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WEIGHTED HERMITE-HADAMARD TYPE INEQUALITIES FOR DIFFERENTIABLE PREINVEX AND PREQUASIINVEX MAPPINGS

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Abstract. In this paper, new weighted Hermite-Hadamard type inequalities for differentiable preinvex and prequasiinvex functions are proved. These results generalize many results proved in earlier works for these classes of functions.

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1. INTRODUCTION

The theory of convex functions depends on the definition of convex functions as stated below:

A function $\varphi : \varnothing \neq I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on a convex set I if inequality

$$\varphi(\eta s + (1-\eta)y) \leq \eta\varphi(s) + (1-\eta)\varphi(y)$$

holds for all s, $y \in I$ and $\eta \in [0,1]$.

Now, we evoke that the concept of quasi-convex functions generalizes the concept of convex functions. More accurately, a function $\varphi : \varnothing \neq I \subseteq \mathbb{R} \to \mathbb{R}$ is said quasi-convex on I if

$$\varphi(\eta s + (1-\eta)y) \leq \max\{\varphi(s), \varphi(y)\}$$

for all $s, y \in I$ and $\eta \in [0,1]$. Evidently, any convex function is a quasi-convex function.

Many researchers have developed a substantial material by using the convex and quasi-convex functions in the past three decades but the most profoundly studied inequalities for convex functions is the Hermite-Hadamard inequality.

The Hermite-Hadamard states that a function $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex on $[z,u]$, where $z, u \in I$ with $z < u$ if anonly if the inequalities (see [6])

$$\varphi\left(\frac{z+u}{2}\right) \leq \frac{1}{u-z}\int_z^u \varphi(s)ds \leq \frac{\varphi(z)+\varphi(u)}{2} \tag{1.1}$$

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hold. The inequalities in (1.1) are reversed if φ is a concave function.

The subject of Hermite-Hadamard type inequalities has been in the focus of young and experienced scholars world-wide working in this domain. A gargantuan study has also been accomplished by a number of mathematicians all over the world.

The notions convex sets and convex functions have been continuously generalized and extended in diverse directions. As a result of unceasing advances in convex sets and convex functions, many new proofs, stimulating extensions, noticeable generalizations, refinements, new Hermite-Hadamard-type inequalities and plentiful applications of the inequalities (1.1) have appeared in the literature of mathematical inequalities and in different branches of mathematics, see for instance [2–5, 7–13, 15, 16, 16–18] and the references quoted in these papers.

Let us now recall the important generalizations of the convex sets, convex functions and quasi-convex functions which are acknowledged as invex sets, preinvex functions and prequasiinvex functions respectively.

Definition 1.1. *[19] Let K be a non-empty subset in $\mathbb{R}^n$ and $\beta : K \times K \to \mathbb{R}^n$. Let $s \in K$, then the set K is said to be invex at s with respect to $\beta(\cdot,\cdot)$, if*

$$s + \eta\beta(y,s) \in K, \forall s, y \in K, \eta \in [0,1].$$

The set K is said to be an invex set with respect to β if K is invex at each $s \in K$. The invex set K is also called an β-connected set.

Definition 1.2. *[19] A function $\varphi : K \to \mathbb{R}$ on an invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to β, if*

$$\varphi(s + \eta\beta(y,s)) \le (1-\eta)\varphi(s) + \eta\varphi(y), \forall s, y \in K, \eta \in [0,1].$$

The function φ is said to be preincave if and only if $-\varphi$ is preinvex.

Every convex function is preinvex with respect to the map $\beta(s,y) = s - y$ but the converse is not true see for instance [1].

Definition 1.3. *[20] A function $\varphi : K \to \mathbb{R}$ on an invex set $K \subseteq \mathbb{R}^n$ is said to be prequasiinvex with respect to β, if*

$$\varphi(s + \eta\beta(y,s)) \le \max\{\varphi(s), \varphi(y)\}, \forall s, y \in K, \eta \in [0,1].$$

Quasi-convex functions are prequasiinvex with respect to the map $\beta(y,s) = y - s$ but the converse does not hold, see for example [20].

Hermite-Hadamard type inequalities for the preinvex functions were initially proved by Noor [15]. Later on, many results on inequalities for preinvex and prequasiinvex functions which are connected to Hermite-Hadamard type inequalities proved in [15] are established, see for instance the articles [2, 9–13, 17, 18] and the references mentioned in them.

In section 2, we prove some integral identities involving a differentiable mapping and a symmetric function with respect to $z + \frac{1}{2}\beta(u,z)$ defined over an interval

$[z, z+\beta(u,z)]$. In section 2, an important inequality for positive linear functional on $C([z, z+\beta(u,z)])$ and a preinvex function are proved in order to obtain some very fascinating results of this manuscript. Section 3 contains some new weighted Hermite-Hadamard type integral inequalities related with the left and right parts of Hermite-Hadamard type inequalities for the preinvex given in [15]. The results of section 3 provide weighted generalization of the results proved so far in the field of mathematical inequalities for differentiable preinvex and prequasiinvex functions.

2. Some Auxiliary Results

Let $w(s) : [z, z+\beta(u,z)] \to [0, \infty)$ be a continuous function such that

$$\int_z^{z+\beta(u,z)} w(s)\, ds = 1.$$

Let us denote the integral $\int_z^{z+\beta(u,z)} sw(s)$ by z_1, i.e.

$$z_1 = \int_z^{z+\beta(u,z)} sw(s)\, ds.$$

Note that $w(s) : [z, z+\beta(u,z)] \to [0, \infty)$ is symmetric with respect to $z + \frac{1}{2}\beta(u,z)$ if the equality holds

$$w(2z + \beta(u,z) - s) = w(s).$$

Now we present a result in which the function $w(s)$ is symmetric with respect to $z + \frac{1}{2}\beta(u,z)$.

Lemma 2.1. *If $w(s) : [z, z+\beta(u,z)] \to [0, \infty)$ is symmetric with respect to $z + \frac{1}{2}\beta(u,z)$. Then*

$$z_1 = z + \frac{1}{2}\beta(u,z).$$

Proof. Since w is symmetric with respect to $z + \frac{1}{2}\beta(u,z)$, we have

$$\begin{aligned}\int_z^{z+\beta(u,z)} sw(s)\, ds &= \int_{z+\beta(u,z)}^{z} (2z+\beta(u,z)-s)\, w(2z+\beta(u,z)-s)\,(-ds) \\ &= \int_z^{z+\beta(u,z)} (2z+\beta(u,z)-s)\, w(s)\, ds.\end{aligned}$$

That is

$$2\int_z^{z+\beta(u,z)} sw(s)\, ds = (2z+\beta(u,z)) \int_z^{z+\beta(u,z)} w(s)\, ds$$

which gives the desired result since

$$\int_z^{z+\beta(u,z)} w(s)\, ds = 1.$$

□

Lemma 2.2. *Let* $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I° *and* $\varphi' \in L([z, z+\beta(u,z)])$*, where* $[z, z+\beta(u,z)] \subseteq I^{\circ}$ *with* $\beta(u,z) > 0$*. Let* $w : [z, z+\beta(u,z)] \to [0, \infty)$ *be a continuous mapping. Then*

$$\begin{aligned}\frac{1}{\beta(u,z)}&\left(\varphi(z)\int_z^{z+\beta(u,z)}(z-s+\beta(u,z))\,w(s)\,ds\right.\\&\left.\quad+\varphi(z+\beta(u,z))\int_z^{z+\beta(u,z)}(s-z)\,w(s)\,ds\right)-\int_z^{z+\beta(u,z)}\varphi(s)\,w(s)\,ds\\&=\frac{\beta(z_1,z)}{\beta(u,z)}\int_0^1 H(w,z,z_1;y)\,\varphi'(z+y\beta(z_1,z))\,dy\\&\quad+\frac{\beta(u,z_1)}{\beta(u,z)}\int_0^1 H(w,z_1,u;y)\,\varphi'(z_1+y\beta(u,z_1))\,dy,\end{aligned}\tag{2.1}$$

where

$$G(w,\alpha,\beta;y)=\int_z^{\alpha+y\beta(\beta,\alpha)}(s-z)\,w(s)\,ds-\int_{\alpha+y\beta(\beta,\alpha)}^{z+y\beta(u,z)}(z+\beta(u,z)-s)\,w(s)\,ds$$

and $\alpha, \beta \in [z, z+\beta(u,z)]$.

Proof. The following identities hold

$$\varphi(s)-\varphi(z)=\int_z^{z+\beta(u,z)}\sigma(s-\eta)\,\varphi'(\eta)\,d\eta,\tag{2.2}$$

$$\varphi(s)-\varphi(z+\beta(u,z))=-\int_z^{z+\beta(u,z)}\sigma(\eta-s)\,\varphi'(\eta)\,d\eta,\tag{2.3}$$

where $\sigma(\cdot)$ is the Heavyside function defined by

$$\sigma(s)=\begin{cases}0, & s<0\\ 1 & s>0.\end{cases}$$

From (2.2), we have

$$\begin{aligned}\int_z^{z+\beta(u,z)}&(z-s+\beta(u,z))\,w(s)\,\varphi(s)\,ds-\varphi(z)\int_z^{z+\beta(u,z)}(z-s+\beta(u,z))\,w(s)\,ds\\&=\int_z^{(z+\beta(u,z))}\left(\int_\eta^{(z+\beta(u,z))}(z-s+\beta(u,z))\,w(s)\,ds\right)\varphi'(\eta)\,d\eta.\end{aligned}\tag{2.4}$$

Similarly, we also have

$$\begin{aligned}\int_z^{z+\beta(u,z)}&(s-z)\,w(s)\,\varphi(s)\,ds-\varphi(u)\int_z^{z+\beta(u,z)}(s-z)\,w(s)\,ds\\&=-\int_z^{z+\beta(u,z)}\left(\int_z^{\eta}(s-z)\,w(s)\,ds\right)\varphi'(\eta)\,d\eta.\end{aligned}\tag{2.5}$$

From (2.4) and (2.5), we get

$$
\begin{aligned}
&\frac{1}{\beta(u,z)}\Bigg(\varphi(z)\int_z^{z+\beta(u,z)}(z-s+\beta(u,z))\,w(s)\,ds \\
&\qquad +\varphi(z+\beta(u,z))\int_z^{z+\beta(u,z)}(s-z)\,w(s)\,ds\Bigg)-\int_z^{z+\beta(u,z)}\varphi(s)\,w(s)\,ds \\
&\quad=\frac{1}{\beta(u,z)}\int_z^{z+\beta(u,z)}\left[\int_z^{\eta}(s-z)\,w(s)\,ds-\int_{\eta}^{z+\beta(u,z)}(z-s+\beta(u,z))\,w(s)\,ds\right] \\
&\qquad\times\varphi'(\eta)\,d\eta \\
&\quad=\frac{1}{\beta(u,z)}\int_z^{z_1}\left[\int_z^{\eta}(s-z)\,w(s)\,ds-\int_{\eta}^{z+\beta(u,z)}(z-s+\beta(u,z))\,w(s)\,ds\right] \\
&\qquad\times\varphi'(\eta)\,d\eta+\frac{1}{\beta(u,z)}\int_{z_1}^{z+\beta(u,z)}\left[\int_z^{\eta}(s-z)\,w(s)\,ds\right. \\
&\qquad\left.-\int_{\eta}^{z+\beta(u,z)}(z-s+\beta(u,z))\,w(s)\right]\varphi'(\eta)\,d\eta. \qquad (2.6)
\end{aligned}
$$

In the last identity, we set $\eta=z+y\beta(z_1,z)$ for the first integral and $\eta=z_1+y\beta(u,z_1)$ by for the second integral and we obtain (2.1). □

Remark 2.3. *If we take $w(s)=\frac{1}{\beta(u,z)}$, for all $s\in[z,z+\beta(u,z)]$, then (2.1) reduces to the identity*

$$
\begin{aligned}
&\frac{\varphi(z)+\varphi(z+\beta(u,z))}{2}-\frac{1}{\beta(u,z)}\int_z^{z+\beta(u,z)}\varphi(s)\,ds=\frac{\beta(u,z)}{4} \\
&\quad\times\left[\int_0^1 y\varphi'\left(z+\frac{1+y}{2}\beta(u,z)\right)dy-\int_0^1 y\varphi'\left(z+\frac{1-y}{2}\beta(u,z)\right)dy\right]. \qquad (2.7)
\end{aligned}
$$

Corollary 2.4. *If the function $w(s)$ is symmetric with respect to $z+\frac{1}{2}\beta(u,z)$ on $[z,z+\beta(u,z)]$, then*

$$
\begin{aligned}
&\frac{\varphi(z)+\varphi(z+\beta(u,z))}{2}-\int_z^{z+\beta(u,z)}\varphi(s)\,w(s)\,ds \\
&\quad=\int_0^1 G(w,z,u;y)\,\varphi'(z+(1-y)\,\beta(u,z))\,dy, \qquad (2.8)
\end{aligned}
$$

where

$$
G(w,z,u;y)=\int_z^{z+\frac{1+y}{2}\beta(u,z)}(s-z)\,w(s)\,ds-\int_{z+\frac{1+y}{2}\beta(u,z)}^{z+\beta(u,z)}(z-s+\beta(u,z))\,w(s)\,ds
$$

for all $y\in[0,1]$.

Proof. Since the function $w(s)$ is symmetric with respect to $z+\frac{1}{2}\beta(u,z)$ on $[z,z+\beta(u,z)]$, we have

$$\int_z^{z+\beta(u,z)} w(s)\,ds = 1 \text{ and } \int_z^{z+\beta(u,z)} sw(s)\,ds = z+\frac{1}{2}\beta(u,z).$$

Moreover

$$G(w,z,z_1;y) = G(w,z_1,u;y) = G(w,z,u;y) = \int_z^{z+\frac{1+y}{2}\beta(u,z)} (s-z)\,w(s)\,ds - \int_{z+\frac{1+y}{2}\beta(u,z)}^{z+\beta(u,z)} (z-s+\beta(u,z))\,w(s)\,ds.$$

Hence from (2.1), we get the required identity (2.8). □

In the next result we need the following fact for the function β proved in [14].

Condition C. Let $\beta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$. The function β is said to satisfy Condition C if for any $s,y\in\mathbb{R}^n$ and $\eta\in[0,1]$, the following equalities hold

$$\beta(s,s+\eta\beta(y,s)) = -\eta\beta(y,s)$$

and

$$\beta(y,s+\eta\beta(y,s)) = (1-\eta)\,\beta(y,s).$$

Lemma 2.5. *Let $A: C([z,z+\beta(u,z)])\to\mathbb{R}$ be a positive linear functional on $C([z,z+\beta(u,z)])$ and let e_i be monomials $e_i(s)=s^i$, $s\in[z,z+\beta(u,z)]$, $i\in\mathbb{N}$. Let g be a preinvex function on $[z,z+\beta(u,z)]$, then the following inequality holds*

$$A(g(e_1)) \le \left(\frac{z+\beta(u,z)-e_1}{\beta(u,z)}\right) g(z) + \left(\frac{e_1-z}{\beta(u,z)}\right) g(z+\beta(u,z)). \tag{2.9}$$

Proof. By using the preinvexity of g on $[z,z+\beta(u,z)]$ and Condition C, we can write

$$e_1 = z+\left(\frac{e_1-z}{\beta(u,z)}\right)\beta(z+\beta(u,z),z).$$

Thus

$$\begin{aligned} g(e_1) &= g\left(z+\left(\frac{e_1-z}{\beta(u,z)}\right)\beta(z+\beta(u,z),z)\right) \\ &\le \left(\frac{z+\beta(u,z)-e_1}{\beta(u,z)}\right) g(z) + \left(\frac{e_1-z}{\beta(u,z)}\right) g(z+\beta(u,z)). \end{aligned} \tag{2.10}$$

Since A is a positive linear functional, we get the inequality (2.9) by applying A on both sides of (2.10). □

3. Main Results

The following theorem generalizes the result given by

Theorem 3.1. *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $\varphi' \in L([z, z+\beta(u,z)])$, where $[z, z+\beta(u,z)] \subseteq I^\circ$ with $\beta(u,z) > 0$. If $w : [z, z+\beta(u,z)] \to [0,\infty)$ is a continuous mapping and $|\varphi'|$ is preinvex on $[z, z+\beta(u,z)]$, then the following inequality holds*

$$\left| \int_z^{z+\beta(u,z)} \varphi(s) w(s) ds - \varphi(z_1) \right| \leq \frac{\mathcal{A}_\beta(z,z_1,u) |\varphi'(z)| + \mathcal{B}_\beta(z,z_1,u) |\varphi'(z+\beta(u,z))|}{\beta(u,z)}, \quad (3.1)$$

where

$$\mathcal{A}_\beta(z,z_1,u) = \frac{1}{2}\int_z^{z_1} (z_1-s)^2 w(s) ds - \frac{1}{2}\int_{z_1}^{z+\beta(u,z)} (z_1-s)^2 w(s) ds + (z+\beta(u,z)-z_1)\left[\int_z^{z_1} (z_1-s) w(s) ds - \int_{z_1}^{z+\beta(u,z)} (z_1-s) w(s) ds\right]$$

and

$$\mathcal{B}_\beta(z,z_1,u) = \frac{1}{2}\int_{z_1}^{z+\beta(u,z)} (s-z_1)^2 w(s) ds - \frac{1}{2}\int_z^{z_1} (s-z_1)^2 w(s) ds + (z_1-z)\left[\int_{z_1}^{z+\beta(u,z)} (s-z_1) w(s) ds - \int_z^{z_1} (s-z_1) w(s) ds\right].$$

Proof. We can write

$$\varphi(s) - \varphi(z_1) = \int_z^{z+\beta(u,z)} [\sigma(s-\eta) - \sigma(z_1-\eta)] \varphi'(\eta) d\eta \quad (3.2)$$

From (3.2), we obtain

$$\int_z^{z+\beta(u,z)} \varphi(s) w(s) ds - \varphi(z_1) = \int_z^{z+\beta(u,z)} \left(\int_\eta^{z+\beta(u,z)} w(s) ds - \sigma(z_1-\eta) \right) \varphi'(\eta) d\eta \quad (3.3)$$

Taking absolute value on both sides of (3.3) and applying Lemma 2.5,we have

$$\left| \int_z^{z+\beta(u,z)} \varphi(s) w(s) ds - \varphi(z_1) \right| \leq \int_z^{z+\beta(u,z)} \left| \int_\eta^{z+\beta(u,z)} w(s) ds - \sigma(z_1-\eta) \right| |\varphi'(\eta)| d\eta$$

$$\leq \frac{|\varphi'(z)| \int_z^{z+\beta(u,z)} \left| \int_\eta^{z+\beta(u,z)} w(s)\,ds - \sigma(z_1-\eta) \right| (z+\beta(u,z)-\eta)\,d\eta}{\beta(u,z)}$$
$$+ \frac{|\varphi'(z+\beta(u,z))| \int_z^{z+\beta(u,z)} \left| \int_\eta^{z+\beta(u,z)} w(s)\,ds - \sigma(z_1-\eta) \right| (\eta - z)\,d\eta}{\beta(u,z)}. \quad (3.4)$$

We notice that

$$\int_z^{z+\beta(u,z)} \left| \int_\eta^{z+\beta(u,z)} w(s)\,ds - \sigma(z_1-\eta) \right| (z+\beta(u,z)-\eta)\,d\eta$$
$$= \int_z^{z_1} \left| \int_\eta^{z+\beta(u,z)} w(s)\,ds - \int_z^{z+\beta(u,z)} w(s)\,ds \right| (z+\beta(u,z)-\eta)\,d\eta$$
$$+ \int_{z_1}^{z+\beta(u,z)} \left(\int_\eta^{z+\beta(u,z)} w(s)\,ds \right) (z+\beta(u,z)-\eta)\,d\eta$$
$$= \int_z^{z_1} \left(\int_z^{\eta} w(s)\,ds \right) (z+\beta(u,z)-\eta)\,d\eta$$
$$+ \int_{z_1}^{z+\beta(u,z)} \left(\int_\eta^{z+\beta(u,z)} w(s)\,ds \right) (z+\beta(u,z)-\eta)\,d\eta. \quad (3.5)$$

Using integration by parts

$$\int_z^{z_1} \left(\int_z^{\eta} w(s)\,ds \right) (z+\beta(u,z)-\eta)\,d\eta$$
$$= -\frac{(z+\beta(u,z)-z_1)^2}{2} \int_z^{z_1} w(s)\,ds + \frac{1}{2} \int_z^{z_1} (z+\beta(u,z)-s)^2 w(s)\,ds. \quad (3.6)$$

and

$$\int_{z_1}^{z+\beta(u,z)} \left(\int_\eta^{z+\beta(u,z)} w(s)\,ds \right) (z+\beta(u,z)-\eta)\,d\eta \quad (3.7)$$
$$= \frac{(z+\beta(u,z)-z_1)^2}{2} \int_{z_1}^{z+\beta(u,z)} w(s)\,ds$$
$$- \frac{1}{2} \int_{z_1}^{z+\beta(u,z)} (z+\beta(u,z)-s)^2 w(s)\,ds.$$

A combination of (3.5), (3.6) and (3.7), we obtain

$$\mathcal{A}_\beta(z,z_1,u) = \int_z^{z+\beta(u,z)} \left| \int_\eta^{z+\beta(u,z)} w(s)\,ds - \sigma(z_1-\eta) \right| (z+\beta(u,z)-s)\,d\eta \quad (3.8)$$
$$= \frac{(z+\beta(u,z)-z_1)^2}{2} \left[\int_{z_1}^{z+\beta(u,z)} w(s)\,ds - \int_z^{z_1} w(s)\,ds \right]$$
$$+ \frac{1}{2} \int_z^{z_1} (z+\beta(u,z)-s)^2 w(s)\,ds$$

$$
\begin{aligned}
&-\frac{1}{2}\int_{z_1}^{z+\beta(u,z)}(z+\beta(u,z)-s)^2 w(s)\,ds\\
&=\frac{(z+\beta(u,z)-z_1)^2}{2}\left[\int_{z_1}^{z+\beta(u,z)} w(s)\,ds-\int_z^{z_1} w(s)\,ds\right]\\
&\quad+\frac{1}{2}\int_z^{z_1}(z+\beta(u,z)-z_1+z_1-s)^2 w(s)\,ds\\
&\quad-\frac{1}{2}\int_{z_1}^{z+\beta(u,z)}(z+\beta(u,z)-z_1+z_1-s)^2 w(s)\,ds\\
&=\frac{1}{2}\int_z^{z_1}(z_1-s)^2 w(s)\,ds-\frac{1}{2}\int_{z_1}^{z+\beta(u,z)}(z_1-s)^2 w(s)\,ds\\
&\quad+(z+\beta(u,z)-z_1)\left[\int_z^{z_1}(z_1-s)\,w(s)\,ds\right.\\
&\quad\left.-\int_{z_1}^{z+\beta(u,z)}(z_1-s)\,w(s)\,ds\right].
\end{aligned}
$$

In a similar way

$$
\begin{aligned}
\mathcal{B}_\beta(z,z_1,u)&=\int_z^{z+\beta(u,z)}\left|\int_\eta^{z+\beta(u,z)} w(s)\,ds-\sigma(z_1-\eta)\right|(\eta-z)\,d\eta\\
&=\frac{1}{2}\int_{z_1}^{z+\beta(u,z)}(s-z_1)^2 w(s)\,ds-\frac{1}{2}\int_z^{z_1}(s-z_1)^2 w(s)\,ds\\
&\quad+(z_1-z)\left[\int_{z_1}^{z+\beta(u,z)}(s-z_1)\,w(s)\,ds-\int_z^{z_1}(s-z_1)\,w(s)\,ds\right].
\end{aligned}
$$

This concludes the proof of the theorem. □

Corollary 3.2. *Suppose that the assumptions of Theorem 3.1 are satisfied and that $w(s)$ is symmetric with respect to $z+\frac{1}{2}\beta(u,z)$ on $[z,z+\beta(u,z)]$. Then the following inequality holds:*

$$
\begin{aligned}
&\left|\int_z^{z+\beta(u,z)}\varphi(s)\,w(s)\,ds-\varphi\left(z+\frac{1}{2}\beta(u,z)\right)\right|\\
&\qquad\le\left[\left|\varphi'(z)\right|+\left|\varphi'(z+\beta(u,z))\right|\right]\int_z^{z+\frac{1}{2}\beta(u,z)}\left(z+\frac{1}{2}\beta(u,z)-s\right)w(s)\,ds. \qquad (3.9)
\end{aligned}
$$

Proof. Since the function $w(s)$ is symmetric with respect to $z+\frac{1}{2}\beta(u,z)$ on $[z,z+\beta(u,z)]$ so $z_1=z+\frac{1}{2}\beta(u,z)$ and the function $(s-z_1)^2 w(s)$ is symmetric with respect $z+\frac{1}{2}\beta(u,z)$ on $[z,z+\beta(u,z)]$. This fact gives

$$
\int_z^{z_1}(s-z_1)^2 w(s)\,ds=\int_{z_1}^{z+\beta(u,z)}(s-z_1)^2 w(s)\,ds=\frac{1}{2}\int_z^{z+\beta(u,z)}(s-z_1)^2 w(s)\,ds.
$$

Thus

$$\mathcal{A}_{\beta}(z,z_1,u)=\mathcal{B}_{\beta}(z,z_1,u)=\beta(u,z)\int_{z}^{z+\frac{1}{2}\beta(u,z)}\left(z+\frac{1}{2}\beta(u,z)-s\right)w(s)\,ds.$$

□

Corollary 3.3. *If we take $w(s)=\frac{g(s)}{\int_z^{z+\beta(u,z)}g(s)ds}$ in (3.1) and $g(s)$ is symmetric with respect to $z+\frac{1}{2}\beta(u,z)$. Then the following inequality holds*

$$\left|\int_{z}^{z+\beta(u,z)}\varphi(s)g(s)\,ds-\varphi\left(z+\frac{1}{2}\beta(u,z)\right)\int_{z}^{z+\beta(u,z)}g(s)\,ds\right| \\ \le\beta(u,z)\left[\frac{|\varphi'(z)|+|\varphi'(z+\beta(u,z))|}{2}\right]\int_{0}^{1}\left(\int_{z}^{z+\frac{1-\eta}{2}\beta(u,z)}g(s)\,ds\right)d\eta. \quad (3.10)$$

Theorem 3.4. *Let $\varphi: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $\varphi'\in L([z,z+\beta(u,z)])$, where $[z,z+\beta(u,z)]\subseteq I^{\circ}$ with $\beta(u,z)>0$. If $w:[z,z+\beta(u,z)]\to[0,\infty)$ is a continuous mapping and $|\varphi'|^q$ is preinvex on $[z,z+\beta(u,z)]$ for $q\ge 1$, then the following inequality holds*

$$\left|\int_{z}^{z+\beta(u,z)}\varphi(s)w(s)\,ds-\varphi(z_1)\right|\le\left(2\int_{z_1}^{z+\beta(u,z)}w(s)\,ds\right)^{1-\frac{1}{q}} \\ \times\left(\frac{\mathcal{A}_{\beta}(z,z_1,u)|\varphi'(z)|^q+\mathcal{B}_{\beta}(z,z_1,u)|\varphi'(\beta(u,z))|^q}{\beta(u,z)}\right)^{\frac{1}{q}}, \quad (3.11)$$

where $\mathcal{A}_{\beta}(z,z_1,u)$ and $\mathcal{B}_{\beta}(z,z_1,u)$ are as defined in Theorem 3.1.

Proof. Application of Hölder inequality in (3.3) yields that

$$\left|\int_{z}^{z+\beta(u,z)}\varphi(s)w(s)\,ds-\varphi(z_1)\right| \\ \le\int_{z}^{z+\beta(u,z)}\left|\int_{\eta}^{z+\beta(u,z)}w(s)\,ds-\sigma(z_1-\eta)\right||\varphi'(\eta)|^q\,d\eta \\ \le\left(\int_{z}^{z+\beta(u,z)}\left|\int_{\eta}^{z+\beta(u,z)}w(s)\,ds-\sigma(z_1-\eta)\right|d\eta\right)^{1-\frac{1}{q}} \\ \times\left(\int_{z}^{z+\beta(u,z)}\left|\int_{\eta}^{z+\beta(u,z)}w(s)\,ds-\sigma(z_1-\eta)\right||\varphi'(\eta)|^q\,d\eta\right)^{\frac{1}{q}}. \quad (3.12)$$

Applying Lemma 2.5, we have

$$\int_{z}^{z+\beta(u,z)}\left|\int_{\eta}^{z+\beta(u,z)}w(s)\,ds-\sigma(z_1-\eta)\right||\varphi'(\eta)|^q\,d\eta \quad (3.13)$$

$$\leq \frac{\mathcal{A}_{\beta}(z,z_1,u)\left|\varphi'(z)\right|^{q}+\mathcal{B}_{\beta}(z,z_1,u)\left|\varphi'(z+\beta(u,z))\right|^{q}}{\beta(u,z)}.$$

On the other hand, we have

$$\begin{aligned}
&\int_{z}^{z+\beta(u,z)}\left|\int_{\eta}^{z+\beta(u,z)} w(s)\,ds-\sigma(z_1-\eta)\right|d\eta\\
&=\int_{z}^{z_1}\left(\int_{z}^{\eta} w(s)\,ds\right)d\eta+\int_{z_1}^{z+\beta(u,z)}\left(\int_{\eta}^{z+\beta(u,z)} w(s)\,ds\right)d\eta\\
&=2\int_{z_1}^{z+\beta(u,z)} w(s)\,ds. \qquad (3.14)
\end{aligned}$$

A combination of (3.12)-(3.14) gives (3.11). This completes the proof of the theorem. □

Corollary 3.5. *If $w(s)$ is symmetric with respect to $z+\frac{1}{2}\beta(u,z)$ on $[z,z+\beta(u,z)]$, then from (3.11), we obtain the following inequality*

$$\begin{aligned}
&\left|\int_{z}^{z+\beta(u,z)}\varphi(s)w(s)\,ds-\varphi\left(z+\frac{1}{2}\beta(u,z)\right)\right|\\
&\leq 2\left(\frac{\left|\varphi'(z)\right|^{q}+\left|\varphi'(z+\beta(u,z))\right|^{q}}{2}\right)^{\frac{1}{q}}\\
&\quad\times\int_{z}^{z+\frac{1}{2}\beta(u,z)}\left(z+\frac{1}{2}\beta(u,z)-s\right)w(s)\,ds. \qquad (3.15)
\end{aligned}$$

Corollary 3.6. *If $w(s)=\frac{g(s)}{\int_{z}^{z+\beta(u,z)}\frac{g(s)}{s}ds}$ and $g(s)$ is symmetric with respect to $z+\frac{1}{2}\beta(u,z)$ on $[z,z+\beta(u,z)]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\int_{z}^{z+\beta(u,z)}\varphi(s)g(s)\,ds-\varphi\left(z+\frac{1}{2}\beta(u,z)\right)\int_{z}^{z+\beta(u,z)} g(s)\,ds\right|\\
&\leq 2\left(\frac{\left|\varphi'(z)\right|^{q}+\left|\varphi'(z+\beta(u,z))\right|^{q}}{\beta(u,z)}\right)^{\frac{1}{q}}\\
&\quad\times\int_{z}^{z+\frac{1}{2}\beta(u,z)}\left(z+\frac{1}{2}\beta(u,z)-s\right)g(s)\,ds. \qquad (3.16)
\end{aligned}$$

For our next results, we use the following notations.

$$\begin{aligned}
\varphi(w,\varphi):=\frac{1}{\beta(u,z)}\Bigg(&\varphi(z)\int_{z}^{z+\beta(u,z)}(z+\beta(u,z)-s)\,ds\\
&+\varphi(z+\beta(u,z))\int_{z}^{z+\beta(u,z)}(s-z)\,w(s)\,ds\Bigg)
\end{aligned}$$

$$-\int_{z}^{z+\beta(u,z)} \varphi(s)\, w(s)\, ds. \tag{3.17}$$

It is clear from (3.17) that

$$\varphi\left(\frac{1}{\beta(u,z)}, \varphi\right) := \frac{\varphi(z)+\varphi(z+\beta(u,z))}{2} - \frac{1}{\beta(u,z)} \int_{z}^{z+\beta(u,z)} \varphi(s)\, ds. \tag{3.18}$$

The next result gives upper bound of $|\varphi(w,\varphi)|$ when the function $\varphi(s)$ is prequasi-invex.

Theorem 3.7. *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $\varphi' \in L([z, z+\beta(u,z)])$, where $[z, z+\beta(u,z)] \subseteq I^{\circ}$ with $\beta(u,z) > 0$. If $w : [z, z+\beta(u,z)] \to [0,\infty)$ be a continuous mapping and $|\varphi'|$ is prequasiinvex on $[z, z+\beta(u,z)]$, then the following inequality holds*

$$\begin{aligned} |\varphi(w,\varphi)| \le{}& \frac{\beta(z_1,z)}{\beta(u,z)} \left(\sup\left\{\left|\varphi'(z)\right|, \left|\varphi'(z+\beta(z_1,z))\right|\right\}\right) \int_0^1 |G(w,z,z_1;y)|\, dy \quad (3.19)\\ &+ \frac{\beta(u,z_1)}{\beta(u,z)} \left(\sup\left\{\left|\varphi'(z_1)\right|, \left|\varphi'(z_1+\beta(u,z_1))\right|\right\}\right) \int_0^1 |G(w,z_1,u;y)|\, dy, \end{aligned}$$

where $G(w,\alpha,\beta;y)$ is defined as in Lemma 2.2.

Proof. Since $|\varphi'|$ is prequasiinvex on $[z, z+\beta(u,z)]$, we have

$$\left|\varphi'(z+y\beta(z_1,z))\right| \le \sup\left\{\left|\varphi'(z)\right|, \left|\varphi'(z+\beta(z_1,z))\right|\right\}$$

and

$$\left|\varphi'(z_1+y\beta(u,z_1))\right| \le \sup\left\{\left|\varphi'(z_1)\right|, \left|\varphi'(z_1+\beta(u,z_1))\right|\right\}.$$

for all $y \in [0,1]$. Hence the inequality (3.19) follows from (2.1). □

Theorem 3.8. *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $\varphi' \in L([z, z+\beta(u,z)])$, where $[z, z+\beta(u,z)] \subseteq I^{\circ}$ with $\beta(u,z) > 0$, then the following inequality holds*

$$\begin{aligned} |\varphi(w,\varphi)| \le{}& \left(\int_{z}^{z+\frac{1}{2}\beta(u,z)} (\eta - z)\, w(\eta)\, d\eta\right) \left[\sup\left\{\left|\varphi'(z)\right|, \left|\varphi'\left(z+\frac{1}{2}\beta(u,z)\right)\right|\right\}\right.\\ &\left.+ \sup\left\{\left|\varphi'\left(z+\frac{1}{2}\beta(u,z)\right)\right|, \left|\varphi'(z+\beta(u,z))\right|\right\}\right] \end{aligned} \tag{3.20}$$

Proof. The symmetry of $w(s)$ with respect to $z+\frac{1}{2}\beta(u,z)$ on $[z, z+\beta(u,z)]$ gives

$$\varphi(w,\varphi) = \frac{\varphi(z)+\varphi(z+\beta(u,z))}{2} - \int_{z}^{z+\beta(u,z)} \varphi(s)\, w(s)\, ds.$$

We also observe that

$$\frac{\beta(z_1,z)}{\beta(u,z)}\int_0^1 |G(w,z,z_1;y)|\,dy = \frac{1}{\beta(u,z)}\int_z^{z_1}\left|\int_z^{\eta}(s-z)\,w(s)\,ds - \int_{\eta}^{z+\beta(u,z)}(z+\beta(u,z)-s)\,w(s)\,ds\right|d\eta$$

and

$$\frac{\beta(u,z_1)}{\beta(u,z)}\int_0^1 |G(w,z_1,u;y)|\,dy = \frac{1}{\beta(u,z)}\int_{z_1}^{z+\beta(u,z)}\left|\int_z^{\eta}(s-z)\,w(s)\,ds - \int_{\eta}^{z+\beta(u,z)}(z+\beta(u,z)-s)\,w(s)\,ds\right|d\eta$$

Consider the function $p:[z,z+\beta(u,z)]\to\mathbb{R}$ defined by

$$p(\eta)=\int_z^{\eta}(s-z)\,w(s)\,ds-\int_{\eta}^{z+\beta(u,z)}(z+\beta(u,z)-s)\,w(s)\,ds.$$

Then

$$p'(\eta)=\frac{\beta(u,z)\,w(\eta)}{\eta}>0, \eta\in[z,z+\beta(u,z)]$$

This shows that $p(\eta)$ is an increasing function on $[z,z+\beta(u,z)]$ and

$$p\left(z+\frac{1}{2}\beta(u,z)\right)=0.$$

Now it is easy to see that

$$\begin{aligned}\frac{\beta(z_1,z)}{\beta(u,z)}\int_0^1 |G(w,z,z_1;y)|\,dy &= \frac{1}{\beta(u,z)}\int_z^{z_1}\left(\int_z^{\eta}(s-z)\,w(s)\,ds - \int_{\eta}^{z+\beta(u,z)}(z+\beta(u,z)-s)\,w(s)\,ds\right)d\eta\\ &=\int_z^{z_1}(\eta-z)\,w(\eta)\,d\eta=\int_z^{z+\frac{1}{2}\beta(u,z)}(\eta-z)\,w(\eta)\,d\eta\end{aligned}$$

and

$$\begin{aligned}\frac{\beta(u,z_1)}{\beta(u,z)}\int_0^1 |G(w,z_1,u;y)|\,dy &= \frac{1}{\beta(u,z)}\int_{z_1}^{z+\beta(u,z)}\left(\int_z^{\eta}(s-z)\,w(s)\,ds - \int_{\eta}^{z+\beta(u,z)}(z+\beta(u,z)-s)\,w(s)\,ds\right)d\eta\\ &=\int_z^{z_1}(\eta-z)\,w(\eta)\,d\eta=\int_z^{z+\frac{1}{2}\beta(u,z)}(\eta-z)\,w(\eta)\,d\eta.\end{aligned}$$

Hence the inequality (3.20) follows from the inequality (3.19), the above equalities and using the Property C. Since

$$\beta\left(z+\frac{1}{2}\beta(u,z),z\right)=\frac{1}{2}\beta(u,z)$$

and

$$z+\frac{1}{2}\beta(u,z)+\beta\left(u,z+\frac{1}{2}\beta(u,z)\right)=z+\beta(u,z).$$

□

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