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STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH p–LAPLACIAN OPERATOR VIA RIESZ-CAPUTO FRACTIONAL DERIVATIVES

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Abstract. In the current work, fractional differential equations (**FDEs**) with Sturm-Liouville boundary conditions (**BCs**) and p–Laplacian operator in terms of the Riesz-Caputo fractional derivatives are considered. With the aid of Schauder's and Banach's fixed point theorems, the existence and uniqueness results of the aforesaid problem are established. An explanatory example is set forth to make efficient the obtained results.

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1. INTRODUCTION

FDEs have acquired plentiful circulation and great significance thanks to their accurate description of some phenomena of real-world problems. For instance, we refer the reader to [21, 23, 30, 31, 33, 35] for detailed interpretations, and we refer to [1, 2, 6, 13–15, 28, 34] for the newest trends in the field of **FDEs**.

Boundary value problems (**BVPs**) for **FDEs** have become an interesting field of research. This is due to their strong presence in many significant problems in mathematical physics. In particular, Sturm-Liouville type **BVPs** have attracted the interest of many mathematicians and physicists due to their numerous applications in various scientific fields; see, for example, [4, 10, 11, 22, 37].

In [24], Leibenson introduced the p–Laplacian differential equation

$$\left(\varphi_p(u'(t))\right)' = f(t,u(t),u'(t)), \;\; t \in (0,1),$$

where $\varphi_p(r) = |r|^{p-2}r, \; p > 1$, for studying the turbulent flow problem in a porous medium. In the last few years, the topic was extended to the case of fractional differential equations using different types of fractional operators; see, for example, [18–20, 25, 29] and the references cited therein.

In 1892, Riesz [21] introduced a two-sided fractional operator including both left and right derivatives, which can reflect both the past and the future memory effects.

As a result of the two-sided feature of the Riesz fractional operator, it has arisen strongly in fractional modeling on finite domains and fractional variational problems; see [3, 5, 9, 17].

In despite of the significant contributions involving the numerical methods of the **FDEs** in terms of the Riesz-Caputo fractional derivatives [32, 36], the analytical results concerning the existence, uniqueness, and stability of the solutions are still scarce. For example, Chen et al. in [7] studied a class of **BVPs** for **FDEs** with the Riesz-Caputo derivative of the form

$$\begin{cases} {}^{RC}_{0}\mathfrak{D}^{\gamma}_{T}y(\tau) = g(\tau, y(\tau)), \ \ t \in [0,T], \ 0 < \gamma \leq 1, \\ y(0) = 0, \quad y(T) = y_T, \end{cases}$$

where ${}^{RC}_{0}\mathfrak{D}^{\gamma}_{T}$ is a Riesz-Caputo derivative and $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. By means of new fractional Gronwall's inequalities and some fixed point theorems, they obtained the main results.

Further, Chen et al. [8] obtained some existence results of the anti-periodic **BVPs** for **FDEs** with Riesz-Caputo derivative

$$\begin{cases} {}^{RC}_{0}\mathfrak{D}^{\gamma}_{T}y(\tau) = g(\tau, y(\tau)), \ \ t \in [0,T], \ 1 < \gamma \leq 2, \\ y(0) + y(T) = 0, \quad y'(0) + y'(T) = 0, \end{cases}$$

where ${}^{RC}_{0}\mathfrak{D}^{\gamma}_{T}$ is a Riesz-Caputo derivative and $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Some existence results are estabilished using Schaefer's and Schauder's fixed point theorems.

In [12], Gu et al. investigated the existence of positive solutions of the **FDEs** with Riesz-Caputo derivative

$$\begin{cases} {}^{RC}_{0}\mathfrak{D}^{\alpha}_{1}x(\tau) = h(\tau, x(\tau)), \ \ t \in [0,1], \ 0 < \alpha \leq 1, \\ x(0) = x_0, \quad x(1) = x_1, \end{cases}$$

where ${}^{RC}_{0}\mathfrak{D}^{\alpha}_{T}$ is a Riesz-Caputo derivative and $h : [0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function. Using Leray-Schauder and Krasnoselskii's fixed point theorems in a cone, they obtained trajectory of the solutions via the numerical discretization of the equivalent fractional integral equations. In this study, inspired by [7, 8], we investigate the existence and uniqueness of solutions of **FDEs** with Sturm-Liouville **BCs** and p–Laplacian operator of the form:

$$\begin{cases} {}^{RC}_{0}\mathfrak{D}^{\alpha}_{T}\left(\varphi_p\left({}^{RC}_{0}\mathfrak{D}^{\beta}_{T}\upsilon(t)\right)\right) = \mathcal{W}(t, \upsilon(t)), \ \ t \in J := [0,T], \ T < \infty, \\ a_1\varphi_p\left({}^{RC}_{0}\mathfrak{D}^{\beta}_{T}\upsilon(t)\right)|_{t=0} - b_1\left(\varphi_p\left({}^{RC}_{0}\mathfrak{D}^{\beta}_{T}\upsilon(t)\right)\right)'|_{t=0} = 0, \\ a_2\varphi_p\left({}^{RC}_{0}\mathfrak{D}^{\beta}_{T}\upsilon(t)\right)|_{t=T} + b_2\left(\varphi_p\left({}^{RC}_{0}\mathfrak{D}^{\beta}_{T}\upsilon(t)\right)\right)'|_{t=T} = 0, \\ c_1\upsilon(0) - d_1\upsilon'(0) = 0, \\ c_2\upsilon(T) + d_2\upsilon'(T) = 0, \end{cases} \tag{1.1}$$

where ${}^{RC}_{0}\mathfrak{D}^{\alpha}_{T}$, ${}^{RC}_{0}\mathfrak{D}^{\beta}_{T}$ denote the Riesz-Caputo fractional derivatives of order $\alpha, \beta \in (0,1]$, $a_i,, b_i, c_i, d_i \in \mathbb{R}, i = 1,2$, $\mathcal{W} : J \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and $\varphi_p(r) = |r|^{p-2}r,\ p > 1$ is a p–Laplacian operator with $(\varphi_p)^{-1} = \varphi_q,\ \frac{1}{p} + \frac{1}{q} = 1$.

To the best knowledge of the author, there are no publications have yet been treated with p–Laplacian **FDEs** with Sturm-Liouville **BCs** in terms of the Riesz-Caputo fractional derivative.

2. Preliminaries

In this section, we recall some definitions, lemmas and properties of fractional calculus [21, 33] and Riesz-Caputo fractional derivative [3, 7].

Definition 1. (Riemann-Liouville fractional integrals)[33]
If $y \in C[0,T]$, then for $t \in [0,T]$, the left and the right Riemann-Liouville fractional integrals of order $\gamma > 0$ are defined by

$$ {}_{0}\mathfrak{I}^{\gamma}_{t} y(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} y(s) ds, \ \ \gamma > 0, \tag{2.1}$$

$$ {}_{t}\mathfrak{I}^{\gamma}_{T} y(t) = \frac{1}{\Gamma(\gamma)} \int_t^T (s-t)^{\gamma-1} y(s) ds, \ \ \gamma > 0, \tag{2.2}$$

respectively, where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2. (Riesz fractional integral)[21]
If $y \in C[0,T]$, then for $t \in [0,T]$, the Riesz fractional integral of order $\gamma > 0$ is defined by

$$ {}_{0}\mathfrak{I}^{\gamma}_{T} y(t) = \frac{1}{\Gamma(\gamma)} \int_0^t |t-s|^{\gamma-1} y(s) ds. \tag{2.3}$$

As a consequence of (2.1) and (2.2), it follows that

$$ {}_{0}\mathfrak{I}^{\gamma}_{T} y(t) = \frac{1}{2} \left({}_{0}\mathfrak{I}^{\gamma}_{t} + {}_{t}\mathfrak{I}^{\gamma}_{T} \right) y(t). \tag{2.4}$$

Definition 3. (Caputo fractional derivatives)[33]
If $y \in C^n[0,T]$, then for $t \in [0,T]$, the left and the right Caputo fractional derivatives of order γ, $n-1 \leq \gamma < n$, are defined by

$$ {}^{C}_{0}\mathfrak{D}^{\gamma}_{t} y(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-s)^{n-\gamma-1} \left(\frac{d}{ds}\right)^n y(s) ds, \tag{2.5}$$

$$ {}^{C}_{t}\mathfrak{D}^{\gamma}_{T} y(t) = \frac{1}{\Gamma(n-\gamma)} \int_t^T (s-t)^{\gamma-1} \left(-\frac{d}{ds}\right)^n y(s) ds, \tag{2.6}$$

respectively, where $n = [\gamma] + 1,\ n \in \mathbb{N}$.

Definition 4. (Riesz-Caputo fractional derivatives)[3]
If $y \in C^n[0,T]$, then for $t \in [0,T]$, the Riesz-Caputo fractional derivative of order γ, $n-1 \leq \gamma < n$, is defined by

$$ {}^{RC}_{0}\mathfrak{D}^{\gamma}_{T} y(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t |t-s|^{n-\gamma-1} \left(\frac{d}{ds}\right)^n y(s)ds, \tag{2.7}$$

where $n = [\gamma]+1$, $n \in \mathbb{N}$.

Using equations (2.5) and (2.6), it follows that

$$ {}^{RC}_{0}\mathfrak{D}^{\gamma}_{T} y(t) = \frac{1}{2}\left({}^{C}_{0}\mathfrak{D}^{\gamma}_{t} + (-1)^{n}\,{}^{C}_{t}\mathfrak{D}^{\gamma}_{T}\right) y(t). \tag{2.8}$$

In particular, for $0 < \gamma < 1$ and $y(t) \in C(0,T)$, one has

$$ {}^{RC}_{0}\mathfrak{D}^{\gamma}_{T} y(t) = \frac{1}{2}\left({}^{C}_{0}\mathfrak{D}^{\gamma}_{t} - {}^{C}_{t}\mathfrak{D}^{\gamma}_{T}\right) y(t). \tag{2.9}$$

Lemma 1. *[21] If $y(t) \in C^n[0,T]$, then*

$$ {}_{0}\mathfrak{I}^{\gamma}_{t}\,{}^{C}_{0}\mathfrak{D}^{\gamma}_{t} y(t) = y(t) - \sum_{m=0}^{n-1} \frac{y^{(m)}(0)}{m!}(t-0)^m, \tag{2.10}$$

$$ {}_{t}\mathfrak{I}^{\gamma}_{T}\,{}^{C}_{t}\mathfrak{D}^{\gamma}_{T} y(t) = (-1)^n \left(y(t) - \sum_{m=0}^{n-1} \frac{(-1)^m y^{(m)}(T)}{m!}(T-t)^m \right). \tag{2.11}$$

In view of [7], equations (2.10) and (2.11) imply

$$ {}_{0}\mathfrak{I}^{\gamma}_{T}\,{}^{RC}_{0}\mathfrak{D}^{\gamma}_{T} y(t) = \frac{1}{2}\left({}_{0}\mathfrak{I}^{\gamma}_{t}\,{}^{C}_{0}\mathfrak{D}^{\gamma}_{t} + (-1)^n \; {}_{t}\mathfrak{I}^{\gamma}_{T}\,{}^{C}_{t}\mathfrak{D}^{\gamma}_{T}\right) y(t). \tag{2.12}$$

In particular, for $0 < \gamma < 1$ and $y(t) \in C(0,T)$, one has

$$ {}_{0}\mathfrak{I}^{\gamma}_{T}\,{}^{RC}_{0}\mathfrak{D}^{\gamma}_{T} y(t) = y(t) - \frac{1}{2}(y(0)+y(T)) \tag{2.13}$$

To end this section, we recall some basic properties of the p–Laplacian operator [26, 27].

Definition 5. The p–Laplacian operator is defined as

$$ \varphi_p(x) = |x|^{p-2}x, \; p > 1, $$

at which $\varphi_p{}^{-1} = \varphi_q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2. *Let $\varphi_p(x)$, $p \geq 2$, be p-Laplacian operator and $|x|, |y| \leq M$, then*

$$ |\varphi_p(x) - \varphi_p(y)| \leq (p-1)M^{p-2}|x-y|. $$

3. Main results

We consider the Banach space $C(J,\mathbb{R})$ of all real and continuous functions from J to $\mathbb{R}$ endowed with the supremum norm

$$\|\upsilon\| = \sup_{t\in J}|\upsilon(t)|.$$

Before investigating the main results, we consider the following auxiliary lemma.

Lemma 3. *Let $0<\alpha,\beta\leq 1$ and $\nu : J\to\mathbb{R}$ be a continuous function. Then the linear p–Laplacian* **BVP**

$$ {}^{RC}_{0}\mathfrak{D}^{\alpha}_{T}\left(\varphi_p\left({}^{RC}_{0}\mathfrak{D}^{\beta}_{T}\upsilon(t)\right)\right) = \nu(t), \quad t\in J \tag{3.1}$$

$$\begin{cases} a_1\varphi_p\left({}^{RC}_{0}\mathfrak{D}^{\beta}_{T}\upsilon(t)\right)|_{t=0} - b_1\left(\varphi_p\left({}^{RC}_{0}\mathfrak{D}^{\beta}_{T}\upsilon(t)\right)\right)'|_{t=0} = 0,\\ a_2\varphi_p\left({}^{RC}_{0}\mathfrak{D}^{\beta}_{T}\upsilon(t)\right)|_{t=T} + b_2\left(\varphi_p\left({}^{RC}_{0}\mathfrak{D}^{\beta}_{T}\upsilon(t)\right)\right)'|_{t=T} = 0, \end{cases} \tag{3.2}$$

$$\begin{cases} c_1\upsilon(0) - d_1\upsilon'(0) = 0,\\ c_2\upsilon(T) + d_2\upsilon'(T) = 0, \end{cases} \tag{3.3}$$

is equivalent to the integral equation

$$\begin{aligned} \upsilon(t) =& \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\varphi_q(h_\nu(s))\,ds + \frac{1}{\Gamma(\beta)}\int_t^T (s-t)^{\beta-1}\varphi_q(h_\nu(s))\,ds\\ &- \frac{d_1}{2c_1\Gamma(\beta-1)}\int_0^T s^{\beta-2}\varphi_q(h_\nu(s))\,ds - \frac{1}{\Gamma(\beta)}\int_0^T s^{\beta-1}\varphi_q(h_\nu(s))\,ds\\ &- \frac{d_2}{2c_2\Gamma(\beta-1)}\int_0^T (T-s)^{\beta-2}\varphi_q(h_\nu(s))\,ds - \frac{1}{\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}\varphi_q(h_\nu(s))\,ds, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} h_\nu(t) =& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\nu(s)ds + \frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1}\nu(s)ds\\ &- \frac{b_1}{2a_1\Gamma(\alpha-1)}\int_0^T s^{\alpha-2}\nu(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^T s^{\alpha-1}\nu(s)ds\\ &- \frac{b_2}{2a_2\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}\nu(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\nu(s)ds. \end{aligned} \tag{3.5}$$

Proof. Let $\omega = \varphi_p\left({}^{RC}_{0}\mathfrak{D}^{\beta}_{T}\upsilon(t)\right)$. Then the BVP (3.1)-(3.2) will be written as

$$\begin{cases} {}^{RC}_{0}\mathfrak{D}^{\alpha}_{T}\omega(t) &= \nu(t), \quad t\in J,\\ a_1\omega(0) - b_1\omega'(0) &= 0,\\ a_2\omega(T) + b_2\omega'(T) &= 0, \end{cases} \tag{3.6}$$

From (2.3), (2.13) and the first equality of (3.6), one has

$$\begin{aligned}\omega(t) &= \frac{1}{2}(\omega(0)+\omega(T)) + \frac{1}{\Gamma(\alpha)}\int_0^T |t-s|^{\alpha-1}\nu(s)ds \\ &= \frac{1}{2}(\omega(0)+\omega(T)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\nu(s)ds + \frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1}\nu(s)ds.\end{aligned} \tag{3.7}$$

Then

$$\omega'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\nu(s)ds - \frac{1}{\Gamma(\alpha-1)}\int_t^T (s-t)^{\alpha-2}\nu(s)ds.$$

Using the boundary conditions $a_1\omega(0)-b_1\omega'(0)=0$ and $a_2\omega(T)+b_2\omega'(T)=0$, we get

$$\omega(T) = -\frac{b_1}{a_1\Gamma(\alpha-1)}\int_0^T s^{\alpha-2}\nu(s)ds - \frac{2}{\Gamma(\alpha)}\int_0^T s^{\alpha-1}\nu(s)ds,$$

$$\omega(0) = -\frac{b_2}{a_2\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}\nu(s)ds - \frac{2}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\nu(s)ds.$$

Substituting the values of $\omega(0)$ and $\omega(T)$ into (3.7), we obtain that

$$\begin{aligned}\omega(t) = \varphi_p\left({}^{RC}_{\ 0}\mathfrak{D}^{\beta}_{T}\upsilon(t)\right) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\nu(s)ds + \frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1}\nu(s)ds \\ &\quad - \frac{b_1}{2a_1\Gamma(\alpha-1)}\int_0^T s^{\alpha-2}\nu(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^T s^{\alpha-1}\nu(s)ds \\ &\quad - \frac{b_2}{2a_2\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}\nu(s)ds \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\nu(s)ds.\end{aligned}$$

Thus,

$${}^{RC}_{\ 0}\mathfrak{D}^{\beta}_{T}\upsilon(t) = \varphi_q(h_\nu(t)),$$

where the function h_ν is defined in (3.5).

With a similar procedure, one can conclude that the **BVP**

$$\begin{cases} {}^{RC}_{\ 0}\mathfrak{D}^{\beta}_{T}\upsilon(t) &= \varphi_q(h_\nu(t)), \quad t\in J, \\ c_1\upsilon(0)-d_1\upsilon'(0) &= 0, \\ c_2\upsilon(T)+d_2\upsilon'(T) &= 0, \end{cases} \tag{3.8}$$

is equivalent to the integral equation

$$\begin{aligned}\upsilon(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\varphi_q(h_\nu(s))ds + \frac{1}{\Gamma(\beta)}\int_t^T (s-t)^{\beta-1}\varphi_q(h_\nu(s))ds \\ &\quad - \frac{d_1}{2c_1\Gamma(\beta-1)}\int_0^T s^{\beta-2}\varphi_q(h_\nu(s))ds - \frac{1}{\Gamma(\beta)}\int_0^T s^{\beta-1}\varphi_q(h_\nu(s))ds\end{aligned}$$

$$- \frac{d_2}{2c_2\Gamma(\beta-1)} \int_0^T (T-s)^{\beta-2} \varphi_q \left(h_\nu(s)\right) ds - \frac{1}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} \varphi_q \left(h_\nu(s)\right) ds.$$

That is the desired integral equation (3.4). The proof is completed. □

In view of Lemma 3, we deduce that the solution of the p–Laplacian **BVP** (1.1) is given by

$$\begin{aligned} \upsilon(t) = & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \varphi_q \left(H_{\mathcal{W},\upsilon}(s)\right) ds + \frac{1}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} \varphi_q \left(H_{\mathcal{W},\upsilon}(s)\right) ds \\ & - \frac{d_1}{2c_1\Gamma(\beta-1)} \int_0^T s^{\beta-2} \varphi_q \left(H_{\mathcal{W},\upsilon}(s)\right) ds - \frac{1}{\Gamma(\beta)} \int_0^T s^{\beta-1} \varphi_q \left(H_{\mathcal{W},\upsilon}(s)\right) ds \\ & - \frac{d_2}{2c_2\Gamma(\beta-1)} \int_0^T (T-s)^{\beta-2} \varphi_q \left(H_{\mathcal{W},\upsilon}(s)\right) ds \\ & - \frac{1}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} \varphi_q \left(H_{\mathcal{W},\upsilon}(s)\right) ds, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} H_{\mathcal{W},\upsilon}(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{W}(s,\upsilon(s))\, ds + \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} \mathcal{W}(s,\upsilon(s))\, ds \\ & - \frac{b_1}{2a_1\Gamma(\alpha-1)} \int_0^T s^{\alpha-2} \mathcal{W}(s,\upsilon(s))\, ds - \frac{1}{\Gamma(\alpha)} \int_0^T s^{\alpha-1} \mathcal{W}(s,\upsilon(s))\, ds \\ & - \frac{b_2}{2a_2\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2} \mathcal{W}(s,\upsilon(s))\, ds \\ & - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \mathcal{W}(s,\upsilon(s))\, ds. \end{aligned} \tag{3.10}$$

For fulfillment the main results, the following assumptions will be imposed.

(A1): The function $\mathcal{W} : J \times \mathbb{R} \to \mathbb{R}$ is continuous.

(A2): Two constants $L_{\mathcal{W}}, K_{\mathcal{W}} > 0$ exist such that

$$|\mathcal{W}(t,\upsilon_1) - \mathcal{W}(t,\upsilon_2)| \leq L_{\mathcal{W}} |\upsilon_1 - \upsilon_2|,$$

for each $t \in J$ and $\upsilon_1, \upsilon_2 \in \mathbb{R}$, along with

$$|\mathcal{W}(t,\upsilon)| \leq K_{\mathcal{W}}.$$

3.1. *Existence result via Schauder's fixed point theorem*

Lemma 4. *(Schauder's fixed point theorem)[16] Let $\mathcal{U}$ be a Banach space with $\mathcal{B} \subset \mathcal{U}$ closed, bounded and convex, and $\mathcal{P} : \mathcal{B} \to \mathcal{B}$ is completely continuous. Then $\mathcal{P}$ has a fixed point in $\mathcal{U}$.*

Theorem 1. *Assume that **(A1)–(A2)** hold. Then the p–Laplacian BVP (1.1) possesses at least one solution on J.*

Proof. The problem (1.1) will be transformed into the fixed point problem $\upsilon = \mathcal{P}\upsilon$, where the operator $\mathcal{P} : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is defined by

$$\begin{aligned}
(\mathcal{P}\upsilon)(t) = {} & \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)ds + \frac{1}{\Gamma(\beta)}\int_t^T (s-t)^{\beta-1}\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)ds \\
& - \frac{d_1}{2c_1\Gamma(\beta-1)}\int_0^T s^{\beta-2}\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)ds - \frac{1}{\Gamma(\beta)}\int_0^T s^{\beta-1}\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)ds \\
& - \frac{d_2}{2c_2\Gamma(\beta-1)}\int_0^T (T-s)^{\beta-2}\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)ds \\
& - \frac{1}{\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)ds,
\end{aligned} \tag{3.11}$$

where the function $H_{\mathcal{W},\upsilon}$ is defined in (3.10).

Define $\mathcal{B}_k = \{\upsilon \in C(J,\mathbb{R}) : \|\upsilon\| \le k\}$ with

$$k \ge \left[\frac{4T^{\beta}}{\Gamma(\beta+1)} + \frac{|d_1|T^{\beta-1}}{2|c_1|\Gamma(\beta)} + \frac{|d_2|T^{\beta-1}}{2|c_2|\Gamma(\beta)}\right]\varphi_q\left(M^*\right). \tag{3.12}$$

It is clear that the set $\mathcal{B}_k$ is a closed, bounded, convex subset of $C(J,\mathbb{R})$. Moreover, for $\upsilon(t) \in \mathcal{B}_k$, the operator $\mathcal{P}$ is well defined.

We first show that $\mathcal{P}(\mathcal{B}_k) \subset \mathcal{B}_k$. For each $t \in J$ and $\upsilon \in \mathcal{B}_k$, we get

$$\begin{aligned}
|(\mathcal{P}\upsilon)(t)| \le {} & \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)|ds + \frac{1}{\Gamma(\beta)}\int_t^T (s-t)^{\beta-1}|\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)|ds \\
& + \frac{|d_1|}{2|c_1|\Gamma(\beta-1)}\int_0^T s^{\beta-2}|\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)|ds + \frac{1}{\Gamma(\beta)}\int_0^T s^{\beta-1}|\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)|ds \\
& + \frac{|d_2|}{2|c_2|\Gamma(\beta-1)}\int_0^T (T-s)^{\beta-2}|\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)| \\
& + \frac{1}{\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}|\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)|ds.
\end{aligned} \tag{3.13}$$

From the assumption **(A2)**, one has

$$\begin{aligned}
|H_{\mathcal{W},\upsilon}(t)| \le {} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\mathcal{W}(s,\upsilon(s))|ds + \frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1}|\mathcal{W}(s,\upsilon(s))|ds \\
& + \frac{|b_1|}{2|a_1|\Gamma(\alpha-1)}\int_0^T s^{\alpha-2}|\mathcal{W}(s,\upsilon(s))|ds + \frac{1}{\Gamma(\alpha)}\int_0^T s^{\alpha-1}|\mathcal{W}(s,\upsilon(s))|ds \\
& + \frac{|b_2|}{2|a_2|\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}|\mathcal{W}(s,\upsilon(s))|ds \\
& + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|\mathcal{W}(s,\upsilon(s))|ds
\end{aligned}$$

$$\leq \left[\frac{t^{\alpha}+(T-t)^{\alpha}+2T^{\alpha}}{\Gamma(\alpha+1)}+\frac{|b_1|T^{\alpha-1}}{2|a_1|\Gamma(\alpha)}+\frac{|b_2|T^{\alpha-1}}{2|a_2|\Gamma(\alpha)}\right]K_{\mathcal{W}}$$
$$\leq \left[\frac{4T^{\alpha}}{\Gamma(\alpha+1)}+\frac{|b_1|T^{\alpha-1}}{2|a_1|\Gamma(\alpha)}+\frac{|b_2|T^{\alpha-1}}{2|a_2|\Gamma(\alpha)}\right]K_{\mathcal{W}}:=M^*. \tag{3.14}$$

Thus, by substitution the estimate M^* in (3.14) to (3.13), we get

$$\begin{aligned}|(\mathcal{P}\upsilon)(t)| &\leq \left[\frac{t^{\beta}+(T-t)^{\beta}+2T^{\beta}}{\Gamma(\beta+1)}+\frac{|d_1|T^{\beta-1}}{2|c_1|\Gamma(\beta)}+\frac{|d_2|T^{\beta-1}}{2|c_2|\Gamma(\beta)}\right]\varphi_q(M^*)\\ &\leq \left[\frac{4T^{\beta}}{\Gamma(\beta+1)}+\frac{|d_1|T^{\beta-1}}{2|c_1|\Gamma(\beta)}+\frac{|d_2|T^{\beta-1}}{2|c_2|\Gamma(\beta)}\right]\varphi_q(M^*)\\ &\leq k.\end{aligned}$$

Thus, (3.12) implies that $\mathcal{P}(\mathcal{B}_k)$ maps into $\mathcal{B}_k$.

Now, we show that $\mathcal{P}$ is completely continuous on $\mathcal{B}_k$.

First, from $\mathcal{P}(\mathcal{B}_k)\subset\mathcal{B}_k$, it follows that $\|\mathcal{P}\upsilon\|\leq k$ for any $\upsilon\in\mathcal{B}_k$, and so $\{\upsilon:\upsilon\in\mathcal{P}\mathcal{B}_k\}$ is uniformly bounded.

Second, we show that $\mathcal{P}(\mathcal{B}_k)$ is equicontinuous. For each $t_1,t_2\in J,\ t_1<t_2$ and $\upsilon\in\mathcal{B}_k$, using (3.14), we get

$$\begin{aligned}&|(\mathcal{P}\upsilon)(t_2)-(\mathcal{P}\upsilon)(t_1)|\\ &\leq \left|\frac{1}{\Gamma(\beta)}\int_0^{t_2}(t_2-s)^{\beta-1}\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)ds\right.\\ &\qquad\left.-\frac{1}{\Gamma(\beta)}\int_0^{t_1}(t_1-s)^{\beta-1}\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)ds\right|\\ &\quad+\left|\frac{1}{\Gamma(\beta)}\int_{t_2}^{T}(s-t_2)^{\beta-1}\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)ds\right.\\ &\qquad\left.-\frac{1}{\Gamma(\beta)}\int_{t_1}^{T}(s-t_1)^{\beta-1}\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)ds\right|\\ &\leq \frac{1}{\Gamma(\beta)}\int_0^{t_2}\left((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right)\left|\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)\right|ds\\ &\quad+\frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}(t_1-s)^{\beta-1}\left|\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)\right|ds\\ &\quad+\frac{1}{\Gamma(\beta)}\int_{t_2}^{T}\left((s-t_2)^{\beta-1}-(s-t_1)^{\beta-1}\right)\left|\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)\right|ds\\ &\quad+\frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}(s-t_1)^{\beta-1}\left|\varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)\right|ds\\ &\leq \frac{\varphi_q(M^*)}{\Gamma(\beta+1)}\left[\left(t_2^{\beta}-t_1^{\beta}\right)+\left((T-t_2)^{\beta}-(T-t_1)^{\beta}\right)+2(t_2-t_1)^{\beta}\right]\end{aligned}$$

The right-hand side of the above inequality tends to zero as $t_2 \to t_1$. So we conclude that $\mathcal{P}(\mathcal{B}_k)$ is equicontinuous. As a consequence of the Arzelà-Ascoli Theorem, we infer that $\mathcal{P}(\mathcal{B}_k)$ is relatively compact in $C(J,\mathbb{R})$.

Finally, we show that $\mathcal{P}$ is continuous. Let $\{\upsilon_n\}$ be a sequence in $\mathcal{B}_k$ such that $\upsilon_n \to \upsilon$, as $n \to \infty$.

$$\begin{aligned}
&|(\mathcal{P}\upsilon_n)(t) - (\mathcal{P}\upsilon)(t)| \\
&\quad \le \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |\varphi_q\left(H_{\mathcal{W},\upsilon_n}(s)\right) - \varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)| ds \\
&\qquad + \frac{1}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} |\varphi_q\left(H_{\mathcal{W},\upsilon_n}(s) - \varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)\right)| ds \\
&\qquad + \frac{|d_1|}{2|c_1|\Gamma(\beta-1)} \int_0^T s^{\beta-2} |\varphi_q\left(H_{\mathcal{W},\upsilon_n}(s)\right) - \varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)| ds \\
&\qquad + \frac{1}{\Gamma(\beta)} \int_0^T s^{\beta-1} |\varphi_q\left(H_{\mathcal{W},\upsilon_n}(s)\right) - \varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)| ds \\
&\qquad + \frac{|d_2|}{2|c_2|\Gamma(\beta-1)} \int_0^T (T-s)^{\beta-2} |\varphi_q\left(H_{\mathcal{W},\upsilon_n}(s)\right) - \varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)| ds \\
&\qquad + \frac{1}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} |\varphi_q\left(H_{\mathcal{W},\upsilon_n}(s)\right) - \varphi_q\left(H_{\mathcal{W},\upsilon}(s)\right)| ds. \qquad (3.15)
\end{aligned}$$

Again, by virtue of the assumption **(A2)**, one has

$$\begin{aligned}
&|H_{\mathcal{W},\upsilon_n}(t) - H_{\mathcal{W},\upsilon}(t)| \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\mathcal{W}(s,\upsilon_n(s)) - \mathcal{W}(s,\upsilon(s))| ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} |\mathcal{W}(s,\upsilon_n(s)) - \mathcal{W}(s,\upsilon(s))| ds \\
&\qquad + \frac{|b_1|}{2|a_1|\Gamma(\alpha-1)} \int_0^T s^{\alpha-2} |\mathcal{W}(s,\upsilon_n(s)) - \mathcal{W}(s,\upsilon(s))| ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_0^T s^{\alpha-1} |\mathcal{W}(s,\upsilon_n(s)) - \mathcal{W}(s,\upsilon(s))| ds \\
&\qquad + \frac{|b_2|}{2|a_2|\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2} |\mathcal{W}(s,\upsilon_n(s)) - \mathcal{W}(s,\upsilon(s))| ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |\mathcal{W}(s,\upsilon_n(s)) - \mathcal{W}(s,\upsilon(s))| ds \\
&\quad \le \left[\frac{t^\alpha + (T-t)^\alpha + 2T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_1| T^{\alpha-1}}{2|a_1|\Gamma(\alpha)} + \frac{|b_2| T^{\alpha-1}}{2|a_2|\Gamma(\alpha)} \right] \\
&\qquad \times \|\mathcal{W}(\cdot,\upsilon_n(\cdot)) - \mathcal{W}(\cdot,\upsilon(\cdot))\|
\end{aligned}$$

$$\leq \left[\frac{4T^{\alpha}}{\Gamma(\alpha+1)} + \frac{|b_1|T^{\alpha-1}}{2|a_1|\Gamma(\alpha)} + \frac{|b_2|T^{\alpha-1}}{2|a_2|\Gamma(\alpha)}\right] \times \|\mathcal{W}(\cdot,\upsilon_n(\cdot)) - \mathcal{W}(\cdot,\upsilon(\cdot))\|. \tag{3.16}$$

Therefore, the continuity of the function $\mathcal{W}$ implies that the function $H_{\mathcal{W},\upsilon}$ is also continuous. Hence, by referring to (3.15), we infer the continuity of $\mathcal{P}$.

In the light of the above analysis, we realize that $\mathcal{P}$ is completely continuous. In consequence of Schauder's fixed point theorem (Lemma 4), $\mathcal{P}$ has a fixed point in $C(J,\mathbb{R})$, that is, it is a solution of the p–Laplacian **BVP** (1.1). The proof is finished. □

3.2. *Uniqueness result via Banach's fixed point theorem*

Theorem 2. *Let* $1 < p < 2$. *Assume that the assumptions* **(A1)** *and* **(A2)** *are satisfied. Then the p–Laplacian* **BVP** *(1.1) possesses a unique solution on J as long as*

$$\Omega_{\alpha,\beta} := \left[\frac{4T^{\beta}}{\Gamma(\beta+1)} + \frac{|d_1|T^{\beta-1}}{2|c_1|\Gamma(\beta)} + \frac{|d_2|T^{\beta-1}}{2|c_2|\Gamma(\beta)}\right] \times \left[\frac{4T^{\alpha}}{\Gamma(\alpha+1)} + \frac{|b_1|T^{\alpha-1}}{2|a_1|\Gamma(\alpha)} + \frac{|b_2|T^{\alpha-1}}{2|a_2|\Gamma(\alpha)}\right](q-1)M^{*q-2}L_{\mathcal{W}} < 1. \tag{3.17}$$

Proof. We consider the operator $\mathcal{P} : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ defined by (3.11). We shall show that $\mathcal{P}$ is a contraction.

For each $t \in J$ and $\upsilon_1,\upsilon_2 \in C(J,\mathbb{R})$, using Lemma 2 and (3.14), we get

$$\begin{aligned}
&|(\mathcal{P}\upsilon_1)(t) - (\mathcal{P}\upsilon_2)(t)| \\
&\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|\varphi_q\left(H_{\mathcal{W},\upsilon_1}(s)\right) - \varphi_q\left(H_{\mathcal{W},\upsilon_2}(s)\right)|ds \\
&\quad + \frac{1}{\Gamma(\beta)}\int_t^T (s-t)^{\beta-1}|\varphi_q\left(H_{\mathcal{W},\upsilon_1}(s)\right) - \varphi_q\left(H_{\mathcal{W},\upsilon_2}(s)\right)|ds \\
&\quad + \frac{|d_1|}{2|c_1|\Gamma(\beta-1)}\int_0^T s^{\beta-2}|\varphi_q\left(H_{\mathcal{W},\upsilon_1}(s)\right) - \varphi_q\left(H_{\mathcal{W},\upsilon_2}(s)\right)|ds \\
&\quad + \frac{1}{\Gamma(\beta)}\int_0^T s^{\beta-1}|\varphi_q\left(H_{\mathcal{W},\upsilon_1}(s)\right) - \varphi_q\left(H_{\mathcal{W},\upsilon_2}(s)\right)|ds \\
&\quad + \frac{|d_2|}{2|c_2|\Gamma(\beta-1)}\int_0^T (T-s)^{\beta-2}|\varphi_q\left(H_{\mathcal{W},\upsilon_1}(s)\right) - \varphi_q\left(H_{\mathcal{W},\upsilon_2}(s)\right)|ds \\
&\quad + \frac{1}{\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}|\varphi_q\left(H_{\mathcal{W},\upsilon_1}(s)\right) - \varphi_q\left(H_{\mathcal{W},\upsilon_2}(s)\right)|ds \\
&\leq \frac{(q-1)M^{*q-2}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|H_{\mathcal{W},\upsilon_1}(s) - H_{\mathcal{W},\upsilon_2}(s)|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(q-1)M^{*q-2}}{\Gamma(\beta)}\int_t^T (s-t)^{\beta-1}|H_{\mathcal{W},\upsilon_1}(s)-H_{\mathcal{W},\upsilon_2}(s)|ds\\
&+\frac{|d_1|(q-1)M^{*q-2}}{2|c_1|\Gamma(\beta-1)}\int_0^T s^{\beta-2}|H_{\mathcal{W},\upsilon_1}(s)-H_{\mathcal{W},\upsilon_2}(s)|ds\\
&+\frac{(q-1)M^{*q-2}}{\Gamma(\beta)}\int_0^T s^{\beta-1}|H_{\mathcal{W},\upsilon_1}(s)-H_{\mathcal{W},\upsilon_2}(s)|ds\\
&+\frac{|d_2|(q-1)M^{*q-2}}{2|c_2|\Gamma(\beta-1)}\int_0^T (T-s)^{\beta-2}|H_{\mathcal{W},\upsilon_1}(s)-H_{\mathcal{W},\upsilon_2}(s)|ds\\
&+\frac{(q-1)M^{*q-2}}{\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}|H_{\mathcal{W},\upsilon_1}(s)-H_{\mathcal{W},\upsilon_2}(s)|ds. \qquad (3.18)
\end{aligned}
$$

Using the assumption **(A2)**, one obtain

$$
\begin{aligned}
&|H_{\mathcal{W},\upsilon_1}(t)-H_{\mathcal{W},\upsilon_2}(t)|\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\mathcal{W}(s,\upsilon_1(s))-\mathcal{W}(s,\upsilon_2(s))|ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1}|\mathcal{W}(s,\upsilon_1(s))-\mathcal{W}(s,\upsilon_2(s))|ds\\
&\quad+\frac{|b_1|}{2|a_1|\Gamma(\alpha-1)}\int_0^T s^{\alpha-2}|\mathcal{W}(s,\upsilon_1(s))-\mathcal{W}(s,\upsilon_2(s))|ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^T s^{\alpha-1}|\mathcal{W}(s,\upsilon_1(s))-\mathcal{W}(s,\upsilon_2(s))|ds\\
&\quad+\frac{|b_2|}{2|a_2|\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}|\mathcal{W}(s,\upsilon_1(s))-\mathcal{W}(s,\upsilon_2(s))|ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|\mathcal{W}(s,\upsilon_1(s))-\mathcal{W}(s,\upsilon_2(s))|ds\\
&\leq \left[\frac{4T^{\alpha}}{\Gamma(\alpha+1)}+\frac{|b_1|T^{\alpha-1}}{2|a_1|\Gamma(\alpha)}+\frac{|b_2|T^{\alpha-1}}{2|a_2|\Gamma(\alpha)}\right]L_{\mathcal{W}}\|\upsilon_1-\upsilon_2\|. \qquad (3.19)
\end{aligned}
$$

By substitution from (3.19) into (3.18), we obtain that

$$\|\mathcal{P}\upsilon_1-\mathcal{P}\upsilon_2\|\leq \Omega_{\alpha,\beta}\|\upsilon_1-\upsilon_2\|.$$

By virtue of (3.17), we infer that $\mathcal{P}$ is a contraction. Hence, by the Banach's fixed point theorem, $\mathcal{P}$ has a unique fixed point which is the unique solution of the p–Laplacian **BVP** (1.1). The proof is completed. □

4. Explanatory example

In this section, we examine the applicability of the theoretical results.

Example 1. Consider the following **BVP**:

$$\begin{cases} {}^{RC}_{0}\mathfrak{D}^{\frac{1}{2}}_{1}\left(\varphi_{\frac{4}{3}}\left({}^{RC}_{0}\mathfrak{D}^{\frac{3}{4}}\upsilon(t)\right)\right) = \frac{1}{(1+5e^{t})^{2}}\frac{|\upsilon(t)|}{1+|\upsilon(t)|}, & t\in[0,1],\\ \varphi_{\frac{4}{3}}\left({}^{RC}_{0}\mathfrak{D}^{\frac{3}{4}}_{1}\upsilon(t)\right)|_{t=0} - \left(\varphi_{\frac{4}{3}}\left({}^{RC}_{0}\mathfrak{D}^{\frac{3}{4}}_{1}\upsilon(t)\right)\right)'|_{t=0} & = 0,\\ \varphi_{\frac{4}{3}}\left({}^{RC}_{0}\mathfrak{D}^{\frac{3}{4}}_{1}\upsilon(t)\right)|_{t=1} + \left(\varphi_{\frac{4}{3}}\left({}^{RC}_{0}\mathfrak{D}^{\beta}_{1}\upsilon(t)\right)\right)'|_{t=1} & = 0,\\ \upsilon(0)-\upsilon'(0) & = 0,\\ \upsilon(1)+\upsilon'(1) & = 0, \end{cases} \tag{4.1}$$

Here, $\alpha=\frac{1}{2}, \beta=\frac{3}{4}, T=1, p=\frac{4}{3}, q=4, a_1=a_2=b_1=b_2=c_1=c_2=d_1=d_2=1$. Set

$$\mathcal{W}(t,\upsilon)=\frac{1}{(1+5e^{t})^{2}}\frac{|\upsilon|}{1+|\upsilon|}.$$

For each $t\in[0,1]$ and $\upsilon\in\mathbb{R}$, we get $|\mathcal{W}(t,\upsilon)|\leq\frac{1}{36}$.

Additionally, for each $t\in[0,1]$ and $\upsilon_1,\upsilon_2\in\mathbb{R}$, one has

$$|\mathcal{W}(t,\upsilon_1)-\mathcal{W}(t,\upsilon_1)|\leq\frac{1}{36}|\upsilon_1-\upsilon_2|.$$

Therefore, the assumptions **(A1)** and **(A2)** hold true with $L_{\mathcal{W}}=K_{\mathcal{W}}=\frac{1}{36}$.

Furthermore, one obtain that $M^{*}\approx 0.1410473959$ and $\Omega_{\alpha,\beta}\approx 0.0435<1$.

Hence, by Theorem 2, the p–Laplacian BVP (4.1) has a unique solution on $[0,1]$.

Finally, by virtue of Definition 5, the inequality (3.12) proves that there exists a real number k satisfies

$$k\geq\left[\frac{4}{\Gamma(7/4)}+\frac{1}{\Gamma(3/4)}\right]\times(0.1410473959)^{3},$$

which implies that $k\geq 0.01450$. Hence, by Theorem 1, the p–Laplacian BVP (4.1) has a solution on $[0,1]$.

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