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A NOTE ON *S*-CURVATURE OF RANDERS MEASURE SPACES

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Abstract. In this paper, we give some characterizations on S-curvature in Randers measure spaces, which shows that only the Busemann-Hausdorff volume form can admit a vanishing S-curvature.

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1. Introduction

In Finsler geometry, the S-curvature, which was introduced by Shen [4] , plays a very important role in the classification of Finsler metrics and the global geometric analysis.

Let $(M, F, d\mu)$ be a Finsler measure space, where $d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$ is an arbitrary volume form. Then the S-curvature is defined by

$$S(x,y) := \frac{d}{dt}[\tau(c(t), \dot{c}(t))]|_{t=0},$$

where $c(t)$ is the geodesic with $c(0) = x$ and $\dot{c}(0) = y$, and τ is the distortion, which is defined by

$$\tau(x,y) := \log \frac{\sqrt{\det(g_{ij}(x,y))}}{\sigma(x)},$$

where $g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F(x,y)^2}{\partial y^i \partial y^j}$. Clearly, the S-curvature demonstrates the rate of change of the distortion along geodesics. In local coordinates, the S-curvature can be written as

$$S(x,y) = \frac{\partial G^i}{\partial y^i} - y^i \frac{\partial}{\partial x^i} \log \sigma(x), \tag{1.1}$$

where G^i is called the spray coefficients of F.

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Let us take a look at the special case: when F is a Riemannian metric and

$$d\mu = e^{-f(x)}dV, \quad \text{where} \quad dV = \sqrt{\det(g_{ij}(x))}dx^1 \wedge \cdots \wedge dx^n,$$

the S-curvature is

$$S(x,y) = \frac{\partial f(x)}{\partial x^i} y^i, \tag{1.2}$$

where f is called the potential function.

Using Ricci curvature, S-curvature and its derivative $\dot{S}$ along geodesics, one can define the Bakry-Emery Ricci curvature (the so called weighted Ricci curvature in the Finsler setting).

In many literatures, S-curvature is defined with respect to the Busemann-Hausdorff volume form. See [2] [1] [4] [5] and the references therein. Now we give the corresponding definitions by using an arbitrary volume form as follows.

Definition 1. Let $(M,F,d\mu)$ be a Finsler measure space, where $d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$ is an arbitrary volume form.

(1) F is of weakly isotropic S-curvature if the S-curvature

$$S = (n+1)cF + \eta,$$

where $c = c(x)$ is a scalar function and $\eta = \eta_i(x)y^i$ is a 1-form on M.

(2) F is of almost isotropic S-curvature if $d\eta = 0$.

(3) F is of isotropic S-curvature if $\eta = 0$.

(4) F is of constant S-curvature if c is a constant and $\eta = 0$.

(5) F is of exact S-curvature if $c = 0$ and $\eta = d\varphi$ for some smooth function on M.

Notice that in Riemannian manifold (M,g,dV), the S-curvature is zero, while for the volume form $d\mu = e^{-f(x)}dV$, M has exact S-curvature (see (1.2)). In the Finsler situation, F is of exact S-curvature if and only if there is a volume form $d\mu$ such that $S_{d\mu} = 0$.

In the following, we focus on the Randers case. Let $F = \alpha + \beta$ be a solution of Zermelo's navigation problem, which is expressed by

$$F = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}, \tag{1.3}$$

where $h := \sqrt{h_{ij}(x)y^i y^j}$ is a Riemannian metric, $W_0 := W_i(x)y^i$ and $\lambda := 1 - \|W_x\|_h^2$.

It is shown that, for the Busemann-Hausdorff volume form, F has vanishing S-curvature if and only if W is a Killing vector field with respect to h (see [5]).

A natural problem is to determine all volume forms such that S-curvature is zero.

In this short note, we study this problem and obtain the following main result.

Theorem 1. *Only (constant multiplications of) the Busemann-Hausdorff volume form can satisfy $S = 0$ on Randers spaces.*

Remark 1. In [3], Ohta has discussed this problem and obtained that, a Randers space $(M, \alpha + \beta)$ admits a measure m with $S = 0$ if and only if β is a Killing form of constant length. However, the "only if" part of Theorem 1.1 in [3] is not necessarily true.

2. The proof of the main result

We first recall some fundamental facts in Randers geometry. For more details, refer to [2]. Let F be a Randers metric in terms of (h, W) as (1.3). Let $d\mu_{BH} = \sigma_{BH}(x)dx^1 \wedge \cdots \wedge dx^n$ denote the Busemann-Hausdorff volume form. Then

$$\sigma_{BH}(x) = \sigma_h(x),$$

where $dV_h = \sigma_h(x)dx^1 \wedge \cdots \wedge dx^n$ is the volume form of h.

From (p.35-36, [2]), we see

$$\begin{aligned} S_{BH} =& \frac{\partial G^i}{\partial y^i} - y^i \frac{\partial}{\partial x^i} \log \sigma_{BH}(x) \\ =& \frac{\partial \mathcal{G}^i}{\partial y^i} + \frac{n+1}{2F}(2F\mathcal{R}_0 - \mathcal{R}_{00} - F^2\mathcal{R}) - y^i \frac{\partial}{\partial x^i} \log \sigma_h(x) \\ =& \frac{n+1}{2F}(2F\mathcal{R}_0 - \mathcal{R}_{00} - F^2\mathcal{R}), \end{aligned}$$

where $\mathcal{G}^i$ is the spray coefficients of h, and

$$\begin{aligned} \mathcal{R}_{ij} :=& \frac{1}{2}(W_{i;j} + W_{j;i}), \quad \mathcal{R}_{00} := \mathcal{R}_{ij} y^i y^j, \\ \mathcal{R}_i :=& \mathcal{R}_{ij} W^j, \quad \mathcal{R}_0 := \mathcal{R}_i y^i, \quad \mathcal{R} := \mathcal{R}_i W^i, \end{aligned}$$

where $W^i := h^{ij} W_j$, and ";" denotes the covariant differentiation with respect to h.

To prove the main result, we give two important lemmas in the following.

Lemma 1. *[1] Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M with the Busemann-Hausdorff volume form. For any $c = c(x)$ on M, the following are equivalent:*

(1) $S_{BH} = (n+1)cF$;
(2) $S_{BH} = (n+1)cF + \eta$, *where η is a 1-form.*

Lemma 2. *[5] Let $F = \alpha + \beta$ be a Randers metric on a manifold M, which is expressed in terms of a Riemannian metric h and a vector field W by (1.3). Then F is of isotropic S-curvature with respect to the Busemann-Hausdorff volume form, $S_{BH} = (n+1)cF$, if and only if W satisfies*

$$\mathcal{R}_{00} = -2ch^2.$$

Theorem 2. *Let $(M,F,d\mu)$ be a Randers measure space, where $d\mu=\sigma(x)dx^1\wedge\cdots\wedge dx^n$ is an arbitrary volume form, and F is expressed in terms of a Riemannian metric h and a vector field W by (1.3). Then F is of weakly isotropic S-curvature with respect to the volume form $d\mu$ if and only if W satisfies*

$$\mathcal{R}_{00}=-2ch^2.$$

Proof. Suppose that $\mathcal{R}_{00}=-2ch^2$. Then by Lemma 2, we have

$$S_{BH}=(n+1)cF. \tag{2.1}$$

Let $d\mu_{BH}=\sigma_{BH}(x)dx^1\wedge\cdots\wedge dx^n$ be the Busemann-Hausdorff volume form. Then there is a positive C^∞ function φ on M such that

$$\sigma(x)=\varphi(x)\sigma_{BH}(x).$$

Thus, from (1.1), we have

$$\begin{aligned} S&=\frac{\partial G^i}{\partial y^i}-y^i\frac{\partial}{\partial x^i}\log\sigma(x)\\ &=\frac{\partial G^i}{\partial y^i}-y^i\frac{\partial}{\partial x^i}\log\sigma_{BH}(x)-y^i\frac{\partial}{\partial x^i}\log\varphi\\ &=S_{BH}-y^i\frac{\partial}{\partial x^i}\log\varphi. \end{aligned} \tag{2.2}$$

Combing (2.1) with (2.2), it follows that S is weakly isotropic.

Conversely, if S is weakly isotropic, we then can write

$$S=(n+1)cF+\eta, \tag{2.3}$$

where η is a 1-form. Then by (2.2), we obtain

$$S_{BH}=(n+1)cF+\eta+y^i\frac{\partial}{\partial x^i}\log\varphi. \tag{2.4}$$

It is obviously to see from (2.4) that S_{BH} is weakly isotropic. By using Lemma 1, it is equivalent to

$$S_{BH}=(n+1)cF. \tag{2.5}$$

Therefore, it holds from Lemma 2 that $\mathcal{R}_{00}=-2ch^2$. □

Corollary 1. *Let $(M,F,d\mu)$ be a Randers measure space, where $d\mu=\sigma(x)dx^1\wedge\cdots\wedge dx^n$ is an arbitrary volume form, and F is expressed in terms of a Riemannian metric h and a vector field W by (1.3). If F is of weakly isotropic S-curvature with respect to volume form $d\mu$, then $d\mu$ is the Busemann-Hausdorff volume form up to a constant multiplication.*

Proof. If F is of weakly isotropic S-curvature with respect to an arbitrary volume form, then from Theorem 2, we have

$$\mathcal{R}_{00}=-2ch^2.$$

Thus, by Lemma 2, we have (2.5). On the other hand, we also have (2.4) from (2.2) and (2.3). Therefore, combing (2.4) with (2.5), we deduce

$$\eta + y^i \frac{\partial}{\partial x^i} \log \varphi = 0,$$

for any η and any y. By the arbitrariness of η and y, we have

$$\frac{\partial \varphi}{\partial x^i} = 0, \quad \forall i,$$

which yields φ is a constant. This ends the proof. □

Proof of Theorem 1. It follows from Corollary 1 directly. □

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