Miskolc Mathematical Notes
Vol. 23 (2022), No. 1, pp. 339–362

HU ISSN 1787-2405
DOI: 10.18514/MMN.2022.3787

NEW GEOMETRIC VIEWPOINTS TO CHEN CHAOTIC SYSTEM

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Received 29 March, 2021

Abstract. This paper presents new geometric viewpoints to Chen chaotic system. Firstly, the existence of two nontransverse homoclinic orbits in Chen system is rigorously proved beyond the classical parameters. Secondly, combined with the theory of tangent bundle, a new geometric viewpoint is given to explore chaos mechanism of Chen system.

The fundamental geometric definition of tangent bundle and the essential role of nonlinear connection between the tangent space and the base space are described. By introducing the geometrical viewpoints of second order system governed by Lie-Poisson equation, some geometric invariants of Chen system can be obtained. Furthermore, the torsion tensor as one of the geometric invariants is obtained, and it gives the geometrical interpretation to the chaotic behaviour of Chen system.

Finally, the torsion tensor of Chen system and Lorenz system are also compared. The obtaining results show that torsion tensor change will lead the Chen system from periodic to chaotic, which is not found in Lorenz system.

2010 *Mathematics Subject Classification:* 34A26; 34A34; 34C37; 53B40; 53C60

Keywords: Chen system, homoclinic orbit, torsion tensor, tangent bundle, chaos

1. INTRODUCTION

In the second half of the 20th century, nonlinear science has achieved unprecedented vigorous development, which is regarded as the ”third revolution” in the history of natural science. Chaos as one of the subjects of nonlinear science has been always widely concerned, which has been widely applied in secure communication, biological science, engineering technology and other fields. The Lorenz system is the first classical chaos system that was proposed by the famous meteorologist E. N. Lorenz in 1963 [19], which has a fundamental significance in the development of chaos. Since then, a increase number of literature are devoted to the study of Lorenz system and Lorenz-like system (e.g. Lorenz system [19], Chen system [4], Lorenz system family [3]). Among them, the most peculiar is the Chen system with a similar

This work is supported by the National Natural Science Foundation of China (Grant No. 11961074), Natural Science Foundation of Guangxi Province (Grant No. 2018GXNSFDA281028), the High Level Innovation Team Program from Guangxi Higher Education Institutions of China (Document No. [2018] 35) and the Senior Talent Research Foundation of Yulin Normal University (Grant No. G2019ZK51).

but topologically nonequivalent to Lorenz system [4, 5, 23], which is the dual system of Lorenz system [3]. It was discovered by G.R. Chen in 1999 and has more complex topological structure and dynamic behaviour than Lorenz system. Chen system is given by

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) \\ \dot{x}_2 = (c-a)x_1 + cx_2 - x_1x_3 \\ \dot{x}_3 = x_1x_2 - bx_3. \end{cases} \tag{1.1}$$

where a, b, c are positive real numbers. When $a = 35$, $b = 3$ and $c = 28$, the corresponding typical numerical chaotic attractor is shown in Fig. 1.

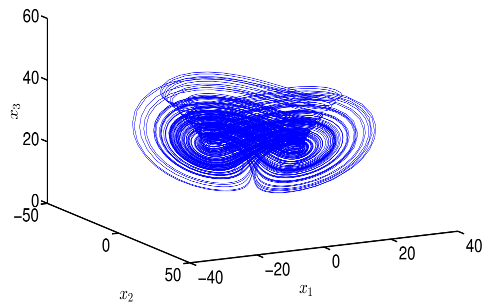


FIGURE 1. The chaotic attractor of Chen system (1.1) with $(a,b,c) = (35,3,28)$ and initial conditions $(x_1(0), x_2(0), x_3(0)) = (-10, 0, 37)$.

There have been already a sea of investigations and studies of Chen system (1.1), for example, bifurcation analyses [14, 20, 23], global analyses [17, 21, 30], the compound topological structure[16, 30], chaotic attractors [23, 31]. In particular, the literature [7], based on KCC-theory, revealed that the underlying chaotic evolution of Chen system by quantitatively described the behavior of the second KCC-invariant.

On the one hand, homoclinic orbits and heteroclinic orbits are important concepts in the study of the bifurcation of vector fields and chaos. Many chaotic behaviours of a complex system are related to the existence or nonexistence of these kinds of orbits in the system. In the literatures [10, 11, 13, 18], the existence of nontransverse homoclinic orbits of Chen system has been well proved. Using Fishing principle, which ideas are from [17], the obtained results showed that when $a > c$, Chen system either has homoclinic orbits with $2a > b$ or has no homoclinic orbits with $b > 2a$. But what happens when $c > a$, which is rarely reported. On the other hand, nowadays, differential geometry has become a practical tool in the study of the complexity of typical chaotic systems. Some good results have been reported in analysing the trajectory behaviour of chaotic systems from the viewpoint of tangent bundle. The literature

[22] has successfully described the chaotic behaviour of Lorenz system on the tangent space. In [27], the aperiodic behaviour of Rikitake system can be expressed by torsion tensor based on the geometrical unified theory. Moreover, the literature [28] shown that a torsion tensor as one of the geometrical invariants, relates to the chaotic behaviour characterized by the Rayleigh number, and results from the decomposition from the second order system to the tangent space (state space) and base space (configuration space), respectively. The above studies indicate that it is feasible and beneficial to discuss the dynamic behaviour of chaotic systems on the tangent bundle. To our knowledge, the dynamic analysis of the Chen system on the tangent bundle has rarely been studied in the literature.

In this paper, on the one hand, the existence of nontransverse homoclinic orbits in Chen system under specific parameter conditions with $c > a$ has been rigorously proven.

On the other hand, we mainly focus on analyzing the relationship between the characteristics of orbit of Chen system and the torsion tensor from the viewpoint of tangent bundle. According to geometric theory, the solution of Chen system can be regarded as a trajectory on the tangent bundle, from which we can calculate the geometric invariants of the Chen system. The advantage of this method is that it uses geometric invariants to qualitatively explore the chaos mechanism of the Chen system in 6+1 dimensional tangent space, and the results obtained are more detailed and intuitive than in original 3+1 dimensional Euclidean space. Moreover, the changing trend and degree of the system trajectory can be observed more clearly in the tangent space. Different from the way of parameter selection of Lorenz system [28], one has been considered selecting three regions near the classical value of Chen system to discuss the relationship between the change of torsion tensor and the dynamic behaviour. The differences of dynamics and torsion tensor between Lorenz system and Chen system has been explored, the obtained results has shown that Chen system has more complex dynamics than Lorenz system, which can be explained geometrically by the change of torsion tensor.

These results may give a contribution in an understanding of the geometric mechanism for chaos of the system.

The present paper is organized as follows. In Section 2, geometric definitions for tangent bundle and geometric quantities in the non-holonomic system are described. In Section 3, the existence of two nontransverse homoclinic orbits in Chen system is rigorously proved. In Section 4, the geometric quantities of Chen system are given and a relation between the chaotic behaviour of Chen system and the torsion tensor is discussed. Moreover, the torsion tensors of Lorenz and Chen systems are compared. In Section 5, the conclusion of this paper is given.

2. TANGENT BUNDLE VIEWPOINT OF THE CHEN SYSTEM

In this section, we introduce the related definitions, notions and formulas of the tangent bundle [28], including nonlinear connection, connection coefficient, torsion tensors, curvature tensors, et.al. One will need them in Section 4. In this article, Einstein's summation convention is used.

In the following, a non-holonomic system and a holonomic system are given by $(\mathbf{x})$-field and $(\dot{\mathbf{p}})$-field, respectively. In addition, the coordinate indices in the non-holonomic system is represented by α, β, γ, δ, ε, $\cdots$, while the coordinate indexes in the holonomic system is represented by i, j, k, $\cdots$.

2.1. *Geometric definitions for tangent bundle*

Let M be the m-dimensional C^k manifold, $T_p(M)$ be the tangent space of M at point p and mark $T(M) = \bigcup_{P\in M} T_p(M) = \{X_p \in T_p(M) \mid P \in M\}$. According to the definition in [1], $(T(M), \pi, M)$, or simply $T(M)$, is called the tangent bundle on M. π is called the natural mapping. $T_p(M)$ is the fiber of $T(M)$ at point p.

Firstly, the coordinate systems of holonomic system are defined on a tangent bundle, and the geometric descriptions are obtained. Then, the coordinates on a base manifold M and the fiber coordinates are denoted by (p^i) and $(\dot{p}^i) = (\mathrm{d}p^i/dt)$, respectively. We also call that the manifold are spanned by the coordinates (p^i) and $(\dot{p}^i)$, which are regarded as $(\mathbf{p})$-field and $(\dot{\mathbf{p}})$-field, respectively. Then, $(p^i,\dot{p}^i)$ represents the coordinates that they are defined on the total space or the tangent bundle TM.

In order to illustrate the essence of the nonlinear connection, a definition is introduced as follows.

Definition 1 ([2]). A non-linear connection on the tangent bundle TM is a subbundle (HTM, τ_H, TM) of the tangent bundle (TTM, τ, TM) such that on fibers, we have

$$T_uT(M) = H_uTM \oplus V_uTM, \forall u \in TM, \tag{2.1}$$

where H_uTM and V_uTM are the horizontal distribution and the vertical distribution, respectively. $\oplus$ is the Whitney sum.

Thus, a non-linear connection N on TM induces the tangent space $T_uT(M)$ to produce as the direct sum decomposition of relation (2.1). For a given non-linear connection (HTM) or N, we always have a natural basis and its dual basis to adapt the relation (2.1). They are given as following

$$(\mathrm{d}p^i, \delta\dot{p}^i) = (\mathrm{d}p^i, \mathrm{d}\dot{p}^i + N^i_j \mathrm{d}p^j), \tag{2.2}$$

$$\left(\frac{\delta}{\delta p^i}, \frac{\partial}{\partial \dot{p}^i}\right) = \left(\frac{\partial}{\partial p^i} - N^j_i \frac{\partial}{\partial \dot{p}^j}, \frac{\partial}{\partial \dot{p}^i}\right), \tag{2.3}$$

where N^i_j is the component of the nonlinear connection.

According to the result in the literature [2], there is canonical isomorphism between two tangent spaces T_pM and T_qM at $p, q \in M$, which will be called a parallel transport it is equivalent to existing a linear connection on the manifold M. This means that we can define a parallel transport between T_uTM and T_vTM for two points $u, v \in TM$ that preserves the above decomposition of (2.1). The linear connection, corresponding to such a parallel transport, is called a Finsler connection (or a N-linear connection) on TM. These functions $N^i_j, F^i_{jk}, C^i_{jk}$ are called the local coefficients of a Finsler connection $D\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$.

With respect to the adapted frames (2.2) and (2.3), the connection coefficients F^i_{jk} and C^i_{jk} are called horizontal and vertical connection coefficients, respectively [2], and are denoted by (F^i_{jk}, C^i_{jk}) on TM. These connections preserve the horizontal and the vertical distributions, and are given by the covariant derivative for an arbitrary vector field V^i [2]:

$$\begin{aligned} \mathrm{D}V^i &= \mathrm{d}V^i + V^j(F^i_{jk}\mathrm{d}p^k + C^i_{jk}\mathrm{D}\dot{p}^k). \\ \mathrm{D}\dot{p}^i &= \mathrm{d}\dot{p}^i + N^i_j\mathrm{d}p^j. \end{aligned} \tag{2.4}$$

In this study, Chen system is regarded as a system of second order differential equations. Then, by using equations (2.7) and (2.8) in section 2.2, the second-order differential equations can be related to the Berwald connection[1]. Thus, in this article, one considers that the connection coefficients F^i_{jk} and C^i_{jk} of the Chen system are the Berwald connection. A Finsler connection $D\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ is reduced to a Berwald connection when $C^i_{jk} = 0$, and sometimes is represented by $(F^i_{jk}, N^i_j, 0)$. Moreover, in this case, F^i_{jk} is a function of p^i or constant.

2.2. *Geometric quantities in the non-holonomic system*

The equation of motion of the differential equation is given by

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = k_{(i)}\dot{p}^i, \tag{2.5}$$

$$\frac{\mathrm{d}\dot{p}^i}{\mathrm{d}t} + 2G^i(p^j, \dot{p}^j) = 0, \tag{2.6}$$

where p^i and $\dot{p}^i$ can be interpreted geometrically as the coordinates of the base space and tangent space, respectively. G^i is a smooth function and $k_{(i)}$ is a constant, where $k_{(i)}\dot{p}^i$ does not sum.

Introducing the non-holonomic transformation defined on TM as follows

$$p^\alpha = p^\alpha(p^i), \tag{2.7}$$

$$x^\alpha = X^\alpha_i \dot{p}^i, \ \dot{p}^i = X^i_\alpha x^\alpha, \tag{2.8}$$

where $X_j^\alpha X_\beta^j = \delta_\beta^\alpha$, $X_\alpha^i = (X_1^i, X_2^i, X_3^i)$, and $X_i^\alpha = (X_i^1, X_i^2, X_i^3)$. Using equations (2.7) and (2.8), equations (2.5) and (2.6) are converted to

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = X_\alpha^i x^\alpha, \tag{2.9}$$

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} + 2\bar{G}^\alpha(p^\beta, x^\gamma) = 0. \tag{2.10}$$

Then, taking the derivation of both sides of Eq. (2.8) with respect to t, and comparing the obtained result with Eq. (2.10) to get

$$\bar{G}^\alpha = X_\alpha^i G^i - \frac{1}{2}\frac{\partial X_\alpha^i}{\partial p^j}\dot{p}^i\dot{p}^j. \tag{2.11}$$

The nonlinear connection and the connection coefficient are defined by the function $\bar{G}^\alpha$ as

$$\bar{N}_\beta^\alpha = \frac{\partial \bar{G}^\alpha}{\partial x^\beta}, \bar{F}_{\beta\gamma}^\alpha = \frac{\partial^2 \bar{G}^\alpha}{\partial x^\beta \partial x^\gamma}. \tag{2.12}$$

The paths of equations(2.9) and (2.10) are defined by

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} + \bar{N}_\beta^\alpha x^\beta + \bar{N}_0^\alpha = 0, \tag{2.13}$$

here, $\bar{N}_0^\alpha = 2\bar{G}^\alpha - \bar{N}_\beta^\alpha x^\beta$ is called the first invariant which indicates external force. If the nonlinear connection is first-order homogeneous in x^α, then the nonlinear connection becomes $\bar{N}_\beta^\alpha = \bar{F}_{\beta\gamma}^\alpha x^\gamma$. The torsion tensor as one of the geometric invariants is defined by

$$\bar{R}_{\beta\gamma}^\alpha = \frac{\delta \bar{N}_\beta^\alpha}{\delta p^\gamma} - \frac{\delta \bar{N}_\gamma^\alpha}{\delta p^\beta}. \tag{2.14}$$

According to the literature [1], similarly to the holonomic system, a Finsler connection $D\Gamma = (\bar{F}_{\beta\gamma}^\alpha, \bar{N}_\beta^\alpha, \bar{C}_{\beta\gamma}^\alpha)$ has only three local components $\bar{R}_{\delta\beta\gamma}^\alpha, \bar{P}_{\delta\beta\gamma}^\alpha$ and $\bar{S}_{\delta\beta\gamma}^\alpha$, which are given by

$$\begin{aligned}
\bar{R}_{\delta\beta\gamma}^\alpha &= \frac{\delta \bar{F}_{\delta\beta}^\alpha}{\delta p^\gamma} - \frac{\delta \bar{F}_{\delta\gamma}^\alpha}{\delta p^\beta} + \bar{F}_{\delta\beta}^\varepsilon \bar{F}_{\varepsilon\gamma}^\alpha - \bar{F}_{\delta\gamma}^\varepsilon \bar{F}_{\varepsilon\beta}^\alpha + \bar{C}_{\delta\varepsilon}^\alpha \bar{R}_{\beta\gamma}^\varepsilon, \\
\bar{P}_{\delta\beta\gamma}^\alpha &= \frac{\partial \bar{F}_{\delta\beta}^\alpha}{\partial x^\gamma} - \frac{\delta \bar{C}_{\delta\gamma}^\alpha}{\delta p^\beta} + \bar{F}_{\delta\beta}^\varepsilon \bar{C}_{\varepsilon\gamma}^\alpha + \bar{F}_{\gamma\beta}^\varepsilon \bar{C}_{\delta\varepsilon}^\alpha + \bar{C}_{\delta\varepsilon}^\alpha \bar{P}_{\alpha\gamma}^\varepsilon, \\
\bar{S}_{\delta\beta\gamma}^\alpha &= \frac{\partial \bar{C}_{\delta\beta}^\alpha}{\partial x^\gamma} - \frac{\partial \bar{C}_{\delta\gamma}^\alpha}{\partial x^\beta} + \bar{C}_{\delta\beta}^\varepsilon \bar{C}_{\varepsilon\gamma}^\alpha - \bar{C}_{\delta\gamma}^\varepsilon \bar{C}_{\varepsilon\beta}^\alpha.
\end{aligned}$$

When $\bar{C}_{\beta\gamma}^\alpha = 0$ in a Finsler connection $D\Gamma = (\bar{F}_{\beta\gamma}^\alpha, \bar{N}_\beta^\alpha, \bar{C}_{\beta\gamma}^\alpha)$, this Finsler connection is reduced to a Berwald connection. Therefore, the Berwald connection $D\Gamma =$

$(\bar{F}^{\alpha}_{\beta\gamma}, \bar{N}^{\alpha}_{\beta}, 0)$ has only two nonzero components of curvature $\bar{P}^{\alpha}_{\delta\beta\gamma}$ and $\bar{R}^{\alpha}_{\delta\beta\gamma}$, and these are given by

$$\bar{P}^{\alpha}_{\delta\beta\gamma} = \frac{\partial \bar{F}^{\alpha}_{\delta\beta}}{\partial x^{\gamma}}, \tag{2.15}$$

$$\bar{R}^{\alpha}_{\delta\beta\gamma} = \frac{\delta \bar{F}^{\alpha}_{\delta\beta}}{\delta p^{\gamma}} - \frac{\delta \bar{F}^{\alpha}_{\delta\gamma}}{\delta p^{\beta}} + \bar{F}^{\varepsilon}_{\delta\beta}\bar{F}^{\alpha}_{\varepsilon\gamma} - \bar{F}^{\varepsilon}_{\delta\gamma}\bar{F}^{\alpha}_{\varepsilon\beta}. \tag{2.16}$$

3. NONTRANSVERSE HOMOCLINIC ORBITS

In this section, one investigates homoclinic orbits in Chen system (1.1) within a specific parameter range by the generalized Melnikov method, which was developed by Wiggins and Holmes [25]. It's important to point out that a rigorous proof that the existence of periodic orbits and homoclinic chaos in the periodic forced Chen system under certain parameter conditions has been studied in literature [6]. There are similar results in Lorenz system [15], the diffusionless Lorenz system [24] and a special system with two stable focal points [29], etc. It should be pointed out that these focus on the case that $c > a$, which is different from the previous rich results [10, 11, 13, 17, 18] that $c < a$.

Under the transformation $z \to z - a + c$, system (1.1) becomes

$$\begin{cases} \dot{x} = a(y-x) \\ \dot{y} = -xz + cy \\ \dot{z} = xy - bz - b(c-a). \end{cases} \tag{3.1}$$

Introducing the rescaling $x \to \frac{ax}{\varepsilon}$, $y \to \frac{ay}{-\varepsilon^2}$, $z \to \frac{az}{-\varepsilon^2}$, $t \to \varepsilon t$, $\varepsilon = \frac{a}{\sqrt{b(c-a)}}$, system (3.1) becomes

$$\begin{cases} \dot{x} = -y - \varepsilon x \\ \dot{y} = -xz + \varepsilon \frac{c}{a} y \\ \dot{z} = xy + \varepsilon(-\frac{b}{a}z + 1). \end{cases} \tag{3.2}$$

To ensure the sufficient smallness of ε, one studys study this dependency here. When the parameters $c \gg a$, one obtains that $0 < \varepsilon \ll 1$. Thus, the approximate homoclinic orbits are obtained as analytical solutions of the Chen system at the value $\varepsilon = 0$. Moreover, unlike the Lorenz system, which is known to be Levinson dissipative (i.e there exists a global bounded absorbing set containing global Lorenz attractor) for all positive values of parameters [9], the Chen system is Levinson dissipative only when $a + b > c > 0$ [12] and in general could have unbounded trajectories tending to infinity. The dissipative condition of the Chen system should be taken into account when studying possible existence of homoclinic orbits, which is a crucial difference between Lorenz system and Chen system. When the parameters $c \gg a$, Chen system is not disspative in the sense of Levinson dissipative, In other words, Chen system has unbounded trajectories tending to infinity. That is, the above definition of ε is reasonable.

When $\varepsilon = 0$, system (3.2) can be seen as a three dimensional generalized Hamiltonian system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & -y \\ -z & 0 & 0 \\ y & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ 0 \\ 1 \end{pmatrix} = J\begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \\ \frac{\partial H}{\partial z} \end{pmatrix}$$

with Hamiltonian function

$$H(x,y,z) = z + \frac{x^2}{2} = A$$

and Casimir function

$$C(x,y,z) = y^2 + z^2 = r^2.$$

Let $r > 0$ and make a polar-coordinate transformation

$$\begin{cases} x = x \\ y = r\cos(\theta - \frac{\pi}{2}) \\ z = r\sin(\theta - \frac{\pi}{2}). \end{cases} \tag{3.3}$$

Then, system (3.2) is transformed into

$$\begin{cases} \dot{\theta} = x + \varepsilon(\frac{c}{a}\sin\theta\cos\theta + \frac{b}{a}\sin\theta\cos\theta + \frac{\sin\theta}{r}) \\ \dot{x} = -r\sin\theta - \varepsilon x \\ \dot{r} = \varepsilon(\frac{rc}{a}\sin^2\theta - \frac{br}{a}\cos^2\theta - \cos\theta). \end{cases} \tag{3.4}$$

Then, the following conclusion holds.

Theorem 1. *Let a, b, c are positive real numbers and* $3a + c > b/2$. *If* $c > a$, *for* ε *sufficiently small near the two homoclinic orbits* $\Gamma_{h_\pm}$ *of system* $(3.4)_{\varepsilon=0}$, *system (3.4) possesses two nontransverse homoclinnic orbits near* $y^2 + z^2 = r_*^2$, *where*

$$r_* = \frac{3a}{2(3a+c) - b}.$$

Proof. System $(3.4)_{\varepsilon=0}$ is a Hamiltonian system, and its Hamiltonian function is $H(x,r,\theta)$, where $H(x,r,\theta) = \frac{x^2}{2} - r\cos\theta = A$. Therefore, when $A = r$, the two homoclinic orbits $\Gamma^1_{h_\pm}$ of the system $(3.4)_{\varepsilon=0}$ are connected to the saddle point $(r,\pi,0)$, and the parameter expression is

$$\begin{cases} \theta_h(t) = \pm 2\arctan(\sinh\sqrt{r}t) \\ x_h(t) = \pm 2\sqrt{r}(\operatorname{sech}\sqrt{r}t). \end{cases} \tag{3.5}$$

From (3.4) and (3.5), Melnikov function of (3.4)

$$M(r) = \int_{-\infty}^{+\infty}\left[-x_h^2(t) + \frac{br}{a}\sin^2\theta_h(t)\cos\theta_h(t) + \sin^2\theta_h(t) + \cos^2\theta_h(t)\right]\mathrm{d}t$$
$$+ \int_{-\infty}^{+\infty}\left[\frac{br}{a}\cos^3\theta_h(t) - \frac{cr}{a}\sin^2\theta_h(t) + \cos\theta_h(t) + \frac{br}{a}\cos^2\theta_h(t)\right]\mathrm{d}t$$

$$= -8\sqrt{r} + \frac{4b\sqrt{r}}{3a} - \frac{8c\sqrt{r}}{3a} + \frac{4}{\sqrt{r}}.$$

Thus, $M(r_*) = 0$, $\frac{\partial M}{\partial r} |_{r=r_*} \neq 0$, where $r = r_* = \frac{3a}{2(3a+c)-b}$. For $3a+c > b/2$, the results can be obtained from Lemma 4.2 in the liberature [25]. Namely, for ε sufficiently small near the two homoclinic orbits $\pm\Gamma^1_{h_\pm}$ of system $(3.4)_{\varepsilon=0}$, system (3.4) possesses two non-transverse homoclinic orbits. □

Remark 1. From [15], it is known that if $r \gg \sigma$, near the two homoclinic orbits $\Gamma^1_{h_\pm}$ of the perturbed system $(2.5)_{\varepsilon=0}$ of the Lorenz system on the phase cylinder $w^2+z^2=1$, there exist two homoclinic orbits $\Gamma^1_{h_\pm}$ connecting the saddle point $(1,0,0)$ of $(2.5)_{\varepsilon=0}$ when $\sigma = \frac{1}{3}$. In this paper, the condition for the existence of non-transverse homoclinic orbit in the Chen system is stronger than that in the Lorenz system, which can be revealed from the fact that the dissipation condition of the Chen system is stronger than that of the Lorenz system.

4. Chaotic behavior and geometric quantities

In this section, from the geometric viewpoint of tangent bundle, we analyze the chaotic complexity of Chen system. The relevant preliminary is briefly introduced in Section 2. These may give a contribution in an understanding of the geometric mechanism for chaotic systems.

4.1. *Convert to a second order differential equation*

Rewrite Chen system into a set of second order differential equations. Firstly, system (2.8) is equivalent to

$$\begin{cases} x^1 = X^1_1\dot{p}^1 + X^1_2\dot{p}^2 + X^1_3\dot{p}^3 = \dot{p}^1 \\ x^2 = X^2_1\dot{p}^1 + X^2_2\dot{p}^2 + X^2_3\dot{p}^3 = \dot{p}^2 \\ x^3 = X^3_1\dot{p}^1 + X^3_2\dot{p}^2 + X^3_3\dot{p}^3 = \dot{p}^3. \end{cases} \tag{4.1}$$

where $X^\alpha_j X^j_\beta = \delta^\alpha_\beta$. Secondly, making the following transformation to system (1.1)

$$x_1 \to x^1, x_2 \to x^2, x_3 \to x^3. \tag{4.2}$$

Then system (1.1) can be rewritten as

$$\begin{cases} \ddot{p}^1 = a(\dot{p}^2 - \dot{p}^1) \\ \ddot{p}^2 = (c-a)\dot{p}^1 + c\dot{p}^2 - \dot{p}^1\dot{p}^3 \\ \ddot{p}^3 = \dot{p}^1\dot{p}^2 - b\dot{p}^3. \end{cases} \tag{4.3}$$

From (4.1), one sees that system (4.3) is equivalent to

$$\begin{cases} \dot{x}^1 = a(x^2 - x^1) \\ \dot{x}^2 = (c-a)x^1 + cx^2 - x^1x^3 \\ \dot{x}^3 = x^1x^2 - bx^3. \end{cases} \tag{4.4}$$

Introduce a linear substitution of variables to system (4.4)

$$x^1 \to x^1, x^2 \to x^2, x^3 \to x^3 + c. \tag{4.5}$$

Then system (4.4) can be rewritten as follows

$$\begin{cases} \frac{dx^1}{dt} = ax^2 - ax^1 \\ \frac{dx^2}{dt} = -ax^1 + cx^2 - x^1x^3 \\ \frac{dx^3}{dt} = x^1x^2 - bx^3 - bc. \end{cases} \tag{4.6}$$

Note that x^1, x^2, x^3 are defined in the tangent space, and $x^1 = \dot{p}^1$, $x^2 = \dot{p}^2$, $x^3 = \dot{p}^3$.

According to literature [8], one sets principle moments of inertia for system (4.6) as follows

$$I = \operatorname{diag}(I_1, I_2, I_3) = \operatorname{diag}(1, (1+e_3)^{-1}, (1+e_1+e_3)^{-1}) = \operatorname{diag}\left(1, \frac{1}{2}, \frac{1}{2}\right).$$

Hamiltonian function are defined as $H = K + U$, $K = \frac{1}{2}((x^1)^2 + 2(x^2)^2 + 2(x^3)^2)$, $U = ax^3$. The Lie-Poisson brackets are given by

$$\{x, H\} = x \times \bigtriangledown H = \begin{pmatrix} ax^2 \\ -x^1x^3 - ax^1 \\ x^1x^2 \end{pmatrix}.$$

Therefore, system (4.6) can be described as the Kolmogorov system. The specific form of the Kolmogorov system is rewritten from Chen system as follows

$$\dot{x} = \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \end{pmatrix} = \begin{pmatrix} ax^2 \\ -x^1x^3 - ax^1 \\ x^1x^2 \end{pmatrix} - \begin{pmatrix} ax^1 \\ -cx^2 \\ bx^3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -bc \end{pmatrix} = \{x, H\} - \Lambda x + f,$$

where $\Lambda = \operatorname{diag}\{a, -c, b\}$ and $f = (0, 0, -bc)^T$.

4.2. *Geometric quantities*

Chen system has been rewritten as the Lie-Poisson equcation in Section 4.1, which means that the Chen system is a non-holonomic system with torsion tensor. Similar to Lorenz system [28], Chen system can be regarded as a unified system of the tangent space and the base space.

By comparing equation (2.10) with equations (4.6), Chen system can be given by three equations of the form

$$\frac{dx^\alpha}{dt} + 2\bar{G}^\alpha(p^\beta, x^\gamma) = 0, \ (\alpha, \beta, \gamma = 1, 2, 3),$$

where

$$\bar{G}^1 = \frac{1}{2}(ax^1 - ax^2), \bar{G}^2 = \frac{1}{2}(x^1x^3 + ax^1 - cx^2)$$

and

$$\bar{G}^3 = \frac{1}{2}(-x^1x^2 + bx^3 + bc).$$

Therefore, the nonlinear connection and the connection coefficient are given by

$$\begin{aligned} &\bar{N}_1^1 = \frac{1}{2}a,\ \bar{N}_2^1 = -\frac{1}{2}a,\ \bar{N}_3^1 = 0,\ \bar{N}_1^2 = \frac{1}{2}(x^3 + a),\ \bar{N}_2^2 = -\frac{1}{2}c, \\ &\bar{N}_3^2 = \frac{1}{2}x^1,\ \bar{N}_1^3 = -\frac{1}{2}x^2,\ \bar{N}_2^3 = -\frac{1}{2}x^1,\ \bar{N}_3^3 = \frac{1}{2}b \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &\bar{F}_{11}^2 = \bar{F}_{12}^2 = \bar{F}_{32}^2 = \bar{F}_{33}^2 = 0,\ \bar{F}_{11}^3 = \bar{F}_{11}^3 = \bar{F}_{13}^3 = \bar{F}_{23}^3 = \bar{F}_{22}^3 = 0,\ \bar{F}_{\beta\gamma}^1 = 0, \\ &\bar{F}_{2\gamma}^2 = 0,\ \bar{F}_{3\gamma}^3 = 0,\ (\beta, \gamma = 1,2,3),\ \bar{F}_{13}^2 = \bar{F}_{31}^2 = \frac{1}{2},\ \bar{F}_{12}^3 = \bar{F}_{21}^3 = -\frac{1}{2}, \end{aligned} \tag{4.8}$$

respectively. The first invariants of Chen system are obtained as follows

$$\bar{N}_0^1 = \frac{1}{2}(ax^1 - ax^2), \quad \bar{N}_0^2 = \frac{1}{2}(ax^1 - cx^2), \quad \bar{N}_0^3 = bc - \frac{1}{2}bx^3.$$

Just to make it easier to compute the torsion tensor, we firstly give the following result.

Theorem 2. *The form of $\bar{G}^\alpha$ in Chen system is given by*

$$\bar{G}^\alpha = \bar{F}_{\beta\gamma}^\alpha x^\beta x^\gamma + \Lambda_\beta^\alpha x^\beta + f^\alpha, \tag{4.9}$$

where Λ_β^α and f^α represent the dissipation term and external force, respectively.

Proof. Noting that $\alpha = 1$, $\bar{N}_1^1 = \frac{1}{2}a$, $\bar{N}_2^1 = -\frac{1}{2}a$, $\bar{N}_3^1 = 0$, $\bar{F}_{\beta\gamma}^1 = 0$, $(\beta, \gamma = 1,2,3)$. The first equation of the system (4.6) can be seen as

$$\frac{\mathrm{d}x^1}{\mathrm{d}t} + 2\bar{F}_{\beta\gamma}^1 x^\beta x^\gamma + 2\bar{N}_1^1 x^1 - 2\bar{N}_2^1 x^2 + 2f^1 = 0.$$

Further,

$$\frac{\mathrm{d}x^1}{\mathrm{d}t} + 2\bar{F}_{\beta\gamma}^1 x^\beta x^\gamma + 2\Lambda_1^1 x^1 + 2\Lambda_2^1 x^2 + 2\Lambda_3^1 x^3 + 2f^1 = 0,$$

where $\Lambda_1^1 = \bar{N}_1^1$, $\Lambda_2^1 = -\bar{N}_2^1$, $\Lambda_3^1 = 0$, $f^1 = 0$. Consequently,

$$\frac{\mathrm{d}x^1}{\mathrm{d}t} + 2(\bar{F}_{\beta\gamma}^1 x^\beta x^\gamma + \Lambda_\beta^1 x^\beta + f^1) = 0.$$

From equation (2.10), one obtains $\bar{G}^1 = \bar{F}_{\beta\gamma}^1 + \Lambda_\beta^1 x^\beta + f^1$.

When $\alpha = 2$ or 3, similarly results are obtained as follows

$$\frac{\mathrm{d}x^2}{\mathrm{d}t} + 2(\bar{F}_{\beta\gamma}^2 x^\beta x^\gamma + \Lambda_\beta^2 x^\beta + f^2) = 0$$

and

$$\frac{dx^3}{dt} + 2(\bar{F}^3_{\beta\gamma} x^\beta x^\gamma + \Lambda^3_\beta x^\beta + f^3) = 0,$$

respectively. Therefore, we obtain $\bar{G}^\alpha$ in Chen system as the following form

$$\bar{G}^\alpha = \bar{F}^\alpha_{\beta\gamma} x^\beta x^\gamma + \Lambda^\alpha_\beta x^\beta + f^\alpha.$$

□

From the above Theorem 2, formula (2.14) can be rewritten as follows

$$\bar{R}^\alpha_{\beta\gamma} = \bar{N}^\delta_\beta \bar{F}^\alpha_{\delta\gamma} - \bar{N}^\delta_\gamma \bar{F}^\alpha_{\delta\beta}. \tag{4.10}$$

The curvature tensors of Chen system are given by

$$\bar{P}^\alpha_{\delta\beta\gamma} = \frac{\partial \bar{F}^\alpha_{\delta\beta}}{\partial x^\gamma} = 0, \ (\alpha, \beta, \delta, \gamma = 1, 2, 3).$$

From equation (2.16), another the curvature tensors of the Chen system are obtained:

$$\bar{R}^\alpha_{\delta\beta\gamma} = \begin{cases} 0, & \text{for } \delta, \beta, \gamma = 1, 2, 3, \text{ and } \alpha = 1; \\ -\frac{1}{4}, & \text{for } \delta = 1, \beta = 2, \gamma = 1, \text{ and } \alpha = 2; \\ 0, & \text{for } \delta \neq 1, \beta \neq 2, \gamma \neq 1, \text{ and } \alpha = 2; \\ -\frac{1}{4}, & \text{for } \delta = 1, \beta = 3, \gamma = 1, \text{ and } \alpha = 3; \\ 0, & \text{for } \delta \neq 1, \beta \neq 3, \gamma \neq 1, \text{ and } \alpha = 3. \end{cases}$$

4.3. *Discuss the chaotic behavior and torsion*

In this subsection, the relations between the chaotic behaviour and the torsion for the Chen system are discussed on the base space and the tangent space, respectively. Some good results have been reported in discussing the relationship between the torsion and the trajectory behaviour of dynamical systems in the base space [26–28].

Firstly, one analyses Chen system on the base space. From equation (2.13), the torsion and nonlinear connection with the first invariant can describe the discrepancy of Chen system as follows

$$\Delta x^\alpha = \oint_c dx^\alpha = -\oint_c (\bar{N}^\alpha_\beta X^\beta_i dp^i + \bar{N}^\alpha_0 dt) = -\oint_c \bar{N}^\alpha_{\hat{\beta}} X^{\hat{\beta}}_{\hat{i}} dp^{\hat{i}} \neq 0, \tag{4.11}$$

where Δx^α expresses the discrepancy along the trajectory $c = (x^\alpha(t))$ in the $(\mathbf{x})$-field. Here, we put $(p^{\hat{i}}) = (p^i, p^0) = (p^i, t)$, $X^{\hat{\alpha}}_{\hat{i}} = (X^\alpha_i, X^0_0) = (X^\alpha_i, 1)$ and $X^\alpha_0 = X^0_i = 0$. Therefore, using the Stokes' theorem, relation (4.11) can be rewritten as

$$\Delta x^\alpha = -\oint_c \bar{N}^\alpha_{\hat{\beta}} X^{\hat{\beta}}_{\hat{i}} dp^{\hat{i}} = \frac{1}{2} \iint_S \bar{R}^\alpha_{\hat{\beta}\hat{\gamma}} dp^{\hat{\beta}} \wedge dp^{\hat{\gamma}} \neq 0, \tag{4.12}$$

where S is a region inside the trajectory c. Here, we put $(p^{\hat{\beta}}) = (p^{\beta}, t)$,

$$\bar{R}^{\alpha}_{0\beta} = \frac{\delta \bar{N}^{\alpha}_{0}}{\delta p^{\beta}} - \frac{\delta \bar{N}^{\alpha}_{\beta}}{\delta p^{0}}, \text{ and } \frac{\delta}{\delta p^{0}} = \frac{\partial}{\partial t} - \bar{N}^{\gamma}_{0} \frac{\partial}{\partial x^{\gamma}}. \tag{4.13}$$

Thus, the torsion tensor $\bar{R}^{\alpha}_{\beta\gamma}$ on the base spaces expresses the discrepancy which implies the trajectory of Chen system is not periodic.

In order to do better next analysis, the components of the torsion tensor are calculated as follows.

Theorem 3. *If the torsion tensor $\bar{R}^{2}_{32}$ satisfy $a \neq 0$, then the torsion tensor $\bar{R}^{\alpha}_{\beta\gamma}$ of Chen system does not vanish.*

Proof. Combine equations (4.7), (4.8) and (4.10), one can calculate that $\bar{R}^{1}_{\beta\gamma} = \bar{N}^{\delta}_{\beta} \bar{F}^{1}_{\delta\gamma} - \bar{N}^{\delta}_{\gamma} \bar{F}^{1}_{\delta\beta} = 0$ holds for all $\delta, \beta, \gamma \in \{1,2,3\}$.

Similarly, one can obtain that $\bar{R}^{2}_{\beta\gamma} = \bar{N}^{\delta}_{\beta} \bar{F}^{2}_{\delta\gamma} - \bar{N}^{\delta}_{\gamma} \bar{F}^{2}_{\delta\beta}$. From $\bar{N}^{3}_{1} \bar{F}^{2}_{32} - \bar{N}^{3}_{2} \bar{F}^{2}_{31} = 0$, $(-\frac{1}{2}x^{2}) - (-\frac{1}{2}x^{1}).\frac{1}{2} = \frac{1}{4}x^{1}$ and $\bar{N}^{\delta}_{1} \bar{F}^{2}_{\delta 2} - \bar{N}^{\delta}_{2} \bar{F}^{2}_{\delta 1} = 0$ for $\delta \in \{1,2\}$, one obtains that $\bar{R}^{2}_{12} = \frac{1}{4}x^{1}$. The others can be given by $\bar{R}^{2}_{13} = \frac{1}{4}(a-b)$ and $\bar{R}^{2}_{32} = \frac{1}{4}a$.

From $\bar{R}^{3}_{\beta\gamma} = \bar{N}^{\delta}_{\beta} \bar{F}^{3}_{\delta\gamma} - \bar{N}^{\delta}_{\gamma} \bar{F}^{3}_{\delta\beta}$, one gains that $\bar{N}^{2}_{1} \bar{F}^{3}_{23} - \bar{N}^{2}_{3} \bar{F}^{3}_{21} = \frac{1}{4}x^{1}$ and $\bar{R}^{3}_{\beta\gamma} = \bar{N}^{\delta}_{\beta} \bar{F}^{3}_{\delta\gamma} - \bar{N}^{\delta}_{\gamma} \bar{F}^{3}_{\delta\beta} = 0$ for $\delta \in \{1,3\}$, thence, $\bar{R}^{3}_{13} = \frac{1}{4}x^{1}$. One also acquires that $\bar{N}^{1}_{2} \bar{F}^{3}_{11} - \bar{N}^{1}_{1} \bar{F}^{3}_{12} = \frac{1}{4}a$, $\bar{N}^{2}_{1} \bar{F}^{3}_{23} - \bar{N}^{2}_{3} \bar{F}^{3}_{21} = \frac{1}{4}x^{1}$ and $\bar{N}^{3}_{2} \bar{F}^{3}_{31} - \bar{N}^{3}_{1} \bar{F}^{3}_{32} = 0$, thus, $\bar{R}^{3}_{21} = \frac{1}{4}(a+c)$. From the above analysis, the torsion tensor $\bar{R}^{\alpha}_{\beta\gamma}$ does not disappear because of $\bar{R}^{2}_{32} = \frac{1}{4}a \neq 0$ $(a \neq 0)$. $\square$

Secondly, one will discuss a relation between the dynamics behavior of Chen system and the torsion under three sets parameter conditions by means of the numerical study. Based on the values of the components of the torsion tensor of under each parameter condition, the corresponding Lyapunov exponent spectrum and bifurcation diagram are drawn to describe complex dynamics characteristics of Chen system. After transformation (4.5), the first two components of one of the non-zero equilibrium points of the original Chen system (1.1) and Chen system (4.6) are the same, i.e., $(x^{1}, x^{2}) = (\bar{x}^{1}, \bar{x}^{1}) = (\sqrt{b(2c-a)}, \sqrt{b(2c-a)})$. So we know from our previous calculations that the torsion tensor doesn't change under coordinate transformation (4.5). In other words, studying the relationship between the dynamics of Chen system (1.1) and the torsion tensor is equivalent to studying the Chen system (4.6). Next, we only consider the torsion around at $(x^{1}, x^{2}) = (\bar{x}^{1}, \bar{x}^{1}) = (\sqrt{b(2c-a)}, \sqrt{b(2c-a)})$ of Chen system (1.1) in the tangent space. From Theorem 3, one knows that $\bar{R}^{2}_{12} = \bar{R}^{3}_{13} = \frac{1}{4}x^{1}$. For brevity, we remark that $4\bar{R}^{\alpha}_{\beta\gamma} = \widehat{R}^{\alpha}_{\beta\gamma}$.

It should be noted that the three set of parameter values considered below are all Levinson dissipative, i.e., all satisfying the parameters $a+b > c > 0$ in the Chen system.

First case, fixing the parameters $a = 35$, $b = 3$ and letting $c \in [20,28]$ in Chen system, the values of the components of the torsion tensor with the change of the parameter c are obtained, as shown in Fig. 2. It reveals that as $\widehat{R}^2_{12}$, $\widehat{R}^3_{13}$ and $\widehat{R}^3_{21}$ go up, $\widehat{R}^2_{13}$ and $\widehat{R}^2_{32}$ remain unchanged with increasing the value of parameter c. And it can also be seen intuitively that the torsion $\widehat{R}^2_{12}$, $\widehat{R}^3_{13}$ and $\widehat{R}^3_{21}$ are proportional to the value of parameter c.

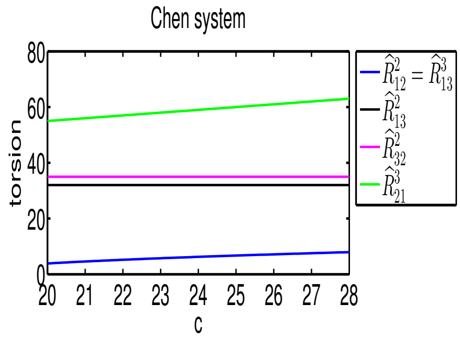


FIGURE 2. The values of the components of the torsion tensor of Chen system (1.1) with $a = 35$, $b = 3$ and $c \in [20,28]$.

In Figs. 4, 6 and 8, computation of the Lyapunov exponent spectrums use wolf algorithm, the period of time is 0.1, start and finish values of time are 0 and 4000, respectively. When $c \in [20,28]$, the corresponding Lyapunov exponent spectrum and the bifurcation diagram of Chen system are shown in Figs. 3(a) and 3(b).

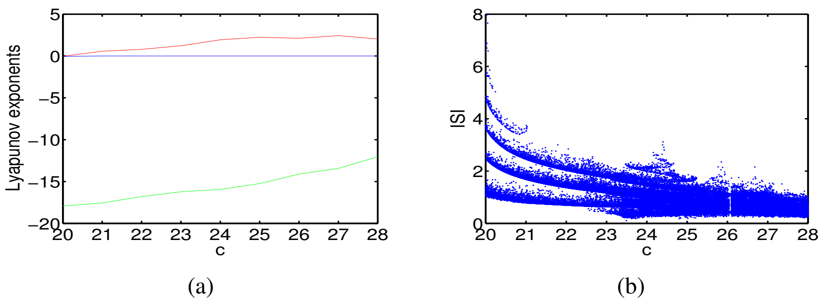


FIGURE 3. (a) Lyapunov exponent spectrum with the initial conditions $(x^1(0),x^2(0),x^3(0)) = (1.3,1.3,0.038)$ and (b) bifurcation diagram with the initial conditions $(x^1(0),x^2(0),x^3(0)) = (1.15,3.5,3.3)$ for Chen system with $a = 35$, $b = 3$ and $c \in [20,28]$.

It can be observed that the bifurcation diagram coincides with the spectrum of Lyapunov exponents. Fig. 3(a) shows that Chen system evolves from the periodic to chaotic for a very wide range of c. Combining Figs. 3(a) and 3(b), the dynamical behaviors of Chen system can be clearly observed. When $c = 20$, the largest Lyapunov exponents is almost equal zero, which implies that Chen system is periodic or quasi-periodic. When $c \in (20, 28]$, the largest Lyapunov exponents is positive, which means that Chen system is chaotic.

Second case, fixing the parameters $a = 35$, $c = 28$, and letting $b \in [3, 15]$, the values of the components of the torsion tensor with the change of the parameter b are given, as shown in Fig. 4. It displays that as $\widehat{R}^2_{12}$ and $\widehat{R}^3_{13}$ go up, $\widehat{R}^2_{13}$ goes down, and $\widehat{R}^2_{13}$, $\widehat{R}^2_{32}$ stay the same with increasing the value of b. And one can see that $\widehat{R}^2_{12}$ and $\widehat{R}^3_{13}$ are proportional to parameter b but $\widehat{R}^2_{13}$ is inversely proportional to parameter b.

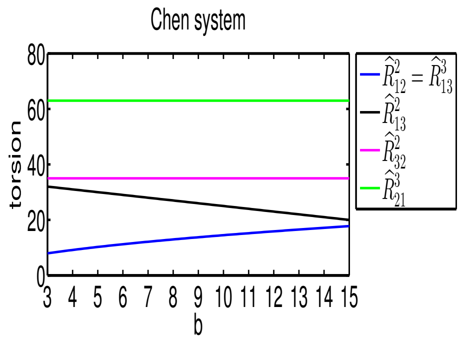


FIGURE 4. The values of the components of the torsion tensor of Chen system with $a = 35$, $c = 28$ and $b \in [3, 15]$.

When $b \in [3, 15]$, the corresponding Lyapunov exponent spectrum and the bifurcation diagram of Chen system are shown in Figs. 5(a) and 5(b). It can be observed that the bifurcation diagram coincides with the spectrum of Lyapunov exponents. Fig. 5(a) shows that Chen system evolves from the periodic to chaotic for a very wide range of b. Combining Figs. 5(a) and 5(b), the dynamical behaviors of Chen system can be clearly observed. When $b \in [4, 15]$, the largest Lyapunov exponents is almost equal zero, which implies that Chen system is periodic or qusi-periodic. The Chen system is chaotic in the sense that the largest Lyapunov exponents is positive when $b \in [3, 4)$.

Third case, fixing the parameters $b = 3$, $c = 28$ in Chen system and letting $a \in [30, 40]$ varies, the values of the components of the torsion tensor with the change of the parameter a are obtained, as shown in Fig. 6. It shows that $\widehat{R}^2_{12}$ and $\widehat{R}^3_{13}$ go down slowly and others go up quickly with increasing the value of a. And it can also be seen intuitively that the torsion $\bar{R}^{\alpha}_{\beta\gamma}$ has a linear relationship with the parameter a.

When $a \in [30, 40]$, the corresponding Lyapunov exponent spectrum and the bifurcation diagram of Chen system are shown in Figs. 7(a) and 7(b). It can be observed

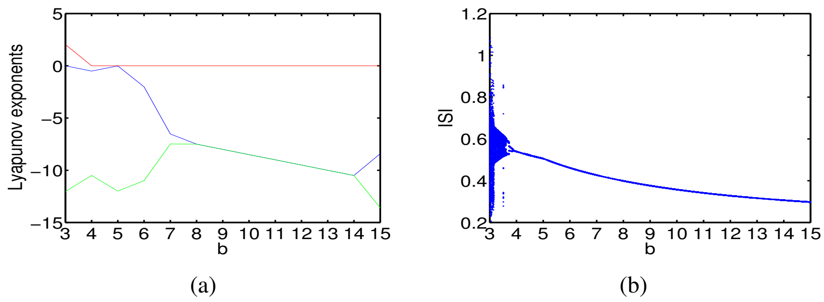


FIGURE 5. (a) Lyapunov exponent spectrum with the initial conditions $(x^1(0), x^2(0), x^3(0)) = (1.3, 1.3, 0.038)$ and (b) bifurcation diagram with the initial conditions $(x^1(0), x^2(0), x^3(0)) = (1.15, 3.5, 3.3)$ for Chen system with $a = 35$, $c = 28$ and $b \in [3, 15]$.

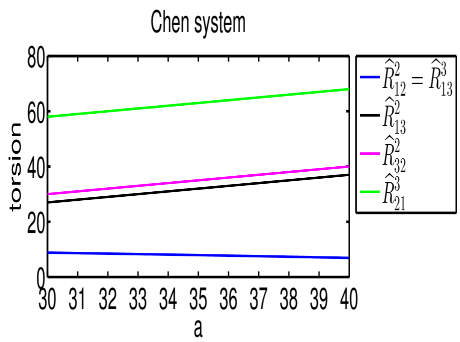


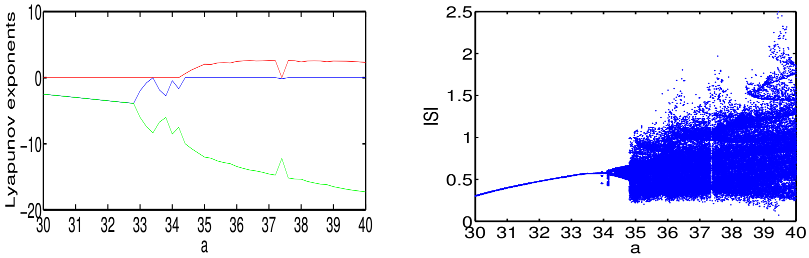
FIGURE 6. The values of the components of the torsion tensor of Chen system with $b = 3$, $c = 28$ and $a \in [30, 40]$.

that the bifurcation diagram coincides with the spectrum of Lyapunov exponents. Fig. 7(a) shows that Chen system evolves from the periodic to chaotic for a very wide range of b. Combining Figs. 7(a) and 7(b), the dynamical behaviours of Chen system can be clearly observed. When $a \in [30, 34.3]$ and $a = 37.4$, the largest Lyapunov exponents is almost equal zero, which implies that Chen system is periodic or quasi-periodic. When $a \in (34.3, 37.4) \bigcup (37.4, 40]$, the largest Lyapunov exponents is positive, which means that Chen system is chaotic.

Therefore, basing on Figs. 3, 5, and 7 and then fixing

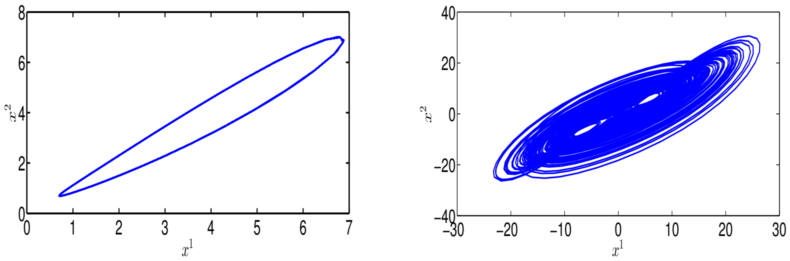
$$(a, b, c) = (35, 3, 20),\ (33, 3, 28),\ (35, 10, 28),\ (35, 3, 28),$$

respectively, the phase diagrams of Chen system are drawn as in Figs. 8, 9 and 10. These figures display that the behaviour of Chen system evolves from the periodic to chaotic coincide with the change of torsion.



(a) (b)

FIGURE 7. (a) Lyapunov exponent spectrum with the initial conditions $(x^1(0), x^2(0), x^3(0)) = (1.3, 1.3, 0.038)$ and (b) bifurcation diagram with with the initial conditions $(x^1(0), x^2(0), x^3(0)) = (1.15, 3.5, 3.3)$ for Chen system with $b = 3$, $c = 28$ and $a \in [30, 40]$.



(a) (b)

FIGURE 8. Trajectory of Chen system with $(a, b) = (35, 3)$ and the initial values $(x^1(0), x^2(0), x^3(0)) = (-2, 3, 8)$: (a) periodic for $c = 20$ and the corresponding the components of the $\widehat{R}^2_{12} = \widehat{R}^3_{13} \approx 3.9$; (b) chaos for $c = 28$ and the corresponding the components of the $\widehat{R}^2_{12} = \widehat{R}^3_{13} \approx 7.9$.

In conclusion, these figures numerically show that the complex dynamic behavior of the Chen system is expressed by the change of torsion tensor which change with the parameters, and among them $\widehat{R}^2_{12}$, $\widehat{R}^3_{13}$ are obtained in the $(\mathbf{x})$-field. That is to say, the torsion tensor $\bar{R}^{\alpha}_{\beta\gamma}$ gives the geometrical interpretation of the complex dynamic behavior of Chen system.

In order to better explore the complexity of dynamics and torsion tensor between Lorenz system and Chen system, one concretely compares that the values of the components of the torsion tensor of two systems in the form of a table as shown in Table 1.

The results show that the four components of the torsion tensor of the two systems are consistent, but the component of the torsion tensor $\widehat{R}^3_{21}$ is particularly different,

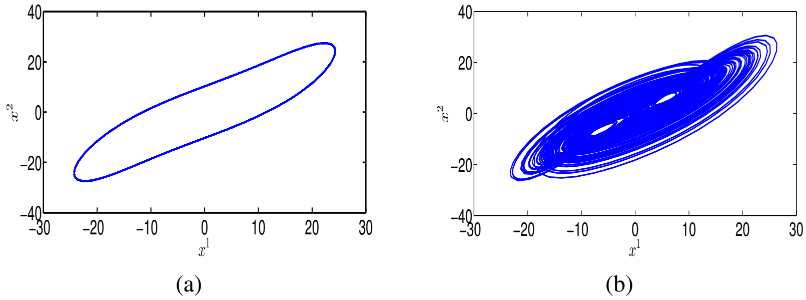


FIGURE 9. Trajectory of the Chen system with $(a,c) = (35,28)$ and initial values $(x^1(0), x^2(0), x^3(0)) = (-2,3,8)$: (a) periodic for $b = 10$ and the corresponding the components of the $\widehat{R}^2_{12} = \widehat{R}^3_{13} \approx 14.5$; (b) chaos for $b = 3$ and the corresponding the components of the $\widehat{R}^2_{12} = \widehat{R}^3_{13} \approx 7.9$.

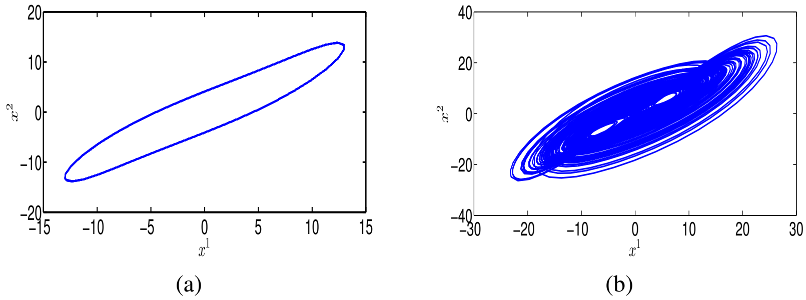


FIGURE 10. Trajectory of the Chen system with $(b,c) = (3,28)$ and initial values $(x^1(0), x^2(0), x^3(0)) = (-2,3,8)$: (a) periodic for $a = 33$ and the corresponding the components of the $\widehat{R}^2_{12} = \widehat{R}^3_{13} \approx 8.3$; (b) chaos for $a = 35$ and the corresponding the components of the $\widehat{R}^2_{12} = \widehat{R}^3_{13} \approx 7.9$.

which is equal to $a-1$ in Lorenz system, and is equal to $a+c$ in Chen system. Therefore, we want to know what happens to the torsion tensor and dynamics of the two systems when a and b are fixed and c takes different values?

Based on the torsion tensor is equal to the first component of the equilibrium point of the two systems, let's compare two systems from two perspectives.

In order to compare with Lorenz system, we fix the parameters $a = 35$ and $b = 3$. With reference to Lorenz system (4.14) in tangent space [28] rewrite the system (4.4)

TABLE 1. The values of the torsion tensor of Lorenz system and Chen system

the components of the torsion tensor	Lorenz system	Chen system
$\widehat{R}^2_{12}$	x^1_0	x^1
$\widehat{R}^3_{13}$	x^2_0	x^2
$\widehat{R}^2_{13}$	$a-b$	$a-b$
$\widehat{R}^2_{32}$	a	a
$\widehat{R}^3_{21}$	$a-1$	$a+c$

into (4.15) as follows

$$\begin{cases} \dot{x}^1 = 35(x^2 - x^1) \\ \dot{x}^2 = c_0x^1 - x^1x^3 - x^2 \\ \dot{x}^3 = x^1x^2 - 3x^3, \end{cases} \tag{4.14}$$

$$\begin{cases} \dot{x}^1 = 35(x^2 - x^1) \\ \dot{x}^2 = c_0x^1 - x^1x^3 - x^2 + (c+1)x^2 + (c-10-c_0)x^1 \\ \dot{x}^3 = x^1x^2 - 3x^3. \end{cases} \tag{4.15}$$

Then the component of the torsion tensor corresponding to Table 1 is changed as follows

TABLE 2. The values of the torsion tensor of Lorenz system and Chen system with $a = 35$, $b = 3$ at the equilibrium point

the components of the torsion tensor	Lorenz system	Chen system
$\widehat{R}^2_{12}$	$x^1_0 = \sqrt{3(c_0-1)}$	$x^1 = \sqrt{3((c_0-1)+(2c-34-c_0))}$
$\widehat{R}^3_{13}$	$x^2_0 = \sqrt{3(c_0-1)}$	$x^2 = \sqrt{3((c_0-1)+(2c-34-c_0))}$
$\widehat{R}^2_{13}$	32	32
$\widehat{R}^2_{32}$	35	35
$\widehat{R}^3_{21}$	34	$35+c$

As we all know, the Chen system is constructed by adding a simple state feedback to the second equation of the Lorenz system [4]. Although the Lorenz system and the Chen system have similar properties, there is difference between the two systems. From Table 2, one obtains that $(x^1)^2 = (x^1_0)^2 + 3(2c-34-c_0)$. The values of the component $\widehat{R}^2_{12} = \widehat{R}^3_{13}$ of the original Lorenz system only have a simple linear relationship with c_0, but in the Chen system their square have one more term, i.e, $3(2c-34-c_0)$ than the Lorenz system. Moreover, in this case, we find that when

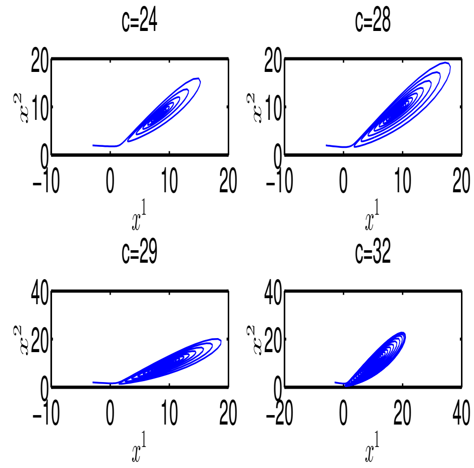
$c_0 = c \in [24, 32]$, the two systems show different dynamic characteristics under the same parameters and initial conditions as shown in Fig. 11. The Lorenz system is always stable, while the Chen system shows more complex dynamic behavior, that is, from chaos to periodic. These indicate that the Chen system exhibits more complex torsion tensor and dynamics than the Lorenz system.

Comparing Chen system with $a = 35$, $b = 3$ and Lorenz system with $a = 10$, $b = 8/3$, the values of the components of the torsion tensor with the change of the parameter c are given as shown in Figs. 2 and 12. And it's clearly observed that the torsion tensor $\widehat{R}^3_{21}$ increases obviously from periodic to chaos of the Chen system in Fig. 2, whereas it is constant of the Lorenz system in Fig. 12. In other words, the Chen system evolves from the periodic to chaos is expressed by the change of torsion $\widehat{R}^3_{21}$, which is not found in Lorenz system. The above shows that the chaotic mechanism of Chen system may be more complex than Lorenz system.

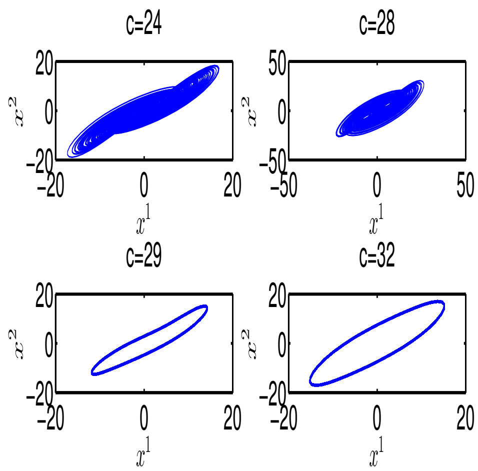
5. CONCLUSION

This paper presents a new perspective of dynamics analysis of Chen system. On the one hand, by using the generalized Meinikov method, the existence of two non-transverse homoclinnic orbits is proved of the Chen system (1.1) for $3a + c > b/2$ and $c > a$.

Based on the theory of tangent bundle, a new geometric method is proposed, and the relationship between the complex dynamic behaviour in Chen system and the torsion tensor has been discussed in this paper. Firstly, Chen system is transformed into Kolmogorov system, which can be regarded as a set of second-order differential equations controlled by Lie-Poisson equations. Then, some geometric invariants of Chen system are obtained by associating non-linear connection and connection coefficients. Furthermore, through numerical study, the results show that the complex dynamic behaviour of Chen system is expressed by the change of torsion tensor which are proportional or inversely proportional to the parameters. And it should be noted that $\widehat{R}^2_{12} = \widehat{R}^3_{13} = x^1 = \sqrt{b(2c-a)}$ are obtained in the $(\mathbf{x})$-field (i.e., tangent space), which indicate that it is meaningful and effective to explore the dynamic characteristic of Chen system on the tangent bundle. These mean that the torsion tensor $\bar{R}^{\alpha}_{\beta\gamma}$ gives the geometrical interpretation of the complex dynamic behaviour of the Chen system. Finally, one concretely compares that the values of the components of the torsion tensor of two systems as shown in Table 1 for exploring the complexity of dynamics and torsion tensor between Lorenz system and Chen system. The results show that torsion tensor change will lead the Chen system from periodic to chaotic, which is not found in Lorenz system. These indicate that the chaotic mechanism of Chen system may be more complex than Lorenz system. It is also hoped that this paper will help to reveal the most essential geometrical structure of the chaotic attractors.



(a)



(b)

FIGURE 11. Trajectory of the system for $(a,b) = (35,3)$ and $c_0 = c \in [24,32]$ with initial values $(x^1(0), x^2(0), x^3(0)) = (-3,2,20)$: (a) stable for Lorenz system; (b) $c \in [24,28]$ chaos and $c \in [29,32]$ periodic for Chen system.

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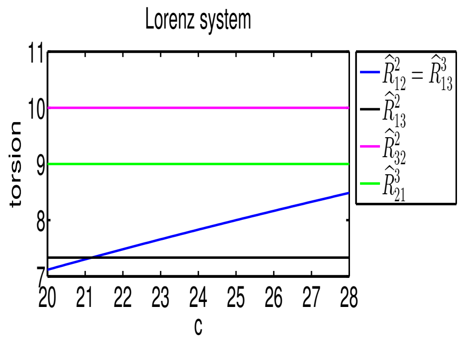

FIGURE 12. The values of the components of the torsion tensor of Lorenz system with $a = 3$, $b = 8/3$ and $c \in [20, 28]$.

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