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SOME NORM INEQUALITIES FOR SOME POSITIVE BLOCK MATRICES

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Abstract. We review Lin's inequality of numerical range widths and prove that if $M=[X_{i,j}]_{i,j=1}^n$ is positive semi-definite then $\bigoplus_{i=1}^n (X_{i,i}-\sum\limits_{j\neq i} X_{j,j}) \leq M^\tau \leq I_n \otimes \sum\limits_{i=1}^n X_{i,i}$, where M^τ is the partial transpose of M. Some classical results are also discussed in terms of permutations.

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1. Introduction and preliminaries

Let $\mathbb{M}_n(\mathbb{M}_m)$ denote the space of $n\times n$ complex matrices with entries in $\mathbb{M}_m(\mathbb{C})$. We write $\mathbb{M}_n^H(\mathbb{M}_m)$ and $\mathbb{M}_n^+(\mathbb{M}_m)$ for the set of Hermitian and positive semi-definite matrices in $\mathbb{M}_n(\mathbb{M}_m)$, respectively. Let $Im(X) := \dfrac{X-X^*}{2\mathrm{i}}$ and $Re(X) := \dfrac{X+X^*}{2}$ be the imaginary part and the real part of a matrix X, respectively. If $W(X)$ denotes the field of values of X then $W(Re(X)) = \Re(W(X))$ and $W(Im(X)) = \Im(W(X))$ see [14]. A positive semi-definite matrix A is denoted by $A\geq 0$ and a norm $\|.\|$ over the space of matrices is a symmetric norm if $\|UAV\| = \|A\|$ for all A and all unitaries U and V.

For a matrix $M=[X_{i,j}]\in\mathbb{M}_n(\mathbb{M}_m)$ we denote M^τ the matrix $[X_{i,j}^*]$ where each block is replaced by its adjoint. We define $\mathrm{Tr}_1(M) := \sum_{i=1}^n X_{i,i}$, $\mathrm{Tr}_2(M) := [\mathrm{Tr}(X_{i,j})]$ for $\mathrm{Tr}_1(M)$ (resp. $\mathrm{Tr}_2(M)$) is an $m\times m$ matrix (resp. an $n\times n$ complex matrix). The matrix M is positive partial transpose (P.P.T.) if $M\geq 0$ and $M^\tau \geq 0$. The matrix I_m (or I) is the identity matrix of dimension m and a 0 block-entry is an all zero submatrix. We finally denote $\lambda_i(M)$ and $\sigma_i(M)$, $i=1,\dots,n$ the eigenvalues and the singular values of M, respectively, arranged in decreasing order.

We need the following notations used latter in the article:

For $X\in\mathbb{M}_n$, the vertical width $\omega_v(X)$ and the width $\omega(X)$ of X are the smallest possible real numbers v and w such that $W(X)$ is contained in a vertical strip of width v and in a strip (rectangle) of width w, respectively. In particular, $\omega(X)\leq\omega_v(X)$ and we may write

$$\min_{(a,b)\in\mathbb{R}^2 \ | W(X)\subset\{x\in\mathbb{C}; a\le\Re(x)\le b\}} |b-a| = \omega_v(X).$$

Notice that if the width of $W(A)$ is ω, then one can find a $\theta \in [0,\pi]$ such that

$$rI_m \le Re(e^{i\theta}A) \le (r+\omega)I_m$$

for some $r\in\mathbb{R}$ and $\omega = \omega_v(e^{i\theta}A)$.

If $S_i = \{M_k, 1\le k\le i, M_k\in\mathbb{M}_m\}$ of cardinality i, we let $\{S_i\}$ denote the set of matrices $\{M_1 \pm \ldots \pm M_i\}$ and an element in $\{S_i\}$ is denoted by $S_i^{[\sigma]}$. We denote by $\omega_v(\{S_i\})$ the 2^{i-1}-tuple of $\omega_v(S_i^{[\sigma]})$ for each $S_i^{[\sigma]}$.

Finally if $M := [X_{i,j}]_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_m)$ with blocks $X_{i,j}$ we write $M = [{}_M[x_{k,l}^{i,j}]]$, where ${}_M[x_{k,l}^{i,j}]$ is the complex number belonging to block $X_{i,j}$ on its line k and column l. For k, l: $1\ldots m$ and i,j: $1\ldots n$ one has

$${}_{M^T}[x_{k,l}^{i,j}] = {}_M[x_{l,k}^{j,i}] \quad ; \quad {}_{M^\tau}[x_{k,l}^{i,j}] = \overline{{}_M[x_{l,k}^{i,j}]}.$$

The matrix $\tilde{M}$ defined in $\mathbb{M}_m(\mathbb{M}_n)$ by ${}_{\tilde{M}}[x_{i,j}^{k,l}] = {}_M[x_{k,l}^{i,j}]$ is known to be congruent to M, that is, there exist a permutation $\mathcal{P}_M$ such that $\mathcal{P}_M^T M \mathcal{P}_M = \tilde{M}$, see [3, Theorem 7] and [9, Theorem 2.4] with $\widetilde{(\tilde{M})^\tau} = (M^\tau)^T$.

We sketch $\mathcal{P}_M$ as follows: An index in $[1,mn]$ is denoted by ι, upon completing M by $m\times m$ zero blocks or extending each block by $0's$ we consider the completed matrix $M_0 \in \mathbb{M}_r(\mathbb{M}_r)$ with $\max(m,n) = r$. Take the transposition T permuting for every s, $0\le s\le r-2$ and $sr+2+s\le\iota\le sr+r$, $\iota \leftrightarrow (\iota - sr-1)r+1+s$, $TM_0T = \widetilde{M_0}$. Applying block swapping (Proposition 1.1) one can show that there is some permutation P such that

$$P^T \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} P = \begin{pmatrix} \tilde{M} & 0 \\ 0 & 0 \end{pmatrix}.$$

Assume that M is invertible then so is $\tilde{M}$, writing P by blocks $P = \begin{pmatrix} L & \star \\ \star & \star \end{pmatrix}$ we see that $L^TML = \tilde{M}$ which implies that L is invertible and thus a permutation matrix. By a continuity argument we can take $\mathcal{P}_M = L$, see [5] and the references therein for further reading.

Proposition 1.1. *Let $A = [A_{i,j}] \in \mathbb{M}_s(\mathbb{M}_p)$ and let $B = [B_{i,j}] \in \mathbb{M}_s(\mathbb{M}_q)$ two complex matrices written by blocks. For $C = [C_{i,j}] \in \mathbb{M}_s(\mathbb{M}_{p+q})$, where $C_{i,j} = \begin{pmatrix} A_{i,j} & 0 \\ 0 & B_{i,j} \end{pmatrix}$ there is a permutation P such that $P^TCP = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.*

Proof. A (u,v,r) block swap B_{uvr} is a permutation matrix (here of dimension $s(p+q)$) of the form $\begin{pmatrix} I_u & 0 & 0 & 0 \\ 0 & 0 & I_v & 0 \\ 0 & I_r & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$. Say $s \geq 2$ for $i = 1 \ldots s-1$, take $P = \prod_{i=1}^{s-1} B_{u_i v_i r_i}$, where $u_i = ip, v_i = iq$ and $r_i = p$. $\square$

2. Main results

If $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_2^H(\mathbb{M}_m)$ (not necessarily ≥ 0) with positive semi-definite diagonal blocks then it is easy to see that $|\lambda_n(M)| \leq \lambda_1(M)$ (to see this consider the eigenvalues of $\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$). In particular if $M \leq I \otimes S$ then $\|M\|_s \leq \|S\|_s$, where $\|.\|_s$ is the spectral norm. This is not true for $n > 2$ with $n = 4$ taking for example $\begin{pmatrix} 0 & 3 & -1 & 0 \\ 3 & 0 & 2 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, where $\lambda_4 \approx -4.1$ and $\lambda_1 \approx 3.22$. In fact if M is an entry wise non-negative matrix we know by Perron-Frobenius theorem that $|\lambda_n(M)| \leq \lambda_1(M)$.

2.1. *Lin's Inequality*

In [1] the following norm inequality is proved:

Theorem 2.1. *[1] Let* $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_2^+(\mathbb{M}_m)$. *Then for all symmetric norms*

$$\|M\| \leq \|A + B + \omega I_m\|,$$

where ω *is the width of a strip containing the numerical range of* X.

The previous theorem has been generalized in [12] for 2×2 block matrices. In [11] a generalization to any number of blocks was presented, here we refine Lin's idea mainly from Theorem's 2.1 proof (the base case):

As M is positive, we may write $M = \begin{pmatrix} X_1^* \\ X_2^* \\ \vdots \\ X_n^* \end{pmatrix} \begin{pmatrix} X_1 & X_2 & \ldots & X_n \end{pmatrix}$ for some $X_i \in \mathbb{M}_{mn \times m}$ so that $M_{i,i} = X_i^* X_i$ and $\|M\|_k = \left\| \sum_{i=1}^{n} X_i X_i^* \right\|_k$ ($\|.\|_k$ are Ky-Fan k-norms). Identifying the n-tuple X to the set $\{X_i, 1 \leq i \leq n\}$ we have (by a direct induction) $\sum_{i=1}^n X_i X_i^* = \frac{1}{2^{n-1}} (\sum_\sigma (X^{[\sigma]})(X^{[\sigma]})^*)$, applying Ky-Fan k-norms triangular inequality we get

$$\|M\|_k = \frac{1}{2^{n-1}} \left\| \sum_\sigma (X^{[\sigma]})(X^{[\sigma]})^* \right\|_k \leq \frac{1}{2^{n-1}} \sum_\sigma \left\| (X^{[\sigma]})^* (X^{[\sigma]}) \right\|_k. \tag{2.1}$$

Theorem 2.2. *[11] Let $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ and let $S_i = \{X_{i,k}, i+1 \le k \le n\}$ for $i = 1, \dots, n-1$. Then for all symmetric norms*

$$\|M\| \le \left\| \sum_{i=1}^{n} X_{i,i} + \sum_{i=1}^{n-1} \upsilon_i I_m \right\|,$$

where υ_i is the average of $\omega_v(\{S_i\})$.

Proof. The proof follows from [11] or one can use (2.1), each term in the σ sum is a positive semi-definite matrix that can be written as

$$\sum_{i=1}^{n} X_i^* X_i + \sum_{i=1}^{n-1} \pm Re(X_i^*(\pm X_{i+1} \cdots \pm X_n)),$$

we can bound each term of the second sum by scalar identities and splitting the quantities we get the result by a counting argument for Ky-Fan k-norms. Ky-Fan dominance theorem completes the proof ([14] -Sec 10.7-). The same last step can be seen in [1, Theorem 2.1's proof]. □

Corollary 2.3. *Let $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$. Then for all symmetric norms*

$$\|M\| \le \left\| \sum_{i=1}^{n} X_{i,i} + \sum_{i=2}^{n} \upsilon_i I_m \right\|,$$

where υ_i ($i = 2, \dots, n$) is the average of $\omega_v(\{S_i\})$ and $S_i = \{X_{k,i}, 1 \le k \le i-1\}$.

Proof. Consider the matrix $\mathcal{L}M\mathcal{L}$ in Theorem 2.2 with $\mathcal{L}$ the anti-diagonal permutation matrix for blocks $\left(\mathcal{L} = \begin{pmatrix} 0 & \cdots & I_m \\ \vdots & \iddots & \vdots \\ I_m & \cdots & 0 \end{pmatrix}\right)$. □

Corollary 2.4. *Let $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ such that for $i < j$, $r_{i,j} I_m \le Re(X_{i,j}) \le r_{i,j} I_m + \delta_{i,j} I_m$. Then $\|M\| \le \left\| \sum_{i=1}^{n} X_{i,i} + (\sum_{i<j} \delta_{i,j}) I_m \right\|$ for all symmetric norms.*

Proof. The proof follows from Theorem 2.2 by writing each $X_{i,j} = Re(X_{i,j}) + \mathrm{i} Im(X_{i,j})$ for $i < j$ in the average expression and bounding the real parts we get the stated inequality. □

The case $Re(X_{i,j}) = 0.I_m$ (skew-Hermitian blocks) is the main result of [13]. We also point out that in Theorem 2.1 of [11] the author replaced (by a miss-justification when $n > 2$ as per our private communication) vertical widths by smallest widths ω and proposed the same replacement in Corollary 2.4 as Question 2.5.

2.2. *Around Hiroshima majorization*

In this section we prove some classical results concerning matrices in $\mathbb{M}_n^H(\mathbb{M}_m)$, related to some permutations and P.P.T. matrices. The approach is close (dual) to that in [4].

Corollary 2.5. *[4] Let $M \in \mathbb{M}_n^H(\mathbb{M}_m)$ such that $M^\tau \leq \mathrm{Tr}_2(M^\tau) \otimes I_m$. Then there exist a permutation $\mathcal{P}_M$ with:*

$$N = \mathcal{P}_M^T M^\tau \mathcal{P}_M \leq \mathcal{P}_M^T(\mathrm{Tr}_2(M^\tau) \otimes I_m)\mathcal{P}_M = I_m \otimes \mathrm{Tr}_1(N) \tag{2.2}$$

for $N \in \mathbb{M}_m^H(\mathbb{M}_n)$, $\mathrm{Tr}_1(N) = \mathrm{Tr}_2(M^\tau)$ is an $n \times n$ matrix. Furthermore if $M \geq 0$ then $N^\tau \geq 0$.

Remark 2.6. *Set $M \in \mathbb{M}_n(\mathbb{M}_m)$, U unitary and $U^*MU \in \mathbb{M}_n(\mathbb{M}_m)$. Define the block $m \times m$ permutation matrix $P_M \in \mathbb{M}_n(\mathbb{M}_m)$ having zero or identity $m \times m$ blocks; if $U = P_M$ is a transposition block matrix then it is easy to see that $(P_M M P_M)^\tau = P_M M^\tau P_M$ and thus upon composition for any block $m \times m$ permutation matrix P_M, $P_M(P_M^T M P_M)^\tau P_M^T = M^\tau$.*

Let $M \in \mathbb{M}_n^H(\mathbb{M}_m)$ such that $M^\tau \geq 0$ and U unitary, for $U^*MU \in \mathbb{M}_n^H(\mathbb{M}_m)$ the matrix $(U^*MU)^\tau$ may not be positive semi-definite in general; consider $M = UNU^* \in \mathbb{M}_3^+(\mathbb{M}_2)$ in Example 2.12 with $U = \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$, however this is true if $n = m$ and $U = \mathcal{P}_M$, this is also true for $M \in \mathbb{M}_2^H(\mathbb{M}_m)$ and $U^* = \begin{pmatrix} e^{i\zeta}\cos(\theta)I & \sin(\theta)I \\ -e^{i\zeta}\sin(\theta)I & \cos(\theta)I \end{pmatrix}$ ([12]).

From Quantum Physics theory we know that if M is P.P.T. then $M \leq I_n \otimes \mathrm{Tr}_1(M)$ and $M \leq \mathrm{Tr}_2(M) \otimes I_m$ see for example [7,8] and more recently [2,4]. Here we present a different proof as follows:

Lemma 2.7. *If $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ then*

$$\bigoplus_{i=1}^{n}(X_{i,i} - \sum_{j\neq i} X_{j,j}) \leq M^\tau \leq I_n \otimes \sum_{i=1}^{n} X_{i,i}.$$

Proof. We prove the right side inequality as the lower bound has a similar proof by noticing that $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0 \Longleftrightarrow \begin{pmatrix} A & -X \\ -X^* & B \end{pmatrix} \geq 0$. We write $E = I_n \otimes \sum_{i=1}^{n} X_{i,i} - M^\tau$ as the sum of positive semi-definite matrices of the form $E_{k,j} = \begin{pmatrix} X_{j,j(k)} & -X_{k,j}^* \\ -X_{k,j} & X_{k,k(j)} \end{pmatrix}$; here the matrix $E_{k,j}$ with $k < j$ refers to the $mn \times mn$ block matrix having all blocks zero except $-X_{k,j}$ and $-X_{k,j}^*$ taken on the same position as $X_{k,j}$ and $X_{k,j}^*$ in E, respectively, $X_{j,j(k)}$ and $X_{k,k(j)}$, where $X_{j,j(k)}$ is the block $X_{j,j}$ on the k^{th} diagonal entry of E (as of $E_{k,j}$). Since $M \geq 0$, up to the congruence $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ each $E_{k,j}$ is a principal block submatrix of M and $E = \sum_{k<j} E_{k,j} \geq 0$.

□

Remark 2.8. *Let $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ one can check following the previous proof that $M \leq \bigoplus_{i=1}^{n} nX_{i,i}$.*

The last lemma and corollary imply the following:

Corollary 2.9. *[4] If $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ then $M^\tau \le \mathrm{Tr}_2(M^\tau) \otimes I_m$.*

In [6]-Theorem 1- the following (Hiroshima majorization) is proved:

Theorem 2.10. *[6] Let $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ such that $M \le \mathrm{Tr}_2(M) \otimes I_m$. Then for all $k \in [1, mn]$:*

$$\sum_{i=1}^{k} \lambda_i(M) \le \sum_{i=1}^{k} \lambda_i(\mathrm{Tr}_2(M)).$$

Corollary 2.11. *Let $M = [X_{i,j}] \in \mathbb{M}_n^H(\mathbb{M}_m)$ such that*

$$0 \le M \le I_n \otimes \sum_{i=1}^{n} X_{i,i}. \tag{2.3}$$

Then for all $k \in [1, m]$:

$$\sum_{i=1}^{k} \sigma_i(M) \le \sum_{i=1}^{k} \sigma_i \left(\sum_{i=1}^{n} X_{i,i} \right)$$

equivalently $\|M\| \le \left\| \sum_{i=1}^{n} X_{i,i} \right\|$ for all symmetric norms.

Upon completing $\sum_{i=1}^{n} X_{i,i}$ by zeros we can take k; $1 \le k \le mn$ to get the last inequality from Ky-Fan dominance theorem. In [10] another direct proof of Corollary 2.11 is given with the fact that Corollary 2.11 and Theorem 2.10 are equivalent (see Corollary 2.5).

Example 2.12. *It is well known that a matrix in $\mathbb{M}_n^+(\mathbb{M}_m)$ $(n > 2)$ satisfying (2.3) need not be P.P.T. as for example $N \in \mathbb{M}_3^+(\mathbb{M}_2) = \begin{pmatrix} 4&0&0&2&0&0\\ 0&4&0&0&1&0\\ 0&0&1&0&0&0\\ 2&0&0&1&0&0\\ 0&1&0&0&1&0\\ 0&0&0&0&0&0 \end{pmatrix}$.*

The next corollary can be proved from the double inequality of Lemma 2.7.

Corollary 2.13. *If $M \in \mathbb{M}_n^+(\mathbb{M}_m)$ then $\|M^\tau\|_s \le \|\mathrm{Tr}_1(M)\|_s$.*

The positive condition in Corollary 2.11 seems necessary as this example shows:

Example 2.14. *We have $M = \begin{pmatrix} 1&0&0&2\\ 0&4.5&0&0\\ 0&0&0.9&0\\ 2&0&0&1 \end{pmatrix} \in \mathbb{M}_2(\mathbb{M}_2)$ and $\sigma_1(M) + \sigma_2(M) =$* $4.5 + 3 > 7.4$

For a matrix $D = [D_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ with all blocks diagonal, it is easily seen that D is P.P.T. as $D^\tau = \bar{D}$ ($\bar{D}$ is D with conjugate entries). Upon considering the rearrangement of D taking $\mathcal{P}_D^T D \mathcal{P}_D$, we see that each diagonal entry of $\mathrm{Tr}_1(D)$ is the sum of n eigenvalues of D which implies: $\|D\| \le \|\mathrm{Tr}_1(D)\|$ for all symmetric norms.

Corollary 2.15. *[15] Let $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$. Then $M \le m\mathrm{Tr}_2(M) \otimes I_m$.*

Proof. Take $M_{\mathcal{P}_M} = \mathcal{P}_M^T M \mathcal{P}_M = [X'_{i,j}] \in \mathbb{M}_m^+(\mathbb{M}_n)$, applying Remark 2.8 $M_{\mathcal{P}_M} \le \bigoplus_{i=1}^m mX'_{i,i}$, reversing the rearrangement $M \le mD$ where $D = [D_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ and $D_{i,j}$ is the (main) diagonal of $X_{i,j}$. Since D is P.P.T. we get the result from Corollary 2.9. □

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