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S-COPURE SUBMODULES OF A MODULE

FARANAK FARSHADIFAR

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Abstract. Let R be a commutative ring with identity, S be a multiplicatively closed subset of R, and M be an R-module. The aim of this paper is to introduce the notion of S-copure submodules and investigate some properties of this class of modules. We say that a submodule N of M is S-*copure* if there exists an $s \in S$ such that $s(N :_M I) \subseteq N + (0 :_M I)$ for every ideal I of R.

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1. **Introduction**

Throughout this paper R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

It is well known that the notions of purity and copurity with their different generalizations play a fundamental role in theory of module categories. In [2], Anderson and Fuller defined a submodule N of an R-module M a *pure submodule* if $IN = N \cap IM$ for every ideal I of R. In [4], H. Ansari-Toroghy and F. Farshadifar introduced the dual notion of pure submodules (that is copure submodules). A submodule N of an R-module M is said to be *copure* if $(N :_M I) = N + (0 :_M I)$ for every ideal I of R [4], where $(N :_M I) = \{x \in M : Ix \subseteq N\}$. An R-module M is said to be *fully pure* (resp. *fully copure*) if every submodule of M is pure (resp. copure) [6].

Let S be a multiplicatively closed subset of R. In [7], F. Farshadifar introduced and investigated the concept of S-pure submodules of modules as a generalization of pure submodules. A submodule N of an R-module M is said to be S-*pure* if there exists an $s \in S$ such that $s(N \cap IM) \subseteq IN$ for every ideal I of R [7]. Also, an R-module M is said to be *fully S-pure* if every submodule of M is S-pure [7].

Let S be a multiplicatively closed subset of R and M be an R-module. In this paper, we introduce the notion of S-copure submodules of M as a generalisation of copure submodules. Also, this notion can be regarded as a dual notion of S-pure submodules. We provide some useful information concerning the is new class of modules.

2. Main results

Throughout this section, S is a multiplicatively closed subset of R, that is, $1 \in S$ and $s_1s_2 \in S$ for any $s_1 \in S$ and any $s_2 \in S$.

Definition 1. We say that a submodule N of an R-module M is *S-copure* if there exists an $s \in S$ such that $s(N :_M I) \subseteq N + (0 :_M I)$ for every ideal I of R.

Definition 2. We say that an R-module M is *fully S-copure* if every submodule of M is S-copure.

Example 1. Let M be an R-module with $Ann_R(M) \cap S \neq \varnothing$. Then clearly, M is a fully S-copure R-module.

Proposition 1. *Every fully copure R-module is a fully S-copure R-module. The converse is true if $S \subseteq U(R)$, where $U(R)$ is the set of units in R.*

Proof. This is clear. □

The following example shows that the converse of Proposition 1 is not true in general.

Example 2. Clearly, for a prime number p, the submodule $p\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}$ is not copure. Take the multiplicatively closed subset $S = \{p^n : n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. Then for each $k \in \mathbb{N}$, $p(p\mathbb{Z} :_{\mathbb{Z}} k\mathbb{Z}) \subseteq p\mathbb{Z} + (0 :_{\mathbb{Z}} k\mathbb{Z})$ implies that $p\mathbb{Z}$ is an S-copure submodule of $\mathbb{Z}$.

Theorem 1. *Let M be an R-module, and let N and K be submodules of M such that $N \subseteq K \subseteq M$. Then we have the following.*

(a) *If K is an S-copure submodule of M and N is a S-copure submodule of K, then N is an S-copure submodule of M.*
(b) *If N is an S-copure submodule of M, then N is an S-copure submodule of K.*
(c) *If K is an S-copure submodule of M, then K/N is an S-copure submodule of M/N.*
(d) *If N is an S-copure submodule of M and K/N is an S-copure submodule of M/N, then K is an S-copure submodule of M.*
(e) *If N is an S-copure submodule of M, then there is a bijection between the S-copure submodules of M containing N and the S-copure submodules of M/N.*

Proof. (a) Let I be an ideal of R. Then since K is an S-copure submodule of M, there exists an $s \in S$ such that

$$\begin{aligned} s(N :_M I) &= s(N \cap K :_M I) = s((N :_M I) \cap (K :_M I)) \\ &\subseteq (N :_M I) \cap (K + (0 :_M I)) = (N :_K I) + (0 :_M I). \end{aligned}$$

Now since N is an S-copure submodule of K, there exists an $t \in S$ such that

$$st(N :_M I) \subseteq t(N :_K I) + t(0 :_M I) \subseteq N + (0 :_K I) + (0 :_M I) = N + (0 :_M I).$$

(b) Let I be an ideal of R. Then as N is an S-copure submodule of M, there exists an $s \in S$ such that

$$\begin{aligned} s(N :_K I) &= s(K \cap (N :_M I)) \subseteq K \cap (N + (0 :_M I)) \\ &= K \cap N + K \cap (0 :_M I) \subseteq N + (0 :_M I). \end{aligned}$$

(c) Let I be an ideal of R. Then there exists an $s \in S$ such that

$$s(K :_K I) \subseteq K + (0 :_M I).$$

Thus

$$\begin{aligned} s(K/N :_{M/N} I) &= s((K :_M I)/N) = s((K :_M I) + K \cap (N :_M I))/N \\ &\subseteq (K + (0 :_M I) + K \cap (N :_M I))/N \\ &= K/N + ((N :_M I) \cap (K + (0 :_M I))/N \\ &\subseteq K/N + ((N :_M I) \cap (K :_M I))/N \\ &= K/N + (N :_M I)/N = K/N + (0 :_{M/N} I). \end{aligned}$$

(d) Let I be an ideal of R. Since N is an S-copure submodule of M, there exists an $s \in S$ such that $s(N :_M I) \subseteq N + (0 :_M I)$. We have

$$s(0 :_{M/N} I) = s(N :_M I)/N \subseteq ((0 :_M I) + N)/N.$$

Now since K/N is an S-copure submodule of M/N, there exists an $t \in S$ such that

$$t(K/N :_{M/N} I) \subseteq K/N + (0 :_{M/N} I).$$

Therefore,

$$\begin{aligned} st(K :_M I)/N &= st(K/N :_{M/N} I) \subseteq sK/N + s(0 :_{M/N} I) \\ &\subseteq K/N + ((0 :_M I) + N)/N = (K + N + (0 :_M I))/N \\ &= (K + (0 :_M I))/N. \end{aligned}$$

Thus $st(K :_M I) \subseteq K + (0 :_M I)$, as desired.

(e) This follows from parts (c) and (d). □

Recall that the saturation S^* of S is defined as $S^* = \{x \in R : x/1 \text{ is a unit of } S^{-1}R\}$. It is obvious that S^* is a multiplicatively closed subset of R containing S [9].

A multiplicatively closed subset S of R is said to satisfy the *maximal multiple condition* if there exists an $s \in S$ such that $t \mid s$ for each $t \in S$.

Proposition 2. *Let M be an R-module. Then we have the following.*

(a) *If $S_1 \subseteq S_2$ are multiplicatively closed subsets of R and M is a fully S_1-copure R-module, then M is a fully S_2-copure R-module.*
(b) *M is a fully S-copure R-module if and only if M is a fully S^*-copure R-module.*
(c) *If N is an S-copure submodule of M, then sN is an S-copure submodule of M for each $s \in S$.*

(d) *If $f : M \to M$ is an endomorphism and there exists an $s \in S$ such that $sf(x) = f^2(x)$ for each $x \in M$, then $Ker(f)$ is an S-copure submodule of M.*

(e) *If N and K are submodules of M such that $N \cap K$ and $N + K$ are S-copure submodules of M. Then N is an S-copure submodule of M.*

(f) *If S is satisfying the maximal multiple condition (e.g., S is finite or $S \subseteq U(R)$) and $\{M_\lambda\}_\Lambda$ is a family of submodules of M with S-copure submodules $N_\lambda \subseteq M_\lambda$, then $\oplus_{\lambda\in\Lambda} N_\lambda$ is an S-copure submodule of $\oplus_{\lambda\in\Lambda} M_\lambda$.*

Proof. (a) This is clear.

(b) Let M be a fully S-copure R-module. Since $S \subseteq S^*$, by part (a), M is a fully S^*-copure R-module. For the converse, assume that M is a fully S^*-copure module, N is a submodule of M, and I is an ideal of R. Then there exists an $x \in S^*$ such that $x(N :_M I) \subseteq N + (0 :_M I)$. As $x \in S^*$, $x/1$ is a unit of $S^{-1}R$ and so $(x/1)(a/s) = 1$ for some $a \in R$ and $s \in S$. This yields that $us = uxa$ for some $u \in S$. Thus we have

$$us(N :_M I) = uxa(N :_M I) \subseteq x(N :_M I) \subseteq N + (0 :_M I).$$

Therefore, M is a fully S-copure R-module.

(c) Let $s \in S$. As N is S-copure, there is an $t \in S$ such that $t(N :_M I) \subseteq N + (0 :_M I)$ for each ideal I of R. Therefore,

$$ts(sN :_M I) \subseteq ts(N :_M I) \subseteq sN + s(0 :_M I) \subseteq sN + (0 :_M I).$$

(d) Let I be an ideal of R and $x \in (Ker(f) :_M I)$. Then $xI \subseteq Ker(f)$. It follows that $f(x) \in (0 :_M I)$. As $sf = f^2$, we have $sx - f(x) \in Ker(f)$. Therefore, $sx = sx - f(x) + f(x) \in Ker(f) + (0 :_M I)$. This implies that

$$s(Ker(f) :_M I) \subseteq Ker(f) + (0 :_M I).$$

(e) Let I be an ideal of R and let $m \in (N :_M I)$. Since $N + K$ is an S-copure submodule of M, there exists an $s \in S$ such that $s(N + K :_M I) \subseteq N + K + (0 :_M I)$. Then $Im \subseteq N + K$ implies that $sm = x + y + t$ for some $x \in N$, $y \in K$ and $t \in (0 :_M I)$. Thus $msI = xI + yI$. This implies that $yI \subseteq N \cap K$. Since $N \cap K$ is an S-copure submodule of M, there exists an $h \in S$ such that $h(N \cap K :_M I) \subseteq N \cap K + (0 :_M I)$. Thus $hy = \acute{x} + \acute{t}$ for some $\acute{x} \in N \cap K$ and $\acute{t} \in (0 :_M I)$. It follows that $shm \in N + (0 :_M I)$. Therefore, $sh(N :_M I) \subseteq N + (0 :_M I)$, as desired.

(f) Let I be an ideal of R. Then there exists an $s \in S$ such that $s(N_\lambda :_{M_\lambda} I) \subseteq N_\lambda + (0 :_{M_\lambda} I)$ for each $\lambda \in \Lambda$. Now one can see that $s(\oplus_{\lambda\in\Lambda} N_\lambda :_{\oplus_{\lambda\in\Lambda} M_\lambda} I) \subseteq \oplus_{\lambda\in\Lambda} N_\lambda + (0 :_{\oplus_{\lambda\in\Lambda} M_\lambda} I)$. □

Definition 3. We say that a submodule N of an R-module M is an *S-direct summand* of M if there exist a submodule K of M and $s \in S$ such that $sM = N + K$ (d.s.).

Definition 4. We say that an R-module M is an *S-semisimple module* if every submodule of M is an S-direct summand of M.

Proposition 3. *Let M be an S-semisimple R-module. Then M is a fully S-copure R-module.*

Proof. Let N be a submodule of M. Then there exist a submodule K of M and $s \in S$ such that $sM = N + K$ (d.s.). Now for each ideal I of R, we have

$$s(N :_M I) = (N :_{sM} I) = (N :_K I) + (N :_N I) \subseteq (0 :_K I) + N \subseteq (0 :_M I) + N.$$

□

Proposition 4. *Let R be a principal ideal domain and M be an R-module. Then every submodule of M is an S-pure submodule if and only if is an S-copure submodule.*

Proof. First suppose that N is an S-pure submodule of M and $r \in R$. Then there exists an $s \in S$ such that $s(N \cap rM) \subseteq rN$. Now let $rm \in N$. Then $srm = srn$, for some $n \in N$. Thus, $sm = s(m-n) + sn \in (0 :_M r) + N$. So, $s(N :_M r) \subseteq N + (0 :_M r)$ and N is S-copure. Now suppose that N is an S-copure submodule of M and $r \in R$.

Then there exists an $s \in S$ such that $s(N :_M r) \subseteq N + (0 :_M r)$. Suppose that $rm \in N$. Then $sm = n_1 + m_1$, where $n_1 \in N$ and $rm_1 = 0$. Thus $srm = rn_1 \in rN$. This shows that N is an S-pure submodule of M. □

Theorem 2. *Let M be a distributive R-module. Then the following hold.*

(a) *A submodule N of M is S-copure if and only if there exists an $s \in S$ such that for each $a \in R$ we have*

$$s(N :_M a) \subseteq N + (0 :_M a).$$

(b) *A submodule N of M is S-pure if and only if there exists an $s \in S$ such that for each $a \in R$ we have*

$$s(N \cap aM) \subseteq aN.$$

(c) *A submodule N of M is an S-pure submodule if and only if it is an S-copure submodule.*

Proof. (a) First assume that there exists an $s \in S$ such that for each $a \in R$ we have $s(N :_M a) \subseteq N + (0 :_M a)$. Suppose that I is an ideal of R. Then we have

$$s(N :_M I) = s(N :_M \sum_{a \in I} Ra) = s \bigcap_{a \in I} (N :_M a) \subseteq \bigcap_{a \in I} (N + (0 :_M a)).$$

Now as M is distributive, we have

$$\bigcap_{a \in I} (N + (0 :_M a)) = N + \bigcap_{a \in I} (0 :_M a) = N + (0 :_M I).$$

Therefore, N is an S-copure submodule of M. The converse is clear.

(b) First suppose that exists an $s \in S$ such that for each $a \in R$ we have $s(N \cap aM) \subseteq aN$. Let I be an ideal of R. Then as M is a distributive R-module, we have

$$\begin{aligned} IN = (\sum_{a \in I} Ra)N &\supseteq \sum_{a \in I} s(RaM \cap N) \supseteq s \sum_{a \in I} (RaM \cap N) \\ &= s((\sum_{a \in I} Ra)M \cap N) = s(IM \cap N). \end{aligned}$$

Hence, N is an S-pure submodule of M. The converse is clear.

(c) This follows from parts (a), (b), and Proposition 4. □

Proposition 5. *Let R be a Noetherian ring and let M be an R-module. Then the following hold.*

(a) *If N is an S-copure submodule of M, then for each prime ideal $\mathfrak{p}$ of R, $N_\mathfrak{p}$ is an $S_\mathfrak{p}$-copure submodule of $M_\mathfrak{p}$ as an $R_\mathfrak{p}$-module.*

(b) *If $N_\mathfrak{m}$ is an $S_\mathfrak{m}$-copure submodule of an $R_\mathfrak{m}$-module $M_\mathfrak{m}$ for each maximal ideal $\mathfrak{m}$ of R, then N is an S-copure submodule of M.*

Proof. (a) This follows from the fact that by [10, 9.13], if I is a finitely generated ideal of R, then $(N :_M I)_\mathfrak{p} = (N_\mathfrak{p} :_{M_\mathfrak{p}} I_\mathfrak{p})$.

(b) Suppose that I is an ideal of R. As R is a Noetherian ring, I is finitely generated ideal of R. Hence by [10, 9.13], for any maximal ideal $\mathfrak{m}$ of R, $(N :_M I)_\mathfrak{m} = (N_\mathfrak{m} :_{M_\mathfrak{m}} I_\mathfrak{m})$. Thus by assumption, for any maximal ideal $\mathfrak{m}$ of R, there is an $h/a \in S_\mathfrak{m}$ such that

$$(h(N :_M I))_\mathfrak{m} = h/a(N_\mathfrak{m} :_{M_\mathfrak{m}} I_\mathfrak{m}) \subseteq N_\mathfrak{m} + (0 :_{M_\mathfrak{m}} I_\mathfrak{m}) = (N + (0 :_M I))_\mathfrak{m}.$$

It follows that

$$h(N :_M I) \subseteq N + (0 :_M I).$$

□

Let M be an R-module. M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$ [3]. M satisfies *the double annihilator conditions* (*DAC* for short) if for each ideal I of R, we have $I = Ann_R((0 :_M I))$. M is said to be a *strong comultiplication module* if M is a comultiplication R-module which satisfies the double annihilator condition [4]. M is said to be an *S-comultiplication module* if for each submodule N of M, there exist $s \in S$ and an ideal I of R such that $s(0 :_M I) \subseteq N \subseteq (0 :_M I)$ [1].

Definition 5. We say that an R-module M satisfies *the S-double annihilator condition* ($S-DAC$ for short) if for each ideal I of R there exists an $s \in S$ such that $sAnn_R((0 :_M I)) \subseteq I$.

Definition 6. We say that an R-module M is an *S-strong comultiplication module* if M is an S-comultiplication R-module which satisfies the S-double annihilator condition.

Lemma 1. *Let M be an S-strong comultiplication R-module. Then we have the following.*

(a) *If I and J are ideals of R with $(0 :_M I) \subseteq (0 :_M J)$, then there exists an $s \in S$ such that $sJ \subseteq I$.*

(b) *For ideals I and J of R there exists an $t \in S$ such that*

$$t(0 :_M I \cap J) \subseteq (0 :_M I) + (0 :_M J).$$

Proof. (a) Let I and J be ideals of R with $(0:_M I) \subseteq (0:_M J)$. Then $Ann_R((0:_M J)) \subseteq Ann_R((0:_M I))$. As M satisfies the S-double annihilator conditions, there exists an $s \in S$ such that $sAnn_R((0:_M I)) \subseteq I$. Thus $sJ \subseteq sAnn_R((0:_M J)) \subseteq I$.

(b) As M satisfies the S-double annihilator condition, there exist $s,t,\in S$ such that $sAnn_R((0:_M I)) \subseteq I$ and $tAnn_R((0:_M J)) \subseteq J$. Thus $sAnn_R((0:_M I)) \cap tAnn_R((0:_M J)) \subseteq I \cap J$. It follows that

$$(0:_M I\cap J) \subseteq (0:_M st(Ann_R((0:_M I)) \cap Ann_R((0:_M J))).$$

Since M is an S-comultiplication module, there exists an $h \in S$ such that

$$h(0:_M Ann_R((0:_M I)+(0:_M J))) \subseteq (0:_M I)+(0:_M J).$$

Therefore, we have

$$hst(0:_M I\cap J) \subseteq h(0:_M Ann_R((0:_M I)+(0:_M J))) \subseteq (0:_M I)+(0:_M J).$$

□

Theorem 3. *Let M ba an S-strong comultiplication R-module. Then we have the following.*

(a) *N is an S-copure submodule of M if and only if $Ann_R(N)$ is an S-pure ideal of R.*
(b) *An ideal I of R is S-pure if and only if $(0:_M I)$ is an S-copure submodule of M.*

Proof. (a) Let N be an S-copure submodule of M and let I be an ideal of R. As M is an S-comultiplication R-module, there exists an $t \in S$ such that $t(0:_M Ann_R(N)) \subseteq N$ and so $(0:_M Ann_R(N)) \subseteq (N:_M t)$. It follows that

$$(0:_M Ann_R(N)I) = ((0:_M Ann_R(N)):_M I) \subseteq (N:_M tI).$$

Since N is an S-copure submodule of M, there exists an $s \in S$ such that

$$s(N:_M tI) \subseteq N+(0:_M tI) \subseteq (0:_M Ann_R(N))+(0:_M tI) \subseteq (0:_M Ann_R(N)\cap tI).$$

Therefore, $(0:_M Ann_R(N)I) \subseteq (0:_M (s(Ann_R(N)\cap tI))$. This in turn implies that $hst(Ann_R(N)\cap I) \subseteq Ann_R(N)I$ for some $h \in S$ by using Lemma 1 (a). Conversely, assume that N is a submodule of M such that $Ann_R(N)$ is an S-pure ideal of R and I be an ideal of R. Then there exists an $s \in S$ such that $s(Ann_R(N)\cap I) \subseteq Ann_R(N)I$. Now we have

$$(N:_M I) \subseteq (0:_M Ann_R(N)I) \subseteq (0:_M s(Ann_R(N)\cap I)) = ((0:_M Ann_R(N)\cap I):_M s).$$

This implies that $s(N:_M I) \subseteq (0:_M Ann_R(N)\cap I)$. By Lemma 1 (b), there exists an $t \in S$ such that $t(0:_M Ann_R(N)\cap I) \subseteq (0:_M Ann_R(N))+(0:_M I)$. As M is an S-comultiplication R-module, there exists an $h \in S$ such that $h(0:_M Ann_R(N) \subseteq N$. Therefore, we have

$$tsh(N:_M I) \subseteq th(0:_M Ann_R(N)\cap I) \subseteq h(0:_M Ann_R(N))+h(0:_M I) \subseteq N+(0:_M I),$$

as desired.

(b) Let I be an S-pure ideal of R. Then there is an $s \in S$ such that $s(I \cap J) \subseteq IJ$ for each ideal J of R. Now we have

$$((0 :_M I) :_M J) = (0 :_M IJ) \subseteq (0 :_M s(I \cap J)).$$

It follows that $s((0 :_M I) :_M J) \subseteq (0 :_M I \cap J)$. By Lemma 1 (b), there exists an $t \in S$ such that $t(0 :_M I \cap J) \subseteq (0 :_M I) + (0 :_M J)$. Therefore,

$$ts((0 :_M I) :_M J) \subseteq (0 :_M I) + (0 :_M J).$$

Conversely, assume that $(0 :_M I)$ is an S-copure submodule of M and J is an ideal of R. Then by part (a), $Ann_R((0 :_M))$ is an S-pure ideal of R. Thus there exists an $s \in S$ such that $s(Ann_R((0 :_M I)) \cap \acute{I}) \subseteq Ann_R((0 :_M I))\acute{I}$ for each ideal $\acute{I}$ of R. Hence we have

$$s(I \cap J) = s(I \cap Ann_R((0 :_M I)) \cap J) \subseteq Ann_R((0 :_M I))(I \cap J)$$

As M satisfies the S-double annihilator condition, there exists an $t \in S$ such that $tAnn_R((0 :_M I)) \subseteq I$. Hence, $st(I \cap J) \subseteq I(I \cap J) \subseteq I^2 \cap IJ \subseteq IJ$, as needed. □

A proper submodule N of an R-module M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [8].

Remark 1. Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$ [5].

A family $\{N_i\}_{i \in I}$ of submodules of an R-module M is said to be an *inverse family of submodules of* M if the intersection of two of its submodules again contains a module in $\{N_i\}_{i \in I}$. Also, M *satisfies the property* $AB5^*$ if for every submodule K of M and every inverse family $\{N_i\}_{i \in I}$ of submodules of M, $K + \cap_{i \in I} N_i = \cap_{i \in I}(K + N_i)$ [11].

Theorem 4. *Let S be a multiplicatively closed subset of R which satisfies the maximal multiple condition (e.g., S is finite or $S \subseteq U(R)$) and M be an R-module which satisfies the property $AB5^*$. Then we have the following.*

(a) *If $\{N_\lambda\}_{\lambda \in \Lambda}$ is a chain of S-copure submodules of M, then $\cap_{\lambda \in \Lambda} N_\lambda$ is S-copure.*

(b) *If N is a submodule of M, then there is a submodule K of M minimal with respect to $N \subseteq K$ and K is an S-copure submodule of M.*

Proof. (a) Let I be an ideal of R. Let L be a completely irreducible submodule of M such that $\cap_{\lambda \in \Lambda} N_\lambda + (0 :_M I) \subseteq L$. Then $\cap_{\lambda \in \Lambda} N_\lambda + (0 :_M I) + L = L$. Since M satisfies the property $AB5^*$, we have

$$\cap_{\lambda \in \Lambda}(N_\lambda + (0 :_M I) + L) = L.$$

Now as L is a completely irreducible submodule of M, there exists an $\alpha \in \Lambda$ such that $N_\alpha + (0 :_M I) + L = L$. Now as S satisfies the maximal multiple condition, there exists an $s \in S$ such that $s(N_\alpha :_M I) + L \subseteq L$ since N_α is an S-copure submodule of M. Thus $s(N_\alpha :_M I) \subseteq L$. Hence, $s(\cap_{\lambda\in\Lambda} N_\lambda :_M I) \subseteq L$. This implies that

$$s(\cap_{\lambda\in\Lambda} N_\lambda :_M I) \subseteq \cap_{\lambda\in\Lambda} N_\lambda + (0 :_M I),$$

by Remark 1.

(b) Let

$$\Sigma = \{N \leq H | H \text{ is a } S-\text{copure submodule of } M\}.$$

Then $M \in \Sigma \neq \varnothing$. Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a totally ordered subset of Σ. Then $N \leq \cap_{\lambda\in\Lambda} N_\lambda$ and by part (a), $\cap_{\lambda\in\Lambda} N_\lambda$ is an S-copure submodule of M. Thus by using Zorn's Lemma, one can see that Σ has a minimal element, K say as needed. □

Let R_i be a commutative ring with identity and M_i be an R_i-module, for $i = 1,2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module. Clearly, every submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2. Also, if S_i is a multiplicatively closed subset of R_i for each $i = 1,2$, then $S = S_1 \times S_2$ is a multiplicatively closed subset of R.

Theorem 5. *Let M_i be an R_i-module and $S_i \subseteq R_i$ be a multiplicatively closed subset for $i = 1,2$. Assume that $M = M_1 \times M_2$, $R = R_1 \times R_2$, and $S = S_1 \times S_2$. Then M is a fully S-copure module if and only if M_i is a fully S_i-copure module for $i = 1,2$.*

Proof. For only if part, without loss of generality we will show M_1 is a fully S_1-copure R_1-module. Take a submodule N_1 of M_1 and ideal I_1 of R_1. Then $N_1 \times \{0\}$ is a submodule of M and $I_1 \times \{0\}$ is an ideal of R. Since M is a fully S-copure R-module, there exists $s = (s_1, s_2) \in S_1 \times S_2$ such that

$$(s_1, s_2)(N_1 \times \{0\} :_M I_1 \times \{0\}) \subseteq N_1 \times \{0\} + (0 :_M I_1 \times \{0\}).$$

By focusing on first coordinate, we have $s_1(N_1 :_{M_1} I_1) \subseteq N_1 + (0 :_{M_1} I_1)$. So M_1 is a fully S_1-copure R_1-module. Now assume that M_1 is a fully S_1-copure module and M_2 is a fully S_2-copure module. Take a submodule N of M and ideal I of R. Then N must be in the form of $N_1 \times N_2$ and $I = I_1 \times I_2$, where $N_1 \subseteq M_1, N_2 \subseteq M_2$ and $I_1 \subseteq R_1, I_2 \subseteq R_2$. Since M_1 is a fully S_1-copure R_1-module, there exists an $s_1 \in S_1$ such that $s_1(N_1 :_{M_1} I_1) \subseteq N_1 + (0 :_{M_1} I_1)$. Similarly, there exists an element $s_2 \in S_2$ such that $s_2(N_2 :_{M_2} I_2) \subseteq N_2 + (0 :_{M_2} I_2)$. Now, put $s = (s_1, s_2) \in S$. Then we get

$$\begin{aligned}
(s_1, s_2)(N :_M I) &= (s_1, s_2)(N_1 \times N_2 :_{M_1 \times M_2} I_1 \times I_2) \\
&= s_1(N_1 :_{M_1} I_1) \times s_2(N_2 :_{M_2} I_2) \\
&\subseteq (N_1 + (0 :_{M_1} I_1)) \times (N_2 + (0 :_{M_2} I_2)) \\
&= N_1 \times N_2 + (0 :_{M_1 \times M_2} I_1 \times I_2) = N + (0 :_M I).
\end{aligned}$$

Hence, M is a fully S-copure R-module. □

In the following theorem, we characterize the fully copure R-modules.

Theorem 6. *Let M be an R-module. Then the following statements are equivalent:*

(a) *M is a fully copure R-module;*
(b) *M is a fully $(R-\mathfrak{p})$-copure R-module for each prime ideal $\mathfrak{p}$ of R;*
(c) *M is a fully $(R-\mathfrak{m})$-copure R-module for each maximal ideal $\mathfrak{m}$ of R;*
(d) *M is a fully $(R-\mathfrak{m})$-copure R-module for each maximal ideal $\mathfrak{m}$ of R with $M_{\mathfrak{m}} \neq 0_{\mathfrak{m}}$.*

Proof. $(a) \Rightarrow (b)$ Let M be a fully copure R-module and $\mathfrak{p}$ be a prime ideal of R. Then $R-\mathfrak{p}$ is multiplicatively closed set of R and so M is a fully $(R-\mathfrak{p})$-copure R-module by Proposition 1.

$(b) \Rightarrow (c)$ Since every maximal ideal is a prime ideal, the result follows from the part (b).

$(c) \Rightarrow (d)$ This is clear.

$(d) \Rightarrow (a)$ Let N be a submodule of M and I be an ideal of R. Take a maximal ideal $\mathfrak{m}$ of R with $M_{\mathfrak{m}} \neq 0_{\mathfrak{m}}$. As M is a fully $(R-\mathfrak{m})$-copure module, there exists an $s \notin \mathfrak{m}$ such that $s(N :_M I) \subseteq N + (0 :_M I)$. This implies that

$$(N :_M I)_{\mathfrak{m}} = (s(N :_M I))_{\mathfrak{m}} \subseteq N_{\mathfrak{m}} + (0 :_M I)_{\mathfrak{m}}.$$

Now we have $(N :_M I)_{\mathfrak{m}} \subseteq N_{\mathfrak{m}} + (0 :_M I)_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R. It follows that $(N :_M I) \subseteq N + (0 :_M I)$, as needed. □

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Author's address

Faranak Farshadifar
Department of Mathematics, Farhangian University, Tehran, Iran
E-mail address: f.farshadifar@cfu.ac.ir