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ON EXISTENCE OF BOUNDED SOLUTIONS ON REAL AXIS $\mathbb{R}$ OF LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Effective sufficient conditions are established for existence of bounded solutions satisfying the Nicoletti condition of the systems of linear generalized ordinary differential equations on the real axis. There are given the method of the construction of such solutions. The sufficient conditions of the existence of unique solution and of positiveness of that are established as well. As particular case, there are investigated the problem of existence of bounded solutions.,

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1. Statement of the problem. Basic notation and definitions

For the linear system of the generalized differential equations

$$dx = dA(t)\cdot x + df(t) \quad \text{for} \quad t\in\mathbb{R}, \tag{1.1}$$

consider the problem on the bounded on $\mathbb{R}$ solution

$$\sup\{\|x(t)\| : t\in\mathbb{R}\} < +\infty, \tag{1.2}$$

where $A=(a_{ik})_{i,k=1}^{n}\in \mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^{n\times n})$, and $f=(f_i)_{i=1}^{n}\in \mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^{n})$.

The generalized ordinary differential equations was introduced by J. Kurzweil [11]. To a considerable extent, the interest to the theory has also been stimulated by the fact that this theory enabled one to investigate ordinary differential, impulsive differential and difference equations from a unified point of view (see [1–8], [12] and references therein). So, we can consider the ordinary differential, impulsive differential and difference equations as equations of the same type.

In this paper effective sufficient conditions are established for the existence of solutions of problem (1.1), (1.2). Analogous results are contained in [10], [9] (see also references therein) for the problem for systems of ordinary differential equations.

In the paper the use will be made of the following notation and definitions

$\mathbb{R} =]-\infty,+\infty[$, $\mathbb{R}_+ = [0,+\infty[$; $[a,b]$ and $]a,b[$ $(a,b \in \mathbb{R})$ are, respectively, closed and open intervals. I is an arbitrary finite or infinite interval from $\mathbb{R}$. $[t]$ is the integer part of $t \in \mathbb{R}$.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max_{j=1,\dots,m} \sum\limits_{i=1}^{n} |x_{ij}|$. $\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det(X)$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n \times n$-matrix; δ_{ij} is the Kroneker symbol, i.e. $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$ $(i, j = 1,\dots)$.

The inequalities between the matrices are understood componentwise.

$\overset{b}{\underset{a}{\mathrm{V}}}(X)$ is the sum total variation of the components of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$. If $X = (x_{ij})_{i,j=1}^{n,m} : \mathbb{R} \to \mathbb{R}^{n\times m}$, then $V(X)(t) = \left(\overset{t}{\underset{0}{\mathrm{V}}}(x_{ij})\right)_{i,j=1}^{n,m}$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t ($X(\alpha-) = X(\alpha)$ if $\alpha \in I$ and $X(\beta+) = X(\beta)$ if $\gamma \in I$; if α or β do not belong to I, then $X(t)$ is defined by continuity outside of I). $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$. $\|X\|_\infty = \sup\{\|X(t)\| : t \in I\}$.

$\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$ such that $\overset{b}{\underset{a}{\mathrm{V}}}(X) < \infty$. $\mathrm{BV}_{loc}(\mathbb{R};\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : \mathbb{R} \to \mathbb{R}^{n\times m}$ whose restrictions on every closed interval $[a,b]$ belong to $\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

s_1, s_2, s_c and $J : \mathrm{BV}_{loc}(\mathbb{R};\mathbb{R}) \to \mathrm{BV}_{loc}(\mathbb{R};\mathbb{R})$ are the operators defined by

$$
\begin{aligned}
& s_1(x)(0) = s_2(x)(0) = 0,\\
& s_c(x)(0) = x(0)\\
& s_1(x)(t) = s_1(x)(s) + \sum_{s<\tau\le t} d_1x(\tau),\\
& s_2(x)(t) = s_2(x)(s) + \sum_{s\le\tau<t} d_2x(\tau)\\
& s_c(x)(t) = s_c(x)(s) + x(t) - x(s) - \sum_{j=1}^{2} (s_j(x)(t) - s_j(x)(s)) \ \text{ for } \ s < t;\\
& J(x)(0) = x(0),\\
& J(x)(t) = J(x)(s) + s_c(x)(t) - s_c(x)(s)\\
& \qquad - \sum_{s<\tau\le t} \ln|1 - d_1x(\tau)| + \sum_{s\le\tau<t} \ln|1 + d_2x(\tau)| \ \text{ for } \ s < t.
\end{aligned}
$$

If $g \in \mathrm{BV}([a,b];\mathbb{R})$, $f : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then we assume

$$\int_s^t x(\tau)\,dg(\tau) = (L-S)\int_{]s,t[} x(\tau)\,dg(\tau) + f(t)d_1g(t) + f(s)d_2g(s).$$

where $(L-S)\int_{]s,t[} f(\tau)\,dg(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$. It is known (see, [12]) that if the integral exists, than the right side of the integral equality equals to the Kurzeil–Stieltjes integral $(K-S)\int_s^t f(\tau)\,dg(\tau)$ and, therefore, $\int_s^t f(\tau)\,dg(\tau) = (K-S)\int_s^t f(\tau)\,dg(\tau)$. If $a=b$, then we assume $\int_a^b x(t)\,dg(t) = 0$.

$$\int_{-\infty}^a f(\tau)dg(\tau) = \lim_{t\to\infty}\int_t^a f(\tau)dg(\tau) \text{ and } \int_a^{+\infty} f(\tau)dg(\tau) = \lim_{t\to+\infty}\int_a^t f(\tau)dg(\tau)$$

if the last limits exist (finite or infinite).

If $G = (g_{ik})_{i,k=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ and $x = (x_k)_k^n : [a,b] \to \mathbb{R}^n$, then

$$\int_a^b dG(\tau)\cdot x(\tau) = \left(\sum_{k=1}^n \int_a^b x_k(\tau)\,dg_{ik}(\tau)\right)_i^n.$$

We introduce the operator $\mathcal{A}(X,Y)$ in the following way:

if $X \in \mathrm{BV}_{loc}(\mathbb{R};\mathbb{R}^{n\times n})$, $\det(I_n + (-1)^j d_jX(t)) \ne 0$ for $t \in \mathbb{R}$ $(j=1,2)$, and $Y \in \mathrm{BV}_{loc}(\mathbb{R};\mathbb{R}^{n\times m})$, then

$$\begin{aligned}
&\mathcal{A}(X,Y)(0) = O_{n\times m},\\
&\mathcal{A}(X,Y)(t) = \mathcal{A}(X,Y)(s) + Y(t) - Y(s) + \sum_{s<\tau\le t} d_1X(\tau)(I_n - d_1X(\tau))^{-1}d_1Y(\tau)\\
&\qquad - \sum_{s\le\tau<t} d_2X(\tau)(I_n + d_2X(\tau))^{-1}\,d_2Y(\tau) \ \ (s<t).
\end{aligned}$$

Here the use will be made of the following formulas:

$$\begin{aligned}
\int_a^b f(t)\,dg(t) &= \int_a^b f(t)\,dg(t+) + f(a)d_2g(a)\\
&= \int_a^b f(t)\,dg(t-) + f(b)d_1g(b),\\
\int_a^t x(\tau)\,dg(\tau) &= \int_a^{t-} x(\tau)\,dg(\tau) + x(t)\,d_1g(t)\\
&= \int_a^{t+} x(\tau)\,dg(\tau) - x(t)\,d_2g(t).
\end{aligned}$$

$$\int_a^b f(t)\,dg(t) + \int_a^b g(t)\,df(t) = f(b)g(b) - f(a)g(a) + \sum_{a<t\le b} d_1 f(t)\cdot d_1 g(t)$$
$$- \sum_{a\le t<b} d_2 f(t)\cdot d_2 g(t)$$

(integration-by-parts formula),

$$\int_a^b h(t)\,d\,(f(t)g(t)) = \int_a^b h(t)f(t)\,dg(t) + \int_a^b h(t)g(t)\,df(t)$$
$$- \sum_{a<t\le b} h(t)d_1 f(t)\cdot d_1 g(t)$$
$$+ \sum_{a\le t<b} h(t)d_2 f(t)\cdot d_2 g(t)$$

(general integration-by-parts formula),

$$\int_a^b f(t)ds_1(g)(t) = \sum_{a<t\le b} f(t)\,d_1 g(t), \int_a^b f(t)ds_2(g)(t)$$
$$= \sum_{a<t\le b} f(t)d_2 g(t),$$
$$\int_a^b f(t)\,d\left(\int_a^s g(s)dh(s)\right) = \int_a^b f(t)g(t)\,dh(t),$$
$$d_j\left(\int_a^t f(s)\,dg(s)\right) = f(t)\,d_j g(t) \ \ (j=1,2).$$

The proof of above formulas are given in [5, 6, 12] for example.

By a solution of system (1.1) we mean a vector-function $x \in \mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^n)$ if

$$x(t) = x(s) + \int_s^t dA(\tau)\cdot x(\tau) + f(t) - f(s)) \ \text{ for } \ s<t, \ s,t\in\mathbb{R}.$$

If $\alpha \in \mathrm{BV}_{loc}(\mathbb{R},\mathbb{R})$ and $t_0 \in \mathbb{R}$ are such that $1+(-1)^j d_j\alpha(t) \neq 0$ for $t\in\mathbb{R}$, $t\neq t_0$ $(j=1,2)$. Then it is known the that (see [7, 8] the initial problem

$$d\xi = \xi d\alpha(t), \ \ \xi(0) = 1$$

has the unique solution ξ_α and it is defined by

$$\xi_\alpha(t) = \begin{cases} \exp(s_c(\alpha)(t) - s_c(\alpha)(0)) \prod\limits_{0<\tau\le t}(1-d_1\alpha(\tau))^{-1} \prod\limits_{0\le\tau<t}(1+d_2\alpha(\tau)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \ t>0, \\ \exp(s_c(\alpha)(t) - s_c(\alpha)(0)) \prod\limits_{t<\tau\le 0}(1-d_1\alpha(\tau)) \prod\limits_{t\le\tau<0}(1+d_2\alpha(\tau))^{-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \ t<0. \end{cases}$$

Let $\gamma_\alpha(t,s) \equiv \xi_\alpha(t)\xi_\alpha^{-1}(s)$ be the Cauchy function of the problem. Then

$$\gamma_\alpha(t,s) = \exp\big(J(\alpha)(t) - J(\alpha)(s)\big) \prod_{s<\tau\le t} \operatorname{sgn}\big(1 - d_1\alpha(\tau)\big)$$
$$\times \prod_{s\le\tau<t} \operatorname{sgn}\big(1 + d_2\alpha(\tau)\big) \ \text{ for } \ t > s,$$
$$\gamma_\alpha(t,s) = \gamma_\alpha^{-1}(s,t) \ \text{ for } \ t < s.$$

Note that the following equality holds (see, [5,6]

$$d\xi_\alpha^{-1}(t,t_0) \equiv -\xi_\alpha^{-1}(t,t_0)d\mathcal{A}(\alpha,\alpha).(t) \tag{1.3}$$

Remark 1. Let $\alpha \in \mathrm{BV}([a,b],\mathbb{R})$ be such that $1+(-1)^j d_j\alpha(t) > 0$ for $t \in [a,b]$ $(j = 1,2)$ and let one of the functions α, $J(\alpha)$ and $\mathcal{A}(\alpha,\alpha)$ be nondecreasing (nonincreasing). Then other two functions will be nondecreasing (nonincreasing), as well.

We introduce the operator

$$\nu(\zeta)(t) = \sup\Big\{\tau \ge t: \ \zeta(\tau) \le \zeta(t+) + 1\Big\}$$

if $\xi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function, and

$$\nu(\zeta)(t) = \inf\Big\{\tau \le t: \ \zeta(\tau) \le \zeta(t-) + 1\Big\},$$

if $\zeta : \mathbb{R} \to \mathbb{R}$ is a nonincreasing function.

2. Formulation of the results

For every $t_i \in \mathbb{R}\cup\{-\infty,+\infty\}$ $(i = 1,\dots,n)$ we put $\mathcal{N}_0(t_1,\dots,t_n) = \{i : t_i \in \mathbb{R}\}$. It is evident that $\mathcal{N}_0(t_1,\dots,t_n) = \{1,\dots,n\}$ if $t_i \in \mathbb{R}$ $(i = 1,\dots,n)$, and $\mathcal{N}_0(t_1,\dots,t_n) = \varnothing$ if $t_i \in \{-\infty,+\infty\}$ $(i = 1,\dots,n)$.

In the case, where $t_i = -\infty$ $(t_i = +\infty)$, we assume $\operatorname{sgn}(t-t_i) = 1$ for $t \in \mathbb{R}$ $(\operatorname{sgn}(t-t_i) = -1$ for $t \in \mathbb{R})$.

Theorem 1. *Let*

$$1+(-1)^j d_j a_{ii}(t) \ne 0 \ \textit{for} \ t \in \mathbb{R} \ \ (j = 1,2;\ i = 1,\dots,n) \tag{2.1}$$

and let there exist $t_i \in \mathbb{R}\cup\{-\infty,+\infty\}$ $(i = 1,\dots,n)$ *such that*

$$s_{ik} = \sup\left\{\left|\int_{t_i}^{t} |\gamma_i(t,\tau)|dV\big(\mathcal{A}(a_{ii},a_{ik})\big)(\tau)\right| : t \in \mathbb{R}\right\}$$
$$< +\infty (i \ne k;\ i,k = 1,\dots,n), \tag{2.2}$$

$$\sup\left\{\left|\int_{t_i}^{t} |\gamma_i(t,\tau)|dV\big(\mathcal{A}(a_{ii},f_i)\big)(\tau)\right| : t \in \mathbb{R}\right\} < +\infty \ \ (i = 1,\dots,n) \tag{2.3}$$

and

$$\sup\{|\gamma_i(t,t_i)| : t \in \mathbb{R}\} < +\infty \ \text{for} \ i \in \mathcal{N}_0(t_1,\dots,t_n), \tag{2.4}$$

where $\gamma_i(t,\tau) \equiv \gamma_{a_{ii}}(t,\tau)$ $(i = 1,\dots,n)$. *Let, moreover, the matrix* $S = (s_{ik})_{i,k=1}^n$, *where* $s_{ii} = 0$ $(i = 1,\dots,n)$, *be such that*

$$r(S) < 1. \tag{2.5}$$

Then for every $c_i \in \mathbb{R}$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$ *system* (1.1) *has at last one a bounded on* $\mathbb{R}$ *solution satisfying the condition*

$$x_i(t_i) = c_i \ \text{for} \ i \in \mathcal{N}_0(t_1,\dots,t_n)). \tag{2.6}$$

If the case, where $\mathcal{N}_0(t_1,\dots,t_n) = \varnothing$, conditions (2.4) and (2.6) be eliminated and the theorem has the following form.

Theorem 1′. *Let conditions* (2.1), (2.2) *and* (2.3) *hold for some* $t_i \in \{-\infty,+\infty\}$ $(i = 1,\dots,n)$, *where* $\gamma_i(t,\tau) \equiv \gamma_{a_{ii}}(t,\tau)$ $(i = 1,\dots,n)$, *and the matrix* $S = (s_{ik})_{i,k=1}^n$, *where* $s_{ii} = 0$ $(i = 1,\dots,n)$, *satisfy condition* (2.5). *Then system* (1.1) *has at last one solution bounded on* $\mathbb{R}$.

Corollary 1. *Let*

$$1 + (-1)^j d_j a_{ii}(t) > 0 \ \text{for} \ t \in \mathbb{R}, \ \ (j = 1,2;\, i = 1,\dots,n) \tag{2.7}$$

and let there exist $t_i \in \mathbb{R} \cup \{-\infty,+\infty\}$ $(i = 1,\dots,n)$ *such that conditions* (2.2), (2.3), (2.4) *and* (2.5) *hold, where* $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ $(i = 1,\dots,n)$ *and* $\gamma_i(t,\tau) \equiv \gamma_{a_{ii}}(t,\tau)$ $(i = 1,\dots,n)$. *Let, moreover, the functions*

$$\begin{gathered}\mathcal{A}(a_{ii},a_{ik})(t)\ \mathrm{sgn}(t-t_i), \ \ \mathcal{A}(a_{ii},f_i)(t)\ \mathrm{sgn}(t-t_i) \\ (i \neq k;\, i,k = 1,\dots,n) \ \text{are nondecreasing on} \ \mathbb{R}.\end{gathered} \tag{2.8}$$

Then for every $c_i \in \mathbb{R}_+$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$ *system* (1.1) *has at last one nonnegative and bounded on* $\mathbb{R}$ *solution satisfying condition* (2.6).

If $\mathcal{N}_0(t_1,\dots,t_n) = \varnothing$ then Corollary 1 has the following form.

Corollary 1′. *Let conditions* (2.7) *and* (2.8) *hold and let there exist* $t_i \in \{-\infty,+\infty\}$ $(i = 1,\dots,n)$ *such that conditions* (2.2), (2.3) *and* (2.5) *hold, where* $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ $(i = 1,\dots,n)$ *and* $\gamma_i(t,\tau) \equiv \gamma_{a_{ii}}(t,\tau)$ $(i = 1,\dots,n)$. *Then system* (1.1) *has at last one a nonnegative and bounded on* $\mathbb{R}$ *solution.*

Theorem 2. *Let* (2.1) *hold and let there exist* $t_i \in \mathbb{R} \cup \{-\infty,+\infty\}$ $(i = 1,\dots,n)$ *such that conditions* (2.2), (2.3), (2.4) *and* (2.5) *hold, where* $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ $(i = 1,\dots,n)$ *and* $\gamma_i(t,\tau) \equiv \gamma_{a_{ii}}(t,\tau)$ $(i = 1,\dots,n)$. *'Let, moreover,*

$$\liminf_{t \to t_i} \gamma_i(0,t) = 0 \ \text{for} \ i \in \{1,\dots,n\} \setminus \mathcal{N}_0(t_1,\dots,t_n). \tag{2.9}$$

Then for every $c_i \in \mathbb{R}$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$ *system* (1.1) *has the unique and bounded on* $\mathbb{R}$ *solution* $(x_i)_{i=1}^n$ *satisfying condition* (2.6) *and*

$$\sum_{i=1}^{n} |x_i(t) - x_{im}(t)| \le \rho_0 \alpha^m \ \text{for} \ t \in \mathbb{R} \ \ (m = 1,2,\dots)), \tag{2.10}$$

where ρ_0 *and* α *are the positive numbers independent of m,* $(x_{im})_{i=1}^n$ $(m = 0,1,\dots)$ *is the sequence of the vector-functions the components of which are defined by*

$$x_{i0}(t) \equiv 0,\ x_{im}(t) \equiv u_i(t) + \sum_{k=1,k\ne i}^{n} \int_{t_i}^{t} \gamma_i(t,\tau)\, d\mathcal{A}(a_{ii}, a_{ik}x_{k\,m-1})(\tau) \tag{2.11}$$

$(i = 1,\dots,n;\ m = 1,2,\dots)$*, and the functions* u_i $(i = 1,\dots,n)$ *are defined due to*

$$u_i(t) \equiv c_i\gamma_i(t,t_i) + \int_{t_i}^{t} \gamma_i(t,\tau)\, d\mathcal{A}(a_{ii}, f_i)(\tau) \ \text{for} \ i \in \mathcal{N}_0(t_1,\dots,t_n), \tag{2.12}$$

$$u_i(t) \equiv \int_{t_i}^{t} \gamma_i(t,\tau)\, d\mathcal{A}(a_{ii}, f_i)(\tau) \ \text{for} \ i \in \{1,\dots,n\} \setminus \mathcal{N}_0(t_1,\dots,t_n). \tag{2.13}$$

Corollary 2. *Let* (2.7) *hold and let there exist* $t_i \in \mathbb{R} \cup \{-\infty,+\infty\}$ $(i = 1,\dots,n)$ *such that the functions* $a_{ii}(t)\operatorname{sgn}(t-t_i)$ $(i = 1,\dots,n)$ *are non-increasing on* $\mathbb{R}$,

$$\liminf_{t\to t_i} a_{ii}(t) = +\infty \ \text{for} \ i \in \{1,\dots,n\} \setminus \mathcal{N}_0(t_1,\dots,t_n), \tag{2.14}$$

$$V(\mathcal{A}(a_{ii},a_{ik}))(t) \le -h_{ik}\operatorname{sgn}(t-t_i)\mathcal{A}(a_{ii},a_{ii})(t) \tag{2.15}$$

for $t \in \mathbb{R}$ $(i \ne k;\ i,k = 1,\dots,n)$ *and*

$$r(\mathcal{H}) < 1, \tag{2.16}$$

where h_{ik} $(i,k = 1,\dots,n)$ *are such that* $\mathcal{H} = \big((1-\delta_{ik})h_{ik}\big)_{i,k=1}^n$*. Let, moreover,*

$$\rho_i = \sup\left\{ \left| \bigvee_{t}^{\nu(\zeta_i)(t)} \big(\mathcal{A}(a_{ii}, f_i)\big) \right| :\ t \in \mathbb{R} \right\} < \infty \ \ (i = 1,\dots,n), \tag{2.17}$$

where $\zeta_i(t) \equiv \xi_{a_{ii}}\operatorname{sgn}(t-t_i)$ $(i = 1,\dots,n)$*. Then conclusion of Theorem* 2 *is true.*

Corollary 3. *Let there exist the points* $t_i \in \mathbb{R} \cup \{-\infty,+\infty\}$ $(i = 1,\dots,n)$ *and the functions* $\alpha_i : \mathbb{R} \to \mathbb{R}$ $(i = 1,\dots,n)$ *such that* $\alpha_i(t)\operatorname{sgn}(t-t_i)$ $(i = 1,\dots,n)$ *are non-decreasing on* $\mathbb{R}$ *and conditions*

$$\big(s_c(a_{ii})(t) - s_c(a_{ii})(s)\big)\operatorname{sgn}(t-s) \le \eta_{ii}\big(s_c(\alpha_i)(t) - s_c(\alpha_i)(s)\big) \tag{2.18}$$

for $(t-s)(s-t_i) > 0$ $(i = 1,\dots,n)$,

$$d_1 a_{ii}(t) \le \eta_{ii} d_1\alpha_i(t) < 1,\ \ -1 < d_2 a_{ii}(t) \le \eta_{ii} d_2\alpha_i(t) \ \ (i = 1,\dots,n), \tag{2.19}$$

$$\big|s_c(a_{ik})(t) - s_c(a_{ik})(s)\big|\operatorname{sgn}(t-s) \le \eta_{ik}\big(s_c(\alpha_i)(t) - s_c(\alpha_i)(s)\big) \tag{2.20}$$

for $(t-s)(s-t_i)>0$ $(i\neq k;\, i,k=1,\dots,n)$ *and*

$$|d_j a_{ik}(t)|\le \eta_{ik}|d_j\alpha_i(t)| \quad (j=1,2;\, i\neq k;\, i,k=1,\dots,n) \tag{2.21}$$

hold on $\mathbb{R}$*, where* $\mathcal{H}=\big((1-\delta_{ik})\eta_{ik}|\eta_{ii}|^{-1}\big)_{i,k=1}^n$. *Let, moreover,*

$$\rho_i=\sup\left\{\left|\bigvee_{t}^{\nu(\vartheta_i)(t)}\big(\mathcal{A}(\eta_i\alpha_i,f_i)\big)\right|:\ t\in\mathbb{R}\right\}<\infty \quad (i=1,\dots,n), \tag{2.22}$$

where $\vartheta_i(t)\equiv\xi_{\eta_i a_i}\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$*. Then conclusion of Theorem 2 is true.*

Theorem 2′. *Let (2.1) hold and let there exist* $t_i\in\{-\infty,+\infty\}$ $(i=1,\dots,n)$ *such that conditions (2.2), (2.3) and (2.5) hold, where* $S=(s_{ik})_{i,k=1}^n$, $s_{ii}=0$ $(i=1,\dots,n)$ *and* $\gamma_i(t,\tau)\equiv\gamma_{a_{ii}}(t,\tau)$ $(i=1,\dots,n)$*.'Let, moreover,*

$$\liminf_{t\to t_i}\gamma_i(0,t)=0 \ \ for\ \ i\in\{1,\dots,n\}.$$

Then system (1.1) has the unique and bounded on $\mathbb{R}$ *solution* $(x_i)_{i=1}^n$ *and*

$$\sum_{i=1}^n|x_i(t)-x_{im}(t)|\le\rho_0\alpha^m \ \ for\ \ t\in\mathbb{R}\ \ (m=1,2,\dots),$$

where ρ_0 *and* α *are the positive numbers independent of* m*,* $(x_{im})_{i=1}^n$ $(m=0,1,\dots)$ *is the sequence of the vector-functions the components of which are defined by*

$$x_{i0}(t)\equiv 0,$$

$$x_{im}(t)\equiv\int_{t_i}^{t}\gamma_i(t,\tau)d\mathcal{A}(a_{ii},f_i)(\tau)$$

$$+\sum_{k=1,k\neq i}^{n}\int_{t_i}^{t}\gamma_i(t,\tau)d\mathcal{A}(a_{ii},a_{ik}x_{k\,m-1})(\tau)\ \ (i=1,\dots,n;\ m=1,2,\dots).$$

Corollary 2′. *Let (2.7) hold and let there exist* $t_i\in\{-\infty,+\infty\}$ $(i=1,\dots,n)$ *such that the functions* $a_{ii}(t)\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$ *are non-increasing on* $\mathbb{R}$*, conditions (2.15), (2.16), (2.17) and*

$$\liminf_{t\to t_i}a_{ii}(t)=+\infty \ \ for\ \ i\in\{1,\dots,n\} \tag{2.23}$$

hold, where $\zeta_i(t)\equiv\xi_{a_{ii}}\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$*, and the numbers* h_{ik} $(i,k=1,\dots,n)$ *are such that* $\mathcal{H}=\big((1-\delta_{ik})h_{ik}\big)_{i,k=1}^n$*. Then conclusion of Theorem 2′ is true.*

Corollary 3′. *Let there exist the points* $t_i\in\{-\infty,+\infty\}$ $(i=1,\dots,n)$ *and the functions* $\alpha_i:\mathbb{R}\to\mathbb{R}$ $(i=1,\dots,n)$ *such that* $\alpha_i(t)\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$ *are nondecreasing on* $\mathbb{R}$ *and conditions (2.16), (2.18) – (2.22) hold on* $\mathbb{R}$*, where* $\vartheta_i(t)\equiv\xi_{\eta_i a_i}\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$*, and the numbers* η_{ik}*,* $\eta_{ii}<0$ $(i,k=1,\dots,n)$ *are such that* $\mathcal{H}=\big((1-\delta_{ik})\eta_{ik}|\eta_{ii}|^{-1}\big)_{i,k=1}^n$*. Then conclusion of Theorem 2′ is true.*

Corollary 4. *Let the conditions of Theorem 2 or Corollary 2 or Corollary 3 are fulfilled. Let, in addition, condition (2.8) hold. Then for every $c_i \in \mathbb{R}_+$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$ system (1.1) has the unique and bounded on $\mathbb{R}$ solution satisfying condition (2.6) and it is nonnegative.*

Corollary 4′. *Let the conditions of Theorem 2′ or Corollary 2′ or Corollary 3′ are fulfilled. Let, in addition, condition (2.8) hold. Then system (1.1) has the unique and bounded on $\mathbb{R}$ solution and it is nonnegative.*

3. Proof of the results

Proof of Theorem 1. Let $c_i \in \mathbb{R}$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$ be an arbitrary fixed numbers. Consider the initial problems

$$du = u\,da_{ii}(t) + df_i(t) \ \text{ for } t \in \mathbb{R}, \ \ u(t_i) = c_i$$

$(i \in \mathcal{N}_0(t_1,\dots,t_n))$. By (2.1) the problem has the unique solution $u_i \in \mathrm{BV}(\mathbb{R};\mathbb{R})$ and according to modified variation of constant formulae (see [5]) it has form (2.12).

Consider the system of integral equations

$$x_i(t) = u_i(t) + \sum_{k=1,k\neq i}^{n} \int_{t_i}^{t} \gamma_i(t,\tau)\,d\mathcal{A}(a_{ii}, a_{ik}x_k)(\tau) \ \text{ for } t \in \mathbb{R} \ \ (i = 1,\dots,n). \tag{3.1}$$

Due to modified variation of constant formulae (see [5]) we conclude that the vector-function $(x_i)_{i=1}^n$ is a solution of the system one. Moreover, it is evident the vector-function $(x_i)_{i=1}^n$ satisfy condition (2.6).

The solution of the last integral system we will find in the set of functions of bounded and local bounded variation on the set $\mathbb{R}$, i.e. in $\mathrm{BV}(\mathbb{R};\mathbb{R}^n)$.

Consider the sequences of the vector-functions $(x_{im})_{i=1}^n$ $(m = 0,1,\dots)$ defined by

$$x_{i0}(t) = 0, \ \ x_{im}(t) = u_i(t) + \sum_{k=1,k\neq i}^{n} \int_{t_i}^{t} \gamma_i(t,\tau)\,d\mathcal{A}(a_{ii}, a_{ik}x_{k\,m-1})(\tau) \ \text{ for } t \in \mathbb{R} \ \ (i = 1,\dots,n).$$

In view of conditions (2.2) and (2.3), from (2.12) and (2.13) we get

$$(u_i)_{i=1}^n \in \mathrm{BV}(\mathbb{R};\mathbb{R}^n). \tag{3.2}$$

It is clear that $(x_i)_{i=1}^n \in \mathrm{BV}(\mathbb{R};\mathbb{R}^n)$. Now, if we assume that

$$(x_{i\,m-1})_{i=1}^n \in \mathrm{BV}(\mathbb{R};\mathbb{R}^n) \tag{3.3}$$

for some m, then due (2.2) and (3.2), from (3.2) we get $(x_{im})_{i=1}^n \in \mathrm{BV}_{loc}(\mathbb{R};\mathbb{R}^n)$ and

$$\|x_{im}\|_\infty \le \|u_i\|_\infty + \sum_{k=1,k\neq i}^{n} s_{ik}\|x_{k\,m-1}\|_\infty < +\infty \ \ (i = 1,\dots,n).$$

Therefore, condition (3.3) holds for every natural m.

Let us show that the sequence $(x_{im})_{i=1}^{m}$ $(m=1,2.,\dots)$ be uniformly stable on $\mathbb{R}$. For this sufficient to show that the functional series

$$\sum_{m=1}^{\infty} |x_{im}(t)-x_{im-1}(t)| \ \ (i=1,\dots,n) \tag{3.4}$$

are converged uniformly on $\mathbb{R}$.

According (2.2) and (3.3), from (2.11) it follows

$$(\|x_{im}-x_{im-1}\|_\infty)_{i=1}^{n} \le S\,(\|x_{im-1}-x_{im-2}\|_\infty)_{i=1}^{n} \ \ (m=2,3,\dots)$$

and, therefore,

$$(\|x_{im}-x_{im-1}\|_\infty)_{i=1}^{n} \le S^{m-1}\,(\|u_i\|_\infty)_{i=1}^{n} \ \ (m=1,2,\dots).$$

Due to (2.5) there exist numbers $\alpha \in]r(S),1[$ and $\beta>0$ such that

$$\|S^{m-1}\| \le \beta\alpha^{m-1} \ \ (m=1,2,\dots).$$

Therefore,

$$\|x_{im}-x_{im-1}\|_\infty \le \beta_0\alpha^{m} \ \ (i=1,\dots,n;\, m=1,2,\dots),$$

where $\beta_0=\beta\alpha^{-1}\sum_{i=1}^{n}\|u_i\|_\infty$.

So that

$$\sum_{m=0}^{\infty}\beta_0\alpha^{m}$$

is the convergence major numerical series for the functional series (3.4) on $\mathbb{R}$. From this, due to the Weierstrass theorem the sequence $(x_{im})_{i=1}^{n}$ $(m=0,1,\dots)$ is uniformly convergence on $\mathbb{R}$.

Let

$$\lim_{m\to+\infty} x_{im}(t)=x_i(t) \ \text{ for } t\in\mathbb{R} \ \ (i=1,\dots,n). \tag{3.5}$$

Then due to Theorem I.4.17 ([12]) we conclude that $(x_i)_{i=1}^{n}$ will be a solution of system (3.1). Moreover, it is evident that $\|x_i\|_\infty<+\infty$ $(i=1,\dots,n)$ and by the equality (3.1) and estimates (2.2) – (2.4) we have $(x_i)_{i=1}^{n}\in \mathrm{BV}(\mathbb{R};\mathbb{R}^n)$. □

Proof of Corollary 1. As we proof above, in conditions of Theorem 1, system (1.1) has the bounded solution on $\mathbb{R}$ satisfying condition (2.6) and it is obtained as the uniformly limits on $\mathbb{R}$ of the sequence of the vector-functions $(x_{im-1})_{i=1}^{n}\in \mathrm{BV}(\mathbb{R};\mathbb{R}^n)$ $(m=0,1,\dots)$ the components of which are defined by (2.11), and u_i $(i=1,\dots,n)$ are defined by (2.12) and (2.13).

In view of (2.8), because $c_i\in\mathbb{R}_+$ $(i\in\mathcal{N}_0(t_1,\dots,t_n))$, it follows from (2.13) – (2.11) that $x_{im}(t)\ge 0$ and, that, $x_i(t)\ge 0$ for $t\in\mathbb{R}$ $(i=1,\dots,n)$. □

Proof of Theorem 2. First we show that the every bounded on $\mathbb{R}$ solution $(x_i)_{i=1}^{n}$ of system (1.1), satisfying of condition (2.6) will be solution of the system of integral equations (3.1).

By (2.9), there exist a sequences t_{im} $(i=1,\dots,n;\ m=1,2,\dots)$ such that

$$\lim_{m\to+\infty} t_{im}=t_i,\quad \lim_{m\to+\infty}\gamma_i(0,t_{im})=0 \ \ (i=,\dots,n). \tag{3.6}$$

We assume

$$t_{im}=t_i \ \text{ for } \ i\in\mathcal{N}_0(t_1,\dots,t_n) \ \ (m=1,2,\dots). \tag{3.7}$$

By modified variation of constant formula and equalities (2.12), (3.7) we have

$$x_i(t)=u_{im}(t)+\sum_{k=1,k\neq i}^{n}\int_{t_{im}}^{t}\gamma_i(t,\tau)d\mathcal{A}(a_{ii},a_{ik}x_k)(\tau) \ \ (i=1,\dots,n;\ m=1,2,\dots) \tag{3.8}$$

on $\mathbb{R}$, where

$$u_{im}(t)\equiv u_i(t) \ \text{ for } \ i\in\mathcal{N}_0(t_1,\dots,t_n) \ \ (m=1,2,\dots), \tag{3.9}$$

$$u_{im}(t)\equiv x_{im}(t_{im})\gamma_i(t,t_{im})+\int_{t_{im}}^{t}\gamma_i(t,\tau)d\mathcal{A}(a_{ii},f_i)(\tau)$$
$$\text{for } \ i\in\{1,\dots,n\}\setminus\mathcal{N}_0(t_1,\dots,t_n) \ \ (m=1,2,\dots). \tag{3.10}$$

Let $i\in\{1,\dots,n\}\setminus\mathcal{N}_0(t_1,\dots,t_n)$. Then because x_i is bounded, by conditions (2.3) and (3.6) from (2.13) and (3.10) we find

$$\lim_{m\to+\infty}u_{im}(t)=u_i(t) \ \text{ for } \ t\in\mathbb{R}.$$

On the hand, due (2.2) we get

$$\lim_{m\to+\infty}\int_{t_{im}}^{t}\gamma_i(t,\tau)d\mathcal{A}(a_{ii},a_{ik}x_k)(\tau)=\int_{t_i}^{t}\gamma_i(t,\tau)d\mathcal{A}(a_{ii},a_{ik}x_k)(\tau) \ \text{ for } \ t\in\mathbb{R}.$$

Therefore, from (3.8) we have

$$x_i(t)=u_i(t)+\sum_{k=1,k\neq i}^{n}\int_{t_i}^{t}\gamma_i(t,\tau)d\mathcal{A}(a_{ii},a_{ik}x_k)(\tau) \ \text{ for } \ t\in\mathbb{R}.$$

Due to (3.7) – (3.9) the last equality is true for the case where $i\in\mathcal{N}_0(t_1,\dots,t_n)$, as well. So that it is proved that the vector-function $(x_i)_{i=1}^n$ is the solution of system (3.1).

In the proof of Theorem 1 we show that system (3.1) has solution $(x_i)_{i=1}^n$ and

$$\lim_{m\to+\infty}\|x_i-x_{im}\|_\infty=0 \ \ (i=1,\dots,n).$$

In addition,

$$\|x_{im}-x_{im-1}\|_\infty\le\beta_0\alpha^m \ \ (i=1,\dots,n;\ m=1,2,\dots),$$

where β_0 and $\alpha \in]0,1[$ are the numbers independent from m. From this, we get

$$\|x_{im+j} - x_{im}\|_\infty \le \sum_{k=m+1}^{m+j} \|x_{ik} - x_{ik-1}\|_\infty \le \beta_0 \sum_{k=m+1}^{m+j} \alpha^k < \beta_0 \frac{\alpha}{1-\alpha}\alpha^m$$

and

$$\|x_i - x_{im}\|_\infty \le \beta_0 \frac{\alpha}{1-\alpha}\alpha^m \ \ (j = 1,2;\, i = 1,\dots,n;\, m = 1,2,\dots).$$

So that estimate (2.10) holds for $\rho_0 = n\beta_0\alpha(1-\alpha)^{-1}$.

Finally, we show that system (3.1) has the unique solution $(x_i)_{i=1}^n$. Let $(\bar{x}_i)_{i=1}^n \in \mathrm{BV}(\mathbb{R};\mathbb{R}^n)$ and arbitrary solution of the system and let $y_i(t) \equiv \bar{x}_i(t) - x_i(t)$ $(i = 1,\dots,n)$. Then

$$y_i(t) = \sum_{k=1,k\ne i}^{n} \int_{t_i}^{t} \gamma_i(t,\tau)\, d\mathcal{A}(a_{ii}, a_{ik}y_k)(\tau) \ \text{ for } t \in \mathbb{R}.$$

By this, in view of (2.2), $(\|y_i\|_\infty)_{i=1}^n \le S(\|y_i\|_\infty)_{i=1}^n$, i.e., $(I_n - S)(\|y_i\|_\infty)_{i=1}^n \le 0_n$. So that, because the matrix S is nonnegative, by condition (2.5) we have $(\|y_i\|_\infty)_{i=1}^n \le 0_n$ and $\|y_i\|_\infty = 0$ $(i = 1,\dots,n)$. Consequently, $\bar{x}_i(t) \equiv x_i(t)$ $(i = 1,\dots,n)$. □

Proof of Corollary 2. Let $\xi_i(t) \equiv \xi_{a_{ii}}(t)$, $\gamma_i(t,\tau) \equiv \xi_i(t)\xi_i^{-1}(\tau)$ and $\nu_i(t) \equiv \nu(\zeta_i)(t)$ $(i = 1,\dots,n)$. Due condition (2.7) we have $\gamma_i(t,t_i) > 0$ $(i = 1,\dots,n)$.

Let $i \in \{1,\dots,n\}$ be fixed. First, consider the case $t \ge t_i$. Then by (2.15) and equality (1.3) we find $d\xi_i^{-1}(t) \equiv -\xi_i^{-1}(t)\, d\mathcal{A}(a_{ii},a_{ii})(t)$, and

$$\begin{aligned}\left| \int_{t_i}^{t} |\gamma_i(t,\tau)|\, dV\big(\mathcal{A}(a_{ii},a_{ik})\big)(\tau) \right| &\le -h_{ik}\xi_i(t) \int_{t_i}^{t} \xi_i^{-1}(\tau)\, d\mathcal{A}(a_{ii},a_{ii})(\tau) \\ &= h_{ik}\xi_i(t) \int_{t_i}^{t} d\xi_i^{-1}(\tau) \le h_{ik} \ \ (i \ne k;\, i,k = 1,\dots,n). \qquad (3.11)\end{aligned}$$

So that condition (2.2) hold. Beside, we have $s_{ik} \le h_{ik}$ $(i,k = 1,\dots,n)$ and, therefore, by (2.16) condition (2.5) hold.

If $i \in \mathcal{N}_0(t_1,\dots,t_n)$, then due Remark 1 the functions β_i $(i = 1,\dots,n)$ are non-increasing. Hence the functions $\gamma_i(t,t_i) \equiv \gamma_{a_{ii}}(t,t_i)$ $(i = 1,\dots,n)$ are non-increasing and so estimate (2.4) holds for $t \ge t_i$.

Let now $i \in \{1,\dots,n\} \setminus \mathcal{N}_0(t_1,\dots,t_n)$ be such that $t_i = -\infty$. Due to (2.7) we find $\gamma_i(0,t) = \exp\big(1 - \xi_i(t)\big)$ for $t < 0$. Therefore, by (2.23) we conclude

$$\liminf_{t\to t_i} \gamma_i(0,t) = 0.$$

Consequently, condition (2.9) hold for the case.

Let us verify (2.3). Let $i \in \{1,\dots,n\}$, $t_i \in \mathbb{R}\cup\{-\infty,+\infty\}$ and $t > t_i$ be fixed. Then $\zeta_i(\tau) = -\xi_i(\tau)$ for $\tau > t_i$, $\zeta_i(t_i) = 0$. Then due to the conditions of theorem the function ζ_i is nondecreasing on the interval $[t_i,+\infty[$.

$$T_{ij} = \{\tau \in [t_i,t] : j \le \zeta_i(\tau) < (j+1)\} \ \ (j = 0,\dots,k_i(t)+1),$$

where $k_i(t) \equiv \big[\zeta_i(t)\big]$ (the integer part) and let

$$\tau_{i0} = t_i, \ \ \tau_{ij} = \begin{cases} \tau_{i\,j-1} \ \text{ if } \ T_{i\,j-1} = \varnothing, \\ \sup T_{i\,j-1} \ \text{ if } \ T_{i\,j-1} \ne \varnothing \ \ (j = 1,\dots,k_i(t)+1). \end{cases}$$

Let us show that

$$\tau_{i\,j+1} \le \nu_i(\tau_{ij}) \ \ (j = 0,\dots,k_i(t)). \tag{3.12}$$

Let $j \in \{0,\dots,k_i(t)\}$ be fixed. If $T_{ij} = \varnothing$, then (3.12) is evident. Let now $T_{ij} \ne \varnothing$. It suffices to show that

$$T_{ij} \subset Q_{ij}, \tag{3.13}$$

where $Q_{ij} = \big\{\tau \in [t_i,t] : \ \zeta_i(\tau) < \zeta_i(\tau_{i\,j-1}+) + 1\big\}$. It is easy to verify that

$$\zeta_i(\tau_{i\,j-1}+) \ge j. \tag{3.14}$$

Indeed, otherwise there exists $\delta > 0$ such that $\zeta_i(\tau_{i\,j-1}+s) < \zeta_i(\tau_{i0}) + j$ for $0 \le s \le \delta$. Next, by the definition of $\tau_{i\,j-1}$, we have $\zeta_i(\tau_{i0}) + (j-1) \le \zeta_i(\tau_{i\,j-1}-)$ and, therefore, $(j-1) \le \zeta_i(\tau_{i\,j-1}+s) < j$ for $0 \le s \le \delta$. But this contradicts the definition of $\tau_{i\,j-1}$. So that if $\tau \in T_{ij}$, then from (3.14) and the inequality $\zeta_i(\tau) < (j+1)$ we get $\zeta_i(\tau) < \zeta_i(\tau_{i\,j-1}+) + 1$ and hence $\tau \in Q_{ij}$. Therefore (3.12) is proved.

Due to (3.12) we find

$$v_i(t) \le \int_{t_i}^{t} \exp\big(\xi_i(t) - \xi_i(\tau)\big)\, db_i(\tau) \le \exp(\xi_i(t)) \sum_{j=1}^{k_i(t)+1} \int_{\tau_{i\,j-1}}^{\tau_{ij}} \exp(\zeta_i(\tau))\, db_i(\tau), \tag{3.15}$$

where $v_i(t) \equiv \left| \int_{t_i}^{t} |\gamma_i(t,\tau)|\, db_i(\tau) \right|$ and $b_i(\tau) \equiv V(\mathcal{A}(a_{ii},f_i))(\tau)$. On the other hand

$$\begin{aligned} \int_{\tau_{i\,j-1}}^{\tau_{ij}} \exp(\zeta_i(\tau))\, db_i(\tau) &= \lim_{\varepsilon\to 0+} \int_{\tau_{i\,j-1}}^{\tau_{ij}-\varepsilon} \exp(\zeta_i(\tau))\, db_i(\tau) + \exp(\zeta_i(\tau_{ij}))d_1 b_i(\tau_{ij}) \\ &\le \exp(\zeta_i(\tau_{ij}-))\big(b_i(\tau_{ij}-) - b_i(\tau_{i\,j-1})\big) + \exp(\zeta_i(\tau_{i\,j+1}-))d_1 b_i(\tau_{ij}) \\ &\le \exp(j)\big(b_i(\tau_{ij}-) - b_i(\tau_{i\,j-1})\big) + \exp(j+1)d_1 b_i(\tau_{ij}). \end{aligned}$$

Similarly we verify that

$$\begin{aligned} \int_{\tau_{ik_i(t)}}^{t} \exp(\zeta_i(\tau))\, db_i(\tau) &\le \exp(\zeta_i(t-))\big(b_i(t-) - b_i(\tau_{ik_i(t)})\big) + \exp(\zeta_i(t))d_1 b_i(t) \\ &\le \exp(k_i(t)+1)\big(b_i(t-) - b_i(\tau_{ik_i(t)})\big) + \exp(k_i(t)+2)d_1 b_i(t). \end{aligned}$$

Taking account the last two estimates, by (3.15) we find

$$v_i(t) \leq \exp(-k_i(t)) \sum_{j=1}^{k_i(t)+1} \Big(\exp(j)\big(b_i(\tau_{ij}-) - b_i(\tau_{i\,j-1})\big) + \exp(j+1)d_1 b_i(\tau_{ij})\Big)$$

$$\leq \exp(-k_i(t)) \Bigg(\exp(1)\big(b_i(\tau_{i1}-) - b_i(\tau_{i0})\big)$$

$$+ \sum_{j=2}^{k_i(t)+1} (\exp(j)\big(b_i(\tau_{ij}-) - b_i(\tau_{i\,j-1}-)\big) + \exp(k_i(t)+2)d_1 b_i(\tau_{i\,k_l(t)+2})\Bigg).$$

From this, due to (2.17) and (3.12) we have

$$v_i(t) \leq 2\rho_i \exp(-k_i(t)) \sum_{j=1}^{k_i(t)+2} \exp(j)$$

$$= 2\rho_i \exp(-k_i(t)) \exp(k_i(t)+2) \exp(1)(\exp(1)-1)^{-1}.$$

Consequently,

$$v_i(t) \leq \eta\rho \ \text{ for } t \geq t_i. \tag{3.16}$$

where $\eta = 2\exp(3)(\exp(1)-1)^{-1}$ and $\rho = \sum_{i=1}^{n} \rho_i$. So that, estimate (2.3) hold on the set $(-\infty, t]$. Similarly we show estimate (2.3) on the set $[t, +\infty)$. In the case we have $\zeta_i(\tau) = -\xi_i(\tau)$ for $\tau < t_i$, $\zeta_i(t_i) = 0$. and

$$\widetilde{T}_{ij} = \{\tau \in [t, t_i] : j \leq \zeta_i(\tau) < (j+1)\} \ \ (j = 0, \dots, k_i(t)+1).$$

The function ζ_i is non-increasing on $(-\infty, t]$.

Similarly, as above, we conclude that the estimates

$$\widetilde{\tau}_{i\,j+1} \geq \nu_i(\widetilde{\tau}_{ij}) \ \ (j = 0, \dots, k_i(t)). \tag{3.17}$$

holds, where

$$\widetilde{\tau}_{i0} = t_i, \ \ \tau_{ij} = \begin{cases} \widetilde{\tau}_{i\,j-1} \ \text{ if } \ \widetilde{T}_{i\,j-1} = \varnothing, \\ \inf \widetilde{T}_{i\,j-1} \ \text{ if } \ \widetilde{T}_{i\,j-1} \neq \varnothing \ (j = 1, \dots, k_i(t)+1). \end{cases}$$

Similarly, we verify that (3.16) hold. So the corollary follows from Theorem 2. □

Proof of Corollary 3. First, as in the proof of Corollary 2, due to condition (2.17) we show that estimates

$$\sup\left\{\left|\int_{t_i}^{t} |\gamma_{\eta_i\alpha_{ii}}(t,\tau)| dV\big(\mathcal{A}(\eta_i\alpha_{ii}, f_i)\big)(\tau)\right| : t \in \mathbb{R}\right\} < +\infty \ \ (i = 1, \dots, n). \tag{3.18}$$

Let $\xi_i(t) \equiv \xi_{a_{ii}}(t)$ and $\gamma_i(t,\tau) \equiv \xi_i(t)\xi_i^{-1}(\tau)$ $(i = 1, \dots, n)$. Due condition (2.19) we have $\gamma_i(t, t_i) > 0$ $(i = 1, \dots, n)$.

In view of (2.18) and (2.19) it is not difficult to verify that

$$J(a_{ii})(t) - J(a_{ii})(s) \leq \operatorname{sgn}(t-s)\big(J(\eta_{ii}\alpha_i)(t) - J(\eta_{ii}\alpha_i)(s)\big) \tag{3.19}$$

for $(t-s)(s-t_i)>0$ $(i=1,\dots,n)$.

Let us show the estimates

$$V\big(\mathcal{A}(a_{ii},a_{ik})\big)(t)\le\frac{\eta_{ik}}{\eta_{ii}}\operatorname{sgn}(t-t_i)\mathcal{A}(\eta_{ii}\alpha_i,\eta_{ii}\alpha_i)(t)\quad(i\ne k;\ i,k=1,\dots,n).\tag{3.20}$$

hold on $\mathbb{R}$. By definition of the operator $\mathcal{A}$ we have

$$\begin{aligned}\mathcal{A}(a_{ii},a_{ik})(t)-\mathcal{A}(a_{ii},a_{ik})(s)&=s_c(a_{ik})(t)-s_c(a_{ik})(s)\\&+\sum_{s<\tau\le t}d_1a_{ik}(\tau)+\sum_{s\le\tau<t}d_2a_{ik}(\tau)+\sum_{s<\tau\le t}d_1a_{ii}(\tau)(1-d_1a_{ii}(\tau))^{-1}d_1a_{ik}(\tau)\\&-\sum_{s\le\tau<t}d_2a_{ii}(\tau)(1+d_2a_{ii}(\tau))^{-1}d_2a_{ik}(\tau)=s_c(a_{ik})(t)-s_c(a_{ik})(s)\\&+\sum_{s<\tau\le t}(1-d_1a_{ii}(\tau))^{-1}d_1a_{ik}(\tau)+\sum_{s\le\tau<t}(1+d_2a_{ii}(\tau))^{-1}d_2a_{ik}(\tau)\end{aligned}$$

for $s<t$ $(i\ne k;\ k=1,\dots,n)$. From this, thanks to (2.19), (2.20), (2.21), we find

$$\begin{aligned}|\mathcal{A}(a_{ii},a_{ik})(t)-\mathcal{A}(a_{ii},a_{ik})(s)|&\le|s_c(a_{ik})(t)-s_c(a_{ik})(s)|\\&+\sum_{s<\tau\le t}(1-d_1a_{ii}(\tau))^{-1}|d_1a_{ik}(\tau)|+\sum_{s\le\tau<t}(1+d_2a_{ii}(\tau))^{-1}|d_2a_{ik}(\tau)|\\&\le\eta_{ik}(s_c(\alpha_i)(t)-s_c(\alpha_i)(s))+\sum_{s<\tau\le t}(1-\eta_{ii}d_1\alpha_i(\tau))^{-1}\eta_{ik}d_1\alpha_i(\tau)\\&+\sum_{s\le\tau<t}(1+\eta_{ii}d_2\alpha_i(\tau))^{-1}\eta_{ik}d_2\alpha_i(\tau)\\&=\frac{\eta_{ik}}{\eta_{ii}}(\mathcal{A}(\eta_{ii}\alpha_i,\eta_{ii}\alpha_i)(t)-\mathcal{A}(\eta_{ii}\alpha_i,\eta_{ii}\alpha_i)(s))\end{aligned}$$

for $t_i\le s<t$ $(i\ne k;\ k=1,\dots,n)$. So that estimate (3.20) hold for $t_i\le t$. Similarly we show (3.20) for $t\le t_i$ as well.

Moreover, using this way we conclude that for $t\in\mathbb{R}$ we have

$$V\big(\mathcal{A}(a_{ii},f_i)\big)(t)\le\operatorname{sgn}(t-t_i)\mathcal{A}(\eta_{ii}\alpha_i,f_i)(t)\quad(i\ne k;\ i,k=1,\dots,n).\tag{3.21}$$

Hence due to (3.18), (3.19) and (3.21) we conclude condition (2.3) and (2.4) hold.

On the other hand, by (1.3), (3.19) and (3.20) we get

$$d\exp\big(-J(\eta_{ii}\alpha_i)(t)\big)\equiv-\exp\big(-J(\eta_{ii}\alpha_i)(t)\big)d\mathcal{A}(\eta_{ii}\alpha_i,h_{ii}\alpha_i)(t)$$

and

$$\begin{aligned}&\left|\int_{t_i}^{t}|\gamma_i(t,\tau)|dV\big(\mathcal{A}(a_{ii},a_{ik})\big)(\tau)\right|\\&\qquad\le\frac{\eta_{ik}}{\eta_{ii}}\exp\big(J(\eta_{ii}\alpha_i)(t)\big)\int_{t_i}^{t}\exp\big(-J(\eta_{ii}\alpha_i)(\tau)\big)d\mathcal{A}(\eta_{ii}\alpha_i,\eta_{ii}\alpha_i)(\tau)\end{aligned}$$

$$= \frac{\eta_{ik}}{|\eta_{ii}|} \exp\big(J(\eta_{ii}\alpha_i)(t)\big)\Big(\exp\big(-J(\eta_{ii}\alpha_i)(t)\big) - \exp\big(-J(\eta_{ii}\alpha_i)(t_i)\big)\Big)$$
$$\leq \frac{\eta_{ik}}{|\eta_{ii}|} \ \text{ for } t \geq t_i \ \ (i \neq k;\, k = 1,\dots,n).$$

Similarly we show the estimate for $t \leq t_i$ as well. So that we have $s_{ik} \leq \eta_{ik}|\eta_{ii}|^{-1}$ $(i,k = 1,\dots,n)$, where s_{ik} is the left hand of estimate (2.2).

Hence, inequality (2.5) follows from (2.16). By (2.21) condition (2.1) holds.

So conditions of Theorem 1 hold. The corollary follows from the theorem. □

Theorem 2′, Corollaries 2′ and 3′ are, respectively, the particular cases of Theorem 2, Corollaries 2 and 3 if we assume $\mathcal{N}_0(t_1,\dots,t_n) = \varnothing$ therein.

Corollaries 4 and 4′ immediately follow from Theorems 2 and 2′ and Corollaries 2, 3 and 2′, 3′.

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