)x(t)$ and $J_N = (1/N)1_N 1_N^T$. Note that $(I_N - J_N)\mathcal{L}(t) = \mathcal{L}(t) = \mathcal{L}(t)(I_N - J_N)$,

$$\begin{aligned}\mathrm{d}\delta(t) = &-c(t)\mathcal{L}(t)\delta(t)\mathrm{d}t + c(t)(I_N - J_N)\alpha^{(t)}\mathrm{d}W_1(t)\\ &+ c(t)\sum_{i=1}^N\sum_{j=1}^N \beta_{ji}(I_N - J_N)S_{ij}^{(t)}\delta(t)\mathrm{d}w_{2ji}(t),\end{aligned}$$

we integrate both sides to the equality $\mathrm{d}[\delta(t)\delta(t)^T\chi_{\{\theta(t)=k\}}] = \chi_{\{\theta(t)=k\}}\mathrm{d}\delta(t)\delta^T(t) + \delta(t)\delta^T(t)\mathrm{d}\chi_{\{\theta(t)=k\}}$, to yield

$$\begin{aligned}\delta(t)\delta(t)^T\chi_{\{\theta(t)=k\}} = \delta^T(0)\delta(0)\chi_{\{\theta(0)=k\}} &+ \int_0^t \chi_{\{\theta(s)=k\}}\mathrm{d}\delta(s)\delta^T(s)\\ &+ \int_0^t \delta(s)\delta^T(s)\mathrm{d}\chi_{\{\theta(s)\}}.\end{aligned}$$

Hence, we obtain

$$\mathbb{E}[\delta^T\delta\chi_{\{\theta(t)=k\}}] = \mathbb{E}[\delta^T(0)\delta(0)\chi_{\{\theta(0)=k\}}] + \int_0^t \mathbb{E}\chi_{\{\theta(s)=k\}}\mathrm{d}\delta(s)\delta^T(s) + \int_0^t \mathbb{E}\delta(s)\delta^T(s)\mathrm{d}\chi_{\{\theta(s)\}}.$$

Denote $V_k(t) = \mathbb{E}[\delta(t)\delta^T(t)\chi_{\{\theta(t)=k\}}]$, we use Ito's formula and Lemma 4.2 in [4] to obtain

$$\begin{aligned}\frac{\mathrm{d}V_k(t)}{\mathrm{d}t} &= \frac{\mathrm{d}\mathbb{E}[\delta(t)\delta^T(t)\chi_{\{\theta(t)=k\}}]}{\mathrm{d}t} \\ &= -c(t)\left(V_k(t)\mathcal{L}(k)^T + \mathcal{L}(k)V_k(t)\right) + c^2(t)(I_N - J_N)\alpha^{(k)}\alpha^{(k)T}(I_N - J_N)p_k(t) \\ &\quad + c^2(t)\sum_{i,j=1}\beta_{ji}^2(I_N - J_N)S_{ij}^{(k)}V_k(t)S_{ij}^{(k)T}(I_N - J_N)^T + \sum_{l=1}\gamma_{lk}V_l(t). \end{aligned} \tag{3.2}$$

But, $V(t) = \sum_{k=1}^{s} V_k(t)$ deduces $V(t) = \mathbb{E}(\delta(t)\delta(t)^T)$. Let $\hat{V}(t) = [V_1(t)\ldots V_s(t)] \in \mathbb{R}^{N\times Ns}$, and $\psi(V)(V \in \mathbb{R}^{N\times N})$ be an N^2 dimensional vector obtained by "stacking" V's columns, that is,

$$\psi(V) = \begin{pmatrix} V(:,1) \\ \vdots \\ V(:,N) \end{pmatrix}, \quad \text{and then,} \quad \psi(\hat{V}(t)) = \begin{pmatrix} \psi(V_1(t)) \\ \vdots \\ \psi(V_s(t)) \end{pmatrix}.$$

Hence,

$$\begin{aligned}\frac{\mathrm{d}\psi(V_k(t))}{\mathrm{d}t} &= -c(t)\psi\left(V_k(t)\mathcal{L}(k)^T + \mathcal{L}(k)V_k(t)\right) + \sum_{l=1}\gamma_{lk}\psi(V_l(t)) \\ &\quad + c^2(t)\psi\Big((I_N - J_N)\alpha^{(k)}\alpha^{(k)T}(I_N - J_N)p_k(t)\Big) \\ &\quad + c^2(t)\sum_{i,j=1}\beta_{ji}^2\psi\Big((I_N - J_N)S_{ij}^{(k)}V_k(t)S_{ij}^{(k)T}(I_N - J_N)^T\Big) \\ &= -c(t)(\mathcal{L}(k)\oplus\mathcal{L}(k))\psi(V_k(t)) + (\Gamma_k^T\otimes I_{N^2})\psi(\hat{V}(t)) + c^2(t)\psi(Z_k(t)) \\ &\quad + c^2(t)\sum_{i,j=1}\beta_{ji}^2\Big((I_N - J_N S_{ij}^{(k)}\otimes(I_N - J_N)S_{ij}^{(k)}))\Big)\psi(V_k(t)), \end{aligned} \tag{3.3}$$

where $Z_k(t) = (I_N - J_N)\alpha^{(k)}\alpha^{(k)T}(I_N - J_N)p_k(t)$. Let $Z(t) = [Z_1(t)\ldots Z_s(t)]$, $\bar{\mathcal{L}} = \mathrm{diag}\,(\mathcal{L}(k)\oplus\mathcal{L}(k), k = 1,\ldots,s)$, $H = \mathrm{diag}(\sum_{i,j=1}\beta_{ji}^2((I_N - J_N)S_{ij}^{(k)}\otimes(I_N - J_N)S_{ij}^{(k)}), k = 1,\ldots,s)$, and $\psi(\hat{V}(0)) = (q_1\psi(V(0))^T,\ldots,q_s\psi(V(0))^T)^T$. Using the above notation, we have

$$\frac{\mathrm{d}\psi(\hat{V}(t))^T\psi(\hat{V}(t))}{\mathrm{d}t} = 2\psi(\hat{V}(t))^T\left(-c(t)\bar{\mathcal{L}} + \Gamma^T\otimes I_{N^2}\right)\psi(\hat{V}(t)) \tag{3.4}$$

$$+2c^2(t)\psi(\hat{V}(t))^T\psi(Z(t))+2c^2(t)\psi(\hat{V}(t))^T\psi(\hat{V}(t)).$$

Let $L := \mathrm{diag}(L(k)\oplus L(k), k=1,2,\ldots,s)$. Applying Assumption 2, we know that $L(k)=\frac{\mathcal{L}^T(k)+\mathcal{L}(k)}{2}$ is symmetric. From Assumption 3, we can see that $Q=\frac{\Gamma^T+\Gamma}{2}$ is irreducible, since $\{\theta(t)\}_{t\geq 0}$ is ergodic and Γ is double stochastic. Therefore, (3.4) could be rewritten as

$$\begin{aligned}\frac{\mathrm{d}\psi(\hat{V}(t))^T\psi(\hat{V}(t))}{\mathrm{d}t} &= 2\psi(\hat{V}(t))^T\left(-c(t)L+Q\otimes I_{N^2}\right)\psi(\hat{V}(t)) \\ &\quad +2c^2(t)\psi(\hat{V}(t))^T\psi(Z(t))+2c^2(t)\psi(\hat{V}(t))^T\psi(\hat{V}(t)).\end{aligned} \tag{3.5}$$

From Assumptions 2, 3 and Lemma 3.13 in [10], we know that matrix $(-c(t)L+Q\otimes I_{N^2})$ has an eigenvalue zero with algebraic multiplicity one, and the matrix can be viewed as the Laplacian matrix corresponding to a strongly undirect graph with the second smallest eigenvalue $v_2>0$, due to Lemma 1. From Assumption 4, there exists t_0 such that $c(t)\leq \min\{1,\frac{v_2}{1+\kappa_H}\}, t\geq t_0$, where $\kappa_H=\max\{0,\lambda_{\max}(H)\}$ and $\lambda_{\max}(H)$ is the maximum eigenvalue of H. The latter combined with Assumption 3 finds that for $t\geq t_0$ it holds

$$\begin{aligned}&\frac{\mathrm{d}\psi(\hat{V}(t))^T\psi(\hat{V}(t))}{\mathrm{d}t} \\ &\quad\leq 2\psi(\hat{V}(t))^T\left(-c(t)L+Q\otimes I_{N^2}\right)\psi(\hat{V}(t))+2c^2(t)\lambda_{\max}(H)\|\psi(\hat{V}(t))\|^2 \\ &\qquad +c^2(t)\left(\|\psi(\hat{V}(t))\|^2+\|\psi(Z(t))\|^2\right) \\ &\quad\leq \left(-2v_2c(t)+c^2(t)+2\lambda_{\max}(H)c^2(t)\right)\|\psi(\hat{V}(t))\|^2+c^2(t)\|\psi(Z(t))\|^2 \\ &\quad\leq -v_2c(t)\|\psi(\hat{V}(t))\|^2+c^2(t)\|\psi(Z(t))\|^2.\end{aligned} \tag{3.6}$$

In (3.6), we use the comparison principle of differential equations, and the becomes

$$\|\psi(\hat{V}(t))\|^2\leq \|\psi(\hat{V}(t_0))\|^2 I_1(t)+I_2(t),$$

where

$$I_1(t)=e^{-v_2\int_{t_0}^t c(s)\mathrm{d}s}, \qquad I_2(t)=\int_{t_0}^t e^{-v_2\int_s^t c(u)\mathrm{d}u}c^2(s)\|\psi(Z(s))\|^2\mathrm{d}s.$$

Applying Assumptions 4, 5 and $v_2>0$ implies

$$\lim_{t\to\infty}\|\psi(\hat{V}(t_0))\|^2 I_1(t)=\lim_{t\to\infty}\|\psi(\hat{V}(t_0))\|^2 e^{-v_2\int_{t_0}^t c(s)\mathrm{d}s}=0.$$

From Assumption 3, we know that the Markov chain $\{\theta(t)\}_{t\geq 0}$ is ergodic. So, there exists a stationary distribution $\{\pi_k>0, k=1,\ldots,s\}$. Let $\bar{Z}=[Z_1\ldots Z_s]$ with $Z_k=(I_N-J_N)\alpha^{(k)}\alpha^{(k)T}(I_N-J_N)\pi_k, k\in\mathbb{S}$. Then, for any given $\varepsilon_1>0$, there is $t_1>0$ such that

$$\left|\|\psi(Z(s))\|^2-\|\psi(\bar{Z}(s))\|^2\right|<\varepsilon_1,\ \ t\geq t_1. \tag{3.7}$$

From Assumption 4, for any given $\varepsilon_2 > 0$, there exists $t_2 > 0$ such that $\int_{t_2}^{\infty} c^2(t)\mathrm{d}t < \varepsilon_2, t \geq t_2$. Set $t_3 = \max\{t_1, t_2\}$. We estimate I_2 to get

$$\begin{aligned}
I_2(t) &= \int_{t_0}^{t_3} e^{-v_2 \int_s^t c(u)\mathrm{d}u} c^2(s)\|\psi(Z(s))\|^2 \mathrm{d}s + \int_{t_3}^{t} e^{-v_2 \int_s^t c(u)\mathrm{d}u} c^2(s)\|\psi(Z(s))\|^2 \mathrm{d}s \\
&\leq e^{-v_2 \int_{t_3}^t c(u)\mathrm{d}u} \int_{t_0}^{t_3} c^2(s)\|\psi(Z(s))\|^2 \mathrm{d}s + \int_{t_3}^{t} c^2(s)\|\psi(Z(s))\|^2 \mathrm{d}s \\
&\leq e^{-v_2 \int_{t_3}^t c(u)\mathrm{d}u} \int_{t_0}^{t_3} c^2(s)\|\psi(Z(s))\|^2 \mathrm{d}s + \left(\|\psi(\bar{Z})\|^2 + \varepsilon_1\right) \int_{t_3}^{t} c^2(s)\mathrm{d}s \\
&\leq e^{-v_2 \int_{t_3}^t c(u)\mathrm{d}u} \int_{t_0}^{t_3} c^2(s)\|\psi(Z(s))\|^2 \mathrm{d}s + \left(\|\psi(\bar{Z})\|^2 + \varepsilon_1\right) \varepsilon_2 \\
&\to \left(\|\psi(\bar{Z})\|^2 + \varepsilon_1\right) \varepsilon_2, \quad t \to \infty. \qquad (3.8)
\end{aligned}$$

Since ε_2 is arbitrary, we have $I_2(t) \to 0$, $t \to \infty$. Therefore, we conclude that $\lim_{t\to\infty} \|\psi(\hat{V}(t))\| = 0$. This leads to $\lim_{t\to\infty} \|V(t))\|_F = 0$. Because of $\mathbb{E}\|\delta(t)\|^2 = \mathrm{tr}(V(t))$, we get $\lim_{t\to\infty} \mathbb{E}\|\delta(t)\|^2 = 0$. Hence, there exists $\kappa_1 > 0$ such that $\mathbb{E}\|\delta(t)\|^2 \leq \kappa_1$, $\forall t \geq 0$.

We use (3.1) and Assumption 2 again to obtain

$$\mathrm{d}\left(J_N x(t) - J_N x(0)\right) = c(t) J_N \alpha^{(\theta(t))} \mathrm{d}W_1(t) + c(t) \sum_{i=1}^{N} \sum_{j=1}^{N} \beta_{ji} J_N S_{ij}^{(\theta(t))} x(t) \mathrm{d}w_{2ji}(t).$$

Integrating the equality above, we have

$$J_N x(t) = J_N x(0) + \int_0^t c(s) J_N \alpha^{(\theta(s))} \mathrm{d}W_1(s) + \int_0^t c(s) \sum_{i=1}^{N} \sum_{j=1}^{N} \beta_{ji} J_N S_{ij}^{(\theta(s))} \delta(s) \mathrm{d}w_{2ji}(s).$$

Let

$$x^* = \frac{1}{N} 1_N^T x(0) + \frac{1}{N} \int_0^\infty c(s) 1_N^T \alpha^{(s)} \mathrm{d}W_1(s) + \frac{1}{N} \int_0^\infty c(s) \sum_{i=1}^{N} \sum_{j=1}^{N} \beta_{ji} 1_N^T S_{ij}^{(\theta(s))} \delta(s) \mathrm{d}w_{2ji}(s).$$

From Lemma 5.4 in [8], we obtain

$$\begin{aligned}
&\mathbb{E}\|J_N x(t) - x^* 1_N\|^2 \\
&= \mathbb{E}\left\| \int_t^\infty c(s) J_N \alpha^{(\theta(s))} \mathrm{d}W_1(s) + \int_t^\infty c(s) \sum_{i=1}^{N} \sum_{j=1}^{N} \beta_{ji} J_N S_{ij}^{(\theta(s))} \delta(s) \mathrm{d}w_{2ji}(s) \right\|^2 \\
&= \mathbb{E}\left\| \int_t^\infty c(s) J_N \alpha^{(\theta(s))} \mathrm{d}W_1(s) \right\|^2 + \mathbb{E}\left\| \int_t^\infty c(s) \sum_{i=1}^{N} \sum_{j=1}^{N} \beta_{ji} J_N S_{ij}^{(\theta(s))} \delta(s) \mathrm{d}w_{2ji}(s) \right\|^2 \\
&= \mathbb{E} \int_t^\infty \|J_N \alpha^{(\theta(s))}\|_F^2 c^2(s) \mathrm{d}s + \sum_{i,j=1}^{N} \beta_{ji}^2 \mathbb{E} \int_t^\infty \|J_N S_{ij}^{(\theta(s))} \delta(s)\|^2 c^2(s) \mathrm{d}s \\
&:= I_3(t) + I_4(t).
\end{aligned}$$

We apply Assumptions 2, 4 and $I_3(t)$ to find

$$I_3(t) \leq \max_{1\leq k\leq s} \|J_N\alpha^{(k)}\|_F^2 \int_t^\infty c^2(s)\mathrm{d}s = o(1), \;\; t\to\infty, \tag{3.9}$$

and

$$\begin{aligned} I_4(t) &= \sum_{i,j=1}^N \beta_{ji}^2 \mathbb{E}\int_t^\infty \|J_N S_{ij}^{(\theta(s))}\delta(s)\|^2 c^2(s)\mathrm{d}s \\ &= \sum_{i,j=1}^N \beta_{ji}^2 \mathbb{E}\int_t^\infty \|a_{ij}(\theta(s))(\delta_j(s)-\delta_i(s))1_N\|^2 c^2(s)\mathrm{d}s \\ &= \sum_{i,j=1}^N \beta_{ji}^2 \mathbb{E}\int_t^\infty a_{ij}^2(\theta(s))(\delta_j(s)-\delta_i(s)^2 N c^2(s)\mathrm{d}s \\ &\leq N\kappa_\beta^2 \int_t^\infty \mathbb{E}\sum_{i,j=1}^N (\delta_j(s)-\delta_i(s)^2 c^2(s)\mathrm{d}s \\ &= N\kappa_\beta^2 \int_t^\infty N\mathbb{E}\|\delta(s)\|^2 c^2(s)\mathrm{d}s \\ &\leq N^2\kappa_1\kappa_\beta^2 \int_t^\infty c^2(s)\mathrm{d}s = o(1), \;\; t\to\infty. \end{aligned} \tag{3.10}$$

Taking account of (3.9) and (3.10), we have

$$\lim_{t\to\infty} \mathbb{E}\|J_N x(t) - x^* 1_N\|^2 = 0.$$

Hence,

$$\mathbb{E}[x^*] = \frac{1}{N} 1_N^T x(0), \;\; \mathrm{Var}[x^*] < \infty.$$

Consequently, by using $\lim_{t\to\infty}\mathbb{E}\|\delta(t)\|^2 = 0$, we complete the proof of the theorem. □

Remark 4. If the considered system in this work is noiseless and $c(t) \equiv 1$, it generates to a linear system with Markov switching topologies whose average consensus has been developed by Matei et al. [10], and the consensus results of [10] were extended to a more general linear systems in [15]. Under assumptions that the systems has only addictive noises working in a stochastic switching topologies, Zhang et al.[16] obtained the sufficient conditions for average consensus. Unlike the consensus results in [16], Li et al. ([7]) proposed weaker necessary and sufficient conditions to obtain weaker consensus results under a fixed topology. If the topology is fixed, then the systems (2.1) with information measurement (2.2) degenerate to a general stochastic differential equations with compound noises, the necessary conditions and sufficient conditions for mean square weak and strong consensus were developed with a weaker time-varying gain comparing to Assumption 4 by Zong et al. [17]. Furthermore,

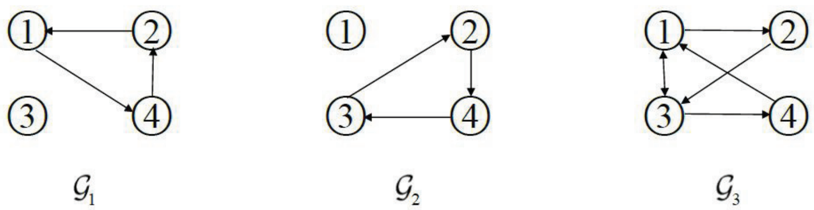

FIGURE 1. All the possible topologies

when a regular time-varying topologies were considered in the systems, they proposed sufficient conditions for weak consensus and average consensus.

4. SIMULATIONS

In this section, in order to analyze the influence of noises and switching topologies to the dynamics of the coupled system (2.1) with information measurement (2.2), we now consider a dynamic of (2.1) with four agents where three different directed graphs are as follows: Obviously, the corresponding topology graphs are balanced and the union graph of the three topology graphs has a spanning tree. If the switching topologies are compelled by a Markov chain in which the generator is chosen as

$$\Gamma = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix},$$

then we know that it will jump between the three topologies and reach every mode in the given time. Given $x(0) = (8, 1, -6, 3)$. Let $\alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{21} = \alpha_{41} = 0.1$, $\beta_{12} = \beta_{13} = \beta_{31} = 1$, and $c(t) = \frac{1}{1+t}$. The dynamic curve of the four agents (see Figure 3) shows a sample path of the consensus seeking process corresponding to a ergodic Markov chain in Figure 2. This also illustrates the average consensus which is achieved and consistent with Theorem 1

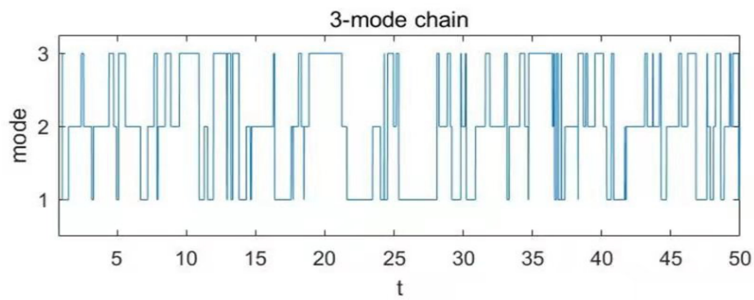

FIGURE 2. Ergodic Markov chain

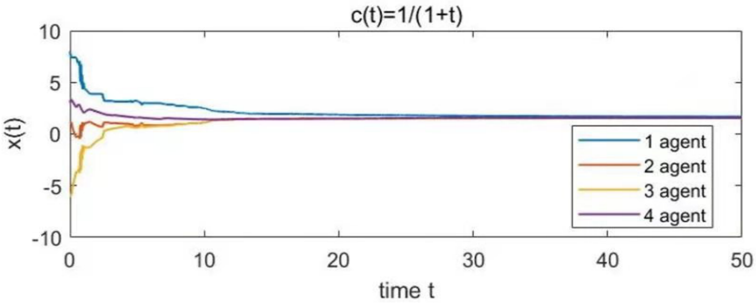

FIGURE 3. dynamic curve of all agents

5. CONCLUSION

We have investigated considered stochastic consensus of the first-order continuous-time multi-agent systems with addictive and multiplicative noises under time-varying switching topologies. To analysis the time-varying topologies, the time-varying topologies is figured by using a continuous-time Markov chain with finite modes which corresponds to different directed graphs in the topological structure of the systems. Besides, for each fixed mode, the noise effecting on an agent is measured under the compound noises in the fixed topology corresponding the mode with fixed noise intensities. By investigating the connectivity of the directed and balanced graphs, conditions on the time-vary control gain and the irreducible Markov chain, we obtain a time-asymptotic mean square average consensus for the considered systems. A simulation example is operated to demonstrate our theoretical results.

ACKNOWLEDGEMENT

The authors gratefully appreciate the two referees whose comments significantly improved the presentation of this paper and acknowledge the Natural Science Foundation of Guangxi Provincec (2018JJA110105), Foundation of Yulin Normal University (G2019ZK12), Middle-aged and Young Teachers' Basic Ability Promotion Project of Guangxi (2019KY0587), for supporting this work as part of postdoctoral work.

REFERENCES

[1] D. X. Chen, W. Li, X. L. Liu, W. W. Yu, and Y. Z. Sun, "Effects of measurement noise on flocking dynamcis of Cucker-Smale systems." *IEEE trans. circults Syst. II*, vol. 67, no. 10, pp. 2064–2068, 2020, doi: 10.1109/TCSII.2019.2947788.

[2] J. G. Dong, S. Y. Ha, and D. Kim, "On the stochastic flocking of the Cucker-Smale flock with randomly switching topologies." *SIAM J. Control Optim.*, vol. 58, no. 4, pp. 2332–2353, 2020, doi: 10.1137/19M1279150.

[3] H. Fourati, *Multisensor data fusion: from algorithm and architectural design to applications.* Boca Raton: CRC press, 2015. doi: 10.1201/b18851.

[4] M. D. Fragoso and O. L. V. Costa, "A unified approach for stochastic and mean square staility of continuous-time linear systems with Markovian jumping parameters and additive disturbances." *SIAM J. Control Optim.*, vol. 44, no. 4, pp. 1165–1191, 2005, doi: 10.1137/S0363012903434753.

[5] M. Li and F. Deng, "Necessary and sufficient conditions for consensus of continuous-time multiagent systems with Markovian switching topologies and communication noise." *IEEE T. Cybernetics*, vol. 50, no. 7, pp. 3264–3270, 2020, doi: 10.1109/TCYB.2019.2919740.

[6] T. Li, F. K. Wu, and J. F. Zhang, "Multi-agent consensus with relative-state-dependent measurement noises." *IEEE Trans. Autom. Control.*, vol. 59, no. 9, pp. 2463–2468, 2014, doi: 10.1109/TAC.2014.2304368.

[7] T. Li and J. F. Zhang, "Mean square average-consensus under measurement noises and fixed topologies: Necessary and sufficient conditions." *Automatica*, vol. 45, pp. 1929–1936, 2009, doi: 10.1016/j.automatica.2009.04.017.

[8] X. Mao, *Stochastic differential quations and applications.* Elsevier, 2007. doi: 10.1533/9780857099402.1.

[9] X. Mao, Y. Shen, and A. Gray, "Almost sure exponential stability of backward Euler-Maruyama discretizations for hybrid stochastic differential equations," *J. Comput. Appl. Math.*, vol. 235, no. 5, pp. 1213–1226, 2011, doi: 10.1016/j.cam.2010.08.006.

[10] I. Matei, J. S. Baras, and C. Somarakis, "Convergence results for the linear consensus problem under Markovian random graphs," *SIAM J. Control. Optim.*, vol. 52, no. 2, pp. 1574–1591, 2013, doi: 10.1137/100816870.

[11] Y.-H. Ni and X. Li, "Consensus seeking in multi-agent systems with multiplicative measurement noises," *Systems & Control Letters*, vol. 62, no. 5, pp. 430–437, 2013, doi: 10.1016/j.sysconle.2013.01.011.

[12] V. P. Tuzlukov, *Signal processing noise.* Boca Raton: CRC press, 2002. doi: 10.1201/9781315220147.

[13] J. Wang and N. Elia, "Distributed averaging under constraints on information exchange: Emergence of Lévy flights." *IEEE Trans. Automat. Control*, vol. 57, no. 10, pp. 2435–2449, 2012, doi: 10.1109/TAC.2012.2186093.

[14] J. Wang and N. Elia, "Mitigation of complex behavior over networked systems: Analysis of spatially invariant structures." *Automatica*, vol. 49, no. 6, pp. 1626–1638, 2013, doi: 10.1016/j.automatica.2013.02.042.

[15] K. You, Z. Li, and L. Xie, " Consensus condition for linear-agent systems over randomly switching topologies." *Automatica*, vol. 19, no. 10, pp. 3125–3132, 2013, doi: 10.1016/j.automatica.2013.07.024.

[16] Q. Zhang and J. F. Zhang, " Distribued consensus of continuous-time multi-agent systems with Markovian switching topologies and stochastic communication noises." *J. Sys. Sci. & Math. Scis.*, vol. 31, no. 9, pp. 1097–1110, 2011, doi: 10.1016/j.jfranklin.2015.08.003.

[17] X. Zong, T. Li, and J. Zhang, "Consensus conditions of continuous-time multi-agent systems with additice and multiplicative measurement noise. ," *SIAM J. Control Optim.*, vol. 56, no. 1, pp. 19–52, 2018, doi: 10.1137/15M1019775.

[18] X. Zong, T. Li, and J. Zhang, " Consensus conditions of continuous-time multi-agent systems with time-delays and measurement noises," *Automatica*, vol. 99, pp. 412–419, 2019, doi: 10.1016/j.automatica.2013.07.024.

Authors' addresses

Xiaojin Huang
National University of Defense Technology, College of Liberal Arts and Sciences, 410000, Hunan, China
Current address: Yulin Normal University, School of Mathematics and Statistics, 537000, Guangxi, China
E-mail address: xjhuangikun@163.com

Yongjian Liu
Yulin Normal University, School of Mathematics and Statistics, 537000, Guangxi, China
E-mail address: liuyongjianj1975@126.com

Jianhua Huang
(corresponding) National University of Defense Technology, College of Liberal Arts and Sciences, 410000, Hunan, China
E-mail address: hhuang32@nudt.edu.cn