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FURTHER IMPROVEMENTS OF GENERALIZED NUMERICAL RADIUS INEQUALITIES FOR SEMI-HILBERTIAN SPACE OPERATORS

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Abstract. Several new improvements of the A-numerical radius inequalities for operators acting on a semi-Hilbertian space, i.e., a space induced by a positive operator A, are proved. In particular, among other inequalities, we show that

$$\frac{1}{4}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A \leq \frac{1}{4}\Big(2\omega_A^2(T)+\gamma(T)\Big) \leq \omega_A^2(T),$$

where

$$\gamma(T)=\sqrt{\left(\|\Re_A(T)\|_A^2-\|\Im_A(T)\|_A^2\right)^2+4\|\Re_A(T)\Im_A(T)\|_A^2}.$$

Here $\omega_A(T)$ and $\|T\|_A$ denote respectively the A-numerical radius and the A-seminorm of an operator T. Also, $\Re_A(T):=\frac{T+T^{\sharp_A}}{2}$ and $\Im_A(T):=\frac{T-T^{\sharp_A}}{2i}$, where $T^{\sharp_A}$ is a distinguished A-adjoint operator of T. Further, some new refinements of the triangle inequality related to $\|\cdot\|_A$ are established.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\|$. The C^*-algebra of all bounded linear operators acting $\mathcal{H}$ will be denoted by $\mathbb{B}(\mathcal{H})$. Let $T\in\mathbb{B}(\mathcal{H})$, the numerical radius and the usual operator norm of T are defined respectively by

$$\omega(T)=\sup_{\|x\|=1}|\langle Tx,x\rangle| \quad\text{and}\quad \|T\|=\sup_{\|x\|=1}\|Tx\|.$$

An operator $T\in\mathbb{B}(\mathcal{H})$ is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in\mathcal{H}$. In all that follows, the range of every operator $T\in\mathbb{B}(\mathcal{H})$ is denoted by $\mathcal{R}(T)$, its null space by $\mathcal{N}(T)$ and T^* is the adjoint of T. If $\mathcal{M}$ is an arbitrary linear subspace of $\mathcal{H}$, then $\overline{\mathcal{M}}$ denotes its closure in the norm topology of $\mathcal{H}$. Given a closed subspace $\mathcal{M}$ of $\mathcal{H}$, $P_{\mathcal{M}}$ stands for the orthogonal projection onto $\mathcal{M}$. For the rest of this paper, by an operator we mean a bounded linear operator and we assume

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that $A \in \mathbb{B}(\mathcal{H})$ is a nonzero positive operator. It is clear that A induces a semi-inner product on $\mathcal{H}$ given by $\langle x,y\rangle_A = \langle Ax,y\rangle$ for all $x,y \in \mathcal{H}$. By $\|\cdot\|_A$ we denote the seminorm induced by $\langle\cdot,\cdot\rangle_A$, i.e. $\|x\|_A = \sqrt{\langle x,x\rangle_A}$ for every $x \in \mathcal{H}$. One can verify that $\|x\|_A = 0$ if and only if $x \in \mathcal{N}(A)$. This implies that $\|\cdot\|_A$ is a norm on $\mathcal{H}$ if and only if A is injective. Further, we observe that the semi-Hibertian space $(\mathcal{H}, \|\cdot\|_A)$ is a complete space if and only if $\mathcal{R}(A)$ is closed in $\mathcal{H}$. For $T \in \mathbb{B}(\mathcal{H})$, an operator $S \in \mathbb{B}(\mathcal{H})$ is called an A-adjoint operator of T if the identity $\langle Tx,y\rangle_A = \langle x,Sy\rangle_A$ holds for every $x,y \in \mathcal{H}$, that is, $AS = T^*A$ (see [2]). In general, the existence of an A-adjoint operator is not guaranteed. The set of all operators that admit A-adjoints will be denoted by $\mathbb{B}_A(\mathcal{H})$. By applying Douglas' theorem [13], we get

$$\mathbb{B}_A(\mathcal{H}) = \left\{T \in \mathbb{B}(\mathcal{H})\,;\, \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\right\}.$$

If $T \in \mathbb{B}_A(\mathcal{H})$, then the "reduced" solution of the equation $AX = T^*A$ is a distinguished A-adjoint operator of T, which will be denoted by $T^{\sharp_A}$. We remark that if $T \in \mathbb{B}_A(\mathcal{H})$, then $T^{\sharp_A} \in \mathbb{B}_A(\mathcal{H})$, $(T^{\sharp_A})^{\sharp_A} = P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}}$ and $((T^{\sharp_A})^{\sharp_A})^{\sharp_A} = T^{\sharp_A}$. Moreover, if $S \in \mathbb{B}_A(\mathcal{H})$, then $TS \in \mathbb{B}_A(\mathcal{H})$ and $(TS)^{\sharp_A} = S^{\sharp_A}T^{\sharp_A}$. An operator $T \in \mathbb{B}(\mathcal{H})$ is said to be A-selfadjoint if AT is selfadjoint, that is, $AT = T^*A$. Obviously if T is A-selfadjoint, then $T \in \mathbb{B}_A(\mathcal{H})$. However, in general, the equality $T = T^{\sharp_A}$ may not hold. More precisely, one can verify that if $T \in \mathbb{B}_A(\mathcal{H})$, then $T = T^{\sharp_A}$ if and only if T is A-selfadjoint and $\mathcal{R}(T) \subseteq \overline{\mathcal{R}(A)}$. Further, we recall that an operator T is called A-positive if $AT \geq 0$ and we write $T \geq_A 0$. Clearly, A-positive operators are always A-selfadjoint. Now, if we denote by $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ the set of all operators admitting $A^{1/2}$-adjoints, then another application of Douglas' theorem [13] gives

$$\mathbb{B}_{A^{1/2}}(\mathcal{H}) = \left\{T \in \mathbb{B}(\mathcal{H})\,;\, \exists c > 0\,;\, \|Tx\|_A \leq c\|x\|_A,\ \forall x \in \mathcal{H}\right\}.$$

It is clear that $\mathbb{B}_A(\mathcal{H})$ and $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ are two subalgebras of $\mathbb{B}(\mathcal{H})$ which are neither closed nor dense in $\mathbb{B}(\mathcal{H})$. Further, the proper inclusions $\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H}) \subseteq \mathbb{B}(\mathcal{H})$ hold with equality if A is injective and has a closed range in $\mathcal{H}$. For more details, we refer the reader to [1,2,18].

Given $T \in \mathbb{B}(\mathcal{H})$. If there exists $\lambda > 0$ such that $\|Tx\|_A \leq \lambda\|x\|_A$, for all $x \in \overline{\mathcal{R}(A)}$, then it holds:

$$\|T\|_A := \sup_{\substack{x\in\overline{\mathcal{R}(A)},\\ x\neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup_{\substack{x\in\overline{\mathcal{R}(A)},\\ \|x\|_A=1}} \|Tx\|_A < \infty.$$

Of course, if $A = I$ we reach the definition of the classical operator norm. It was shown in [14] that for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we have

$$\begin{aligned}\|T\|_A &= \sup\left\{\|Tx\|_A\,;\, x \in \mathcal{H},\, \|x\|_A = 1\right\}\\ &= \sup\left\{|\langle Tx,y\rangle_A|\,;\, x,y \in \mathcal{H},\, \|x\|_A = \|y\|_A = 1\right\}.\end{aligned} \tag{1.1}$$

It is useful to note that for every $T,S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we have

$$\|TS\|_A \leq \|T\|_A\|S\|_A. \tag{1.2}$$

Further, clearly for $T \in \mathbb{B}_A(\mathcal{H})$, the operators $T^{\sharp_A}T$ and $TT^{\sharp_A}$ are A-positive. In addition, it was shown in [1] that

$$\|T^{\sharp_A}T\|_A = \|TT^{\sharp_A}\|_A = \|T\|_A^2 = \|T^{\sharp_A}\|_A^2. \tag{1.3}$$

Recently, the A-spectral radius of A-bounded operators is introduced by the present author in [18] as follows

$$r_A(T) := \inf_{n\geq 1} \|T^n\|_A^{\frac{1}{n}} = \lim_{n\to\infty} \|T^n\|_A^{\frac{1}{n}}. \tag{1.4}$$

We mention that the second equality in (1.4) is also proved by the present author in [18, Theorem 1]. In addition, $r_A(\cdot)$ satisfies the commutativity property, which asserts that

$$r_A(TS) = r_A(ST), \tag{1.5}$$

for every $T,S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ (see [18]). In all that follows, for any arbitrary operator $T \in \mathbb{B}_A(\mathcal{H})$, we denote

$$\Re_A(T) := \frac{T+T^{\sharp_A}}{2} \quad \text{and} \quad \Im_A(T) := \frac{T-T^{\sharp_A}}{2i}.$$

For simplicity, if n is any positive integer, then we will write $\Re_A^n(T)$ and $\Im_A^n(T)$ instead of $[\Re_A(T)]^n$ and $[\Im_A(T)]^n$, respectively. Also, $\omega_A^n(T)$ means $[\omega_A(T)]^n$. For $T \in \mathbb{B}(\mathcal{H})$, the A-numerical radius of an operator T was firstly defined by Saddi in [23] by

$$\omega_A(T) := \sup\left\{|\langle Tx,x\rangle_A|\,;\, x\in\mathcal{H}, \|x\|_A = 1\right\}.$$

Recently, this concept received considerable attention by many authors. For more details, we refer the reader to [3, 5, 7, 9–12, 16, 18] and the references therein. It should be mentioned here that it may happen that $\|T\|_A$ and $\omega_A(T)$ are equal to $+\infty$ for some $T \in \mathbb{B}(\mathcal{H})$ (see [16, 18]). However, it was shown in [3] that the above quantities are equivalent seminorms on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$. More precisely, we have

$$\frac{1}{2}\|T\|_A \leq \omega_A(T) \leq \|T\|_A. \tag{1.6}$$

Recently, several refinements of the inequalities (1.6) have been proved by many authors (e.g., see [6, 9, 16], and the references therein). In particular, it has been shown that for $T \in \mathbb{B}_A(\mathcal{H})$, we have

$$\frac{1}{2}\sqrt{\|T^{\sharp_A}T + TT^{\sharp_A}\|_A} \leq \omega_A(T) \leq \frac{\sqrt{2}}{2}\sqrt{\|T^{\sharp_A}T + TT^{\sharp_A}\|_A}, \tag{1.7}$$

(see [19]). If $A = I$, we get the well-known inequalities proved by Kittaneh in [21, Theorem 1]. One main target of the present paper is to prove some new refinements of the first inequality in (1.7). The inspiration of our investigation comes from the recent works by Moradi et al. [20, 22]. Some of the obtained results are new even in the case that the underlying operator A is the identity operator. In particular, among other inequalities, we prove that for every $T \in \mathbb{B}_A(\mathcal{H})$ we have

$$\frac{1}{2}\sqrt{\|T^{\sharp_A}T+TT^{\sharp_A}\|_A}$$
$$\leq \frac{1}{2}\sqrt{2\omega_A^2(T)+\sqrt{\left(\|\Re_A(T)\|_A^2-\|\Im_A(T)\|_A^2\right)^2+4\|\Re_A(T)\Im_A(T)\|_A^2}} \leq \omega_A(T).$$

In addition, several new refinements of the triangle inequality related to $\|\cdot\|_A$ are proved. Mainly, we prove that for every $T,S \in \mathbb{B}_A(\mathcal{H})$ we have

$$\|T+S\|_A \leq \sqrt{\frac{1}{2}\left(\|T\|_A^2+\|S\|_A^2+\sqrt{\left(\|T\|_A^2-\|S\|_A^2\right)^2+4\|TS^{\sharp_A}\|_A^2}\right)+2\omega_A(S^{\sharp_A}T)}$$
$$\leq \|T\|_A+\|S\|_A.$$

Several applications of the obtained inequalities are also given.

2. Results

In this section, we present our results. In order to achieve the goals of the present section, we need the following lemma.

Lemma 1. *([4, 16]) Let $T \in \mathbb{B}(\mathcal{H})$ be an A-selfadjoint operator. Then, the following assertions hold:*

(i) *$T^{\sharp_A}$ is A-selfadjoint and $(T^{\sharp_A})^{\sharp_A} = T^{\sharp_A}$.*
(ii) *$\|T\|_A = \omega_A(T) = r_A(T)$.*
(iii) *$\|T^n\|_A = \|T\|_A^n$ for any positive integer n.*
(vi) *$T^{2n} \geq_A 0$ for any positive integer n.*

Our first result in this section provides a refinement of the first inequality in (1.7) and reads as follows.

Theorem 1. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\frac{1}{2}\sqrt{\|T^{\sharp_A}T+TT^{\sharp_A}\|_A} \leq \frac{\sqrt{2}}{2}\sqrt{\|\Re_A(T)\|_A^2+\|\Im_A(T)\|_A^2} \leq \omega_A(T). \tag{2.1}$$

Proof. Observe first that T can be decomposed as $T = \Re_A(T) + i\Im_A(T)$. Further, it is not difficult to verify that $\Re_A(T)$ and $\Im_A(T)$ are A-selfadjoint operators. Thus, by applying Lemma 1 (i) we get

$$([\Re_A(T)]^{\sharp_A})^{\sharp_A} = [\Re_A(T)]^{\sharp_A} \quad \text{and} \quad ([\Im_A(T)]^{\sharp_A})^{\sharp_A} = [\Im_A(T)]^{\sharp_A}.$$

So, a short calculations shows that

$$\frac{1}{2}\left(TT^{\sharp_A}+T^{\sharp_A}T\right)^{\sharp_A} = \left([\Re_A(T)]^{\sharp_A}\right)^2 + \left([\Im_A(T)]^{\sharp_A}\right)^2. \tag{2.2}$$

Thus, one observes that

$$\begin{aligned}
\frac{1}{4}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A &= \frac{1}{4}\left\|\left(TT^{\sharp_A}+T^{\sharp_A}T\right)^{\sharp_A}\right\|_A \\
&= \frac{1}{2}\left\|\left([\Re_A(T)]^{\sharp_A}\right)^2+\left([\Im_A(T)]^{\sharp_A}\right)^2\right\|_A \\
&\leq \frac{1}{2}\left\|\left([\Re_A(T)]^{\sharp_A}\right)^2\right\|_A+\frac{1}{2}\left\|\left([\Im_A(T)]^{\sharp_A}\right)^2\right\|_A \\
&= \frac{1}{2}\left\|\Re_A^2(T)\right\|_A+\frac{1}{2}\left\|\Im_A^2(T)\right\|_A,
\end{aligned}$$

where the last equality follows from the fact that $\|X^{\sharp_A}\|_A=\|X\|_A$ for all $X\in\mathbb{B}_A(\mathcal{H})$. So, by applying Lemma 1 (iii), we get

$$\frac{1}{4}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A \leq \frac{1}{2}\|\Re_A(T)\|_A^2+\frac{1}{2}\|\Im_A(T)\|_A^2. \tag{2.3}$$

On the other hand, let $x\in\mathcal{H}$ be such that $\|x\|_A=1$. Then, we verify that

$$\begin{aligned}
\left|\langle Tx,x\rangle_A\right|^2 &= \left|\langle[\Re_A(T)+i\Im_A(T)]x,x\rangle_A\right|^2 \\
&= \left|\langle\Re_A(T)x,x\rangle_A+i\langle\Im_A(T)x,x\rangle_A\right|^2 \\
&= \left|\langle\Re_A(T)x,x\rangle_A\right|^2+\left|\langle\Im_A(T)x,x\rangle_A\right|^2 \\
&\geq \left|\langle\Re_A(T)x,x\rangle_A\right|^2.
\end{aligned}$$

So, by taking the supremum over all $x\in\mathcal{H}$ with $\|x\|_A=1$ in the above inequality and then using Lemma 1 (ii), we obtain

$$\|\Re_A(T)\|_A^2 \leq \omega_A^2(T). \tag{2.4}$$

Similarly, one can prove that

$$\|\Im_A(T)\|_A^2 \leq \omega_A^2(T). \tag{2.5}$$

By combining (2.3) together with (2.4) and (2.5), we get the desired result. This finishes the proof of the theorem. □

In order to derive a new improvement of the first inequality in (1.7), we need the following lemma.

Lemma 2. *Let $T,S\in\mathbb{B}(\mathcal{H})$ be A-selfadjoint operators. Then,*

$$\|T^2+S^2\|_A \leq \frac{1}{2}\left(\|T^2\|_A+\|S^2\|_A+\sqrt{(\|T^2\|_A-\|S^2\|_A)^2+4\|TS\|_A^2}\right).$$

Proof. Since, $T,S\in\mathbb{B}(\mathcal{H})$ are A-selfadjoint operators, then by applying Lemma 1 (vi) one see that $T^2+S^2\geq_A 0$. So, by Lemma 1 (ii) we have

$$\|T^2+S^2\|_A = r_A\left(T^2+S^2\right). \tag{2.6}$$

On the other hand, it can be checked that

$$r_A(T^2+S^2)=r_{\mathbb{A}}\left[\begin{pmatrix}T^2+S^2 & 0\\ 0 & 0\end{pmatrix}\right]=r_{\mathbb{A}}\left[\begin{pmatrix}T & S\\ 0 & 0\end{pmatrix}\begin{pmatrix}T & 0\\ S & 0\end{pmatrix}\right],$$

where $\mathbb{A}=\begin{pmatrix}A & 0\\ 0 & A\end{pmatrix}\in\mathbb{B}(\mathcal{H}\oplus\mathcal{H})$ is the 2×2 positive diagonal operator matrix whose each diagonal entry is the positive operator A. Further, by using (1.5) we get

$$\begin{aligned} r_A(T^2+S^2)&=r_{\mathbb{A}}\left[\begin{pmatrix}T & 0\\ S & 0\end{pmatrix}\begin{pmatrix}T & S\\ 0 & 0\end{pmatrix}\right]\\ &=r_{\mathbb{A}}\left[\begin{pmatrix}T^2 & TS\\ ST & S^2\end{pmatrix}\right]. \end{aligned}\tag{2.7}$$

Further, by [17] we have

$$r_{\mathbb{A}}\left[\begin{pmatrix}T^2 & TS\\ ST & S^2\end{pmatrix}\right]\le r\left[\begin{pmatrix}\|T^2\|_A & \|TS\|_A\\ \|ST\|_A & \|S^2\|_A\end{pmatrix}\right].\tag{2.8}$$

Therefore, by applying (2.6) together with (2.7) and (2.8) we see that

$$\|T^2+S^2\|_A\le r\left[\begin{pmatrix}\|T^2\|_A & \|TS\|_A\\ \|ST\|_A & \|S^2\|_A\end{pmatrix}\right].\tag{2.9}$$

Since $T\ge_A 0$ and $S\ge_A 0$, then T and S are A-selfadjoint. This implies, through Lemma 1 (i) that $T^{\sharp_A}$ and $S^{\sharp_A}$ are also A-selfadjoint. Thus, Lemma 1 (i) gives $(T^{\sharp_A})^{\sharp_A}=T^{\sharp_A}$ and $(S^{\sharp_A})^{\sharp_A}=S^{\sharp_A}$. So, we obtain

$$\|TS\|_A=\|(TS)^{\sharp_A}\|_A=\|S^{\sharp_A}T^{\sharp_A}\|_A=\|(T^{\sharp_A})^{\sharp_A}(S^{\sharp_A})^{\sharp_A}\|_A=\|T^{\sharp_A}S^{\sharp_A}\|_A=\|ST\|_A.$$

Hence, $\begin{pmatrix}\|T^2\|_A & \|TS\|_A\\ \|ST\|_A & \|S^2\|_A\end{pmatrix}$ is a symmetric matrix. This yields that

$$r\left[\begin{pmatrix}\|T^2\|_A & \|TS\|_A\\ \|ST\|_A & \|S^2\|_A\end{pmatrix}\right]=\frac{1}{2}\left(\|T^2\|_A+\|S^2\|_A+\sqrt{(\|T^2\|_A-\|S^2\|_A)^2+4\|TS\|_A^2}\right).$$

This finishes the proof by taking into consideration (2.9). □

Now, we are in a position to prove one of our main results in this paper.

Theorem 2. *Let $T\in\mathbb{B}_A(\mathcal{H})$. Then*

$$\begin{aligned} &\frac{1}{4}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A\\ &\le\frac{1}{4}\left(2\omega_A^2(T)+\sqrt{(\|\Re_A(T)\|_A^2-\|\Im_A(T)\|_A^2)^2+4\|\Re_A(T)\Im_A(T)\|_A^2}\right)\\ &\le\omega_A^2(T). \end{aligned}$$

Proof. Notice first that T can be written as $T = [\Re_A(T)] + i[\Im_A(T)]$. By using an argument similar to that used in proof of Theorem 1, we get

$$\begin{aligned}\frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A &= \frac{1}{2}\left\|\left([\Re_A(T)]^{\sharp_A}\right)^2 + \left([\Im_A(T)]^{\sharp_A}\right)^2\right\|_A \\ &= \frac{1}{2}\left\|\Re_A^2(T) + \Im_A^2(T)\right\|_A.\end{aligned}$$

Since the operators $\Re_A(T)$ and $\Im_A(T)$ are A-selfadjoint, then an application of Lemma 2 gives

$$\begin{aligned}&\frac{1}{4}\left\|T^{\sharp_A}T + TT^{\sharp_A}\right\|_A \\ &\leq \frac{1}{4}\left(\|\Re_A^2(T)\|_A + \|\Im_A^2(T)\|_A + \sqrt{\left(\|\Re_A^2(T)\|_A - \|\Im_A^2(T)\|_A\right)^2 + 4\|\Re_A(T)\Im_A(T)\|_A^2}\right) \\ &= \frac{1}{4}\left(\|\Re_A(T)\|_A^2 + \|\Im_A(T)\|_A^2 + \sqrt{\left(\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\right)^2 + 4\|\Re_A(T)\Im_A(T)\|_A^2}\right),\end{aligned}$$

where the last equality follows by applying Lemma 1 (iii). Further, by applying (2.4) and (2.5) we see that

$$\begin{aligned}&\frac{1}{4}\left\|T^{\sharp_A}T + TT^{\sharp_A}\right\|_A \\ &\leq \frac{1}{4}\left(2\omega_A^2(T) + \sqrt{\left(\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\right)^2 + 4\|\Re_A(T)\Im_A(T)\|_A^2}\right).\end{aligned}$$

This proves the first inequality in Theorem 2. Now, by using (1.2) and then making simple calculations we get

$$\begin{aligned}&\frac{1}{4}\left(2\omega_A^2(T) + \sqrt{\left(\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\right)^2 + 4\|\Re_A(T)\Im_A(T)\|_A^2}\right) \\ &\leq \frac{1}{4}\left(2\omega_A^2(T) + \sqrt{\left(\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\right)^2 + 4\|\Re_A(T)\|_A^2\|\Im_A(T)\|_A^2}\right) \\ &= \frac{1}{4}\left(2\omega_A^2(T) + \sqrt{\left(\|\Re_A(T)\|_A^2 + \|\Im_A(T)\|_A^2\right)^2}\right) \\ &= \frac{1}{4}\left(2\omega_A^2(T) + \|\Re_A(T)\|_A^2 + \|\Im_A(T)\|_A^2\right) \\ &\leq \frac{1}{4}\left(2\omega_A^2(T) + 2\omega_A^2(T)\right) = \omega_A^2(T).\end{aligned}$$

This finishes the proof of the theorem. □

The following lemma plays a crucial role in proving our next result and provides a new refinement of the triangle inequality related to $\|\cdot\|_A$.

Lemma 3. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then*

$$\begin{aligned}\|T+S\|_A &\leq \int_0^1 \left\|\lambda T + (1-\lambda)\frac{T+S}{2}\right\|_A d\lambda + \int_0^1 \left\|\lambda S + (1-\lambda)\frac{T+S}{2}\right\|_A d\lambda \\ &\leq \|T\|_A + \|S\|_A.\end{aligned}$$

Proof. Let $\lambda \in [0,1]$. Then, one observes that

$$\begin{aligned}\|T+S\|_A &= \left\|\lambda T + (1-\lambda)\frac{T+S}{2} + \lambda S + (1-\lambda)\frac{T+S}{2}\right\|_A \\ &\leq \left\|\lambda T + (1-\lambda)\frac{T+S}{2}\right\|_A + \left\|\lambda S + (1-\lambda)\frac{T+S}{2}\right\|_A.\end{aligned}$$

This proves the first inequality in Lemma 3 by taking integral over $\lambda \in [0,1]$. The second inequality in Lemma 3 follows immediately by applying the triangle inequality related to $\|\cdot\|_A$ and then making simple calculations. □

Another improvement of the first inequality in (1.7) can be stated as follows.

Theorem 3. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\begin{aligned}\frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A &\leq \frac{1}{2}\int_0^1 \left\|\lambda \Re_A^2(T) + (1-\lambda)\frac{\Re_A^2(T)+\Im_A^2(T)}{2}\right\|_A d\lambda \\ &\quad + \frac{1}{2}\int_0^1 \left\|\lambda \Im_A^2(T) + (1-\lambda)\frac{\Re_A^2(T)+\Im_A^2(T)}{2}\right\|_A d\lambda \\ &\leq \omega_A^2(T).\end{aligned}$$

Proof. Since $T = \Re_A(T) + i\Im_A(T)$, then by using an argument similar to that used in proof of Theorem 1, we get

$$\begin{aligned}\frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A &= \frac{1}{2}\left\|\left([\Re_A(T)]^{\sharp_A}\right)^2 + \left([\Im_A(T)]^{\sharp_A}\right)^2\right\|_A \\ &= \frac{1}{2}\left\|[\Re_A^2(T)]^{\sharp_A} + [\Im_A^2(T)]^{\sharp_A}\right\|_A \\ &= \frac{1}{2}\left\|\Re_A^2(T) + \Im_A^2(T)\right\|_A.\end{aligned}$$

So, by applying Lemma 3, we see that

$$\begin{aligned}\frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A &\leq \frac{1}{2}\int_0^1 \left\|\lambda \Re_A^2(T) + (1-\lambda)\frac{\Re_A^2(T)+\Im_A^2(T)}{2}\right\|_A d\lambda \\ &\quad + \frac{1}{2}\int_0^1 \left\|\lambda \Im_A^2(T) + (1-\lambda)\frac{\Re_A^2(T)+\Im_A^2(T)}{2}\right\|_A d\lambda \\ &\leq \frac{1}{2}\left(\|\Re_A^2(T)\|_A + \|\Im_A^2(T)\|_A\right)\end{aligned}$$

$$= \frac{1}{2}\left(\|\Re_A(T)\|_A^2 + \|\Im_A(T)\|_A^2\right) \quad \text{(by Lemma 1(iii))}$$
$$\leq \omega_A^2(T),$$

where the last inequality follows by applying (2.4) together with (2.5). Hence the proof is complete. □

In the following theorem, we prove another refinement of the triangle inequality related to $\|\cdot\|_A$.

Theorem 4. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\|T \pm S\|_A \leq \sqrt{\|T^{\sharp_A}T + S^{\sharp_A}S\|_A + \|T^{\sharp_A}S + S^{\sharp_A}T\|_A} \leq \|T\|_A + \|S\|_A. \tag{2.10}$$

Proof. Let $x \in \mathcal{H}$ be such that $\|x\|_A = 1$. Then we see that

$$\begin{aligned}\|Tx + Sx\|_A^2 &= \|Tx\|_A^2 + \langle Tx, Sx\rangle_A + \langle Sx, Tx\rangle_A + \|Sx\|_A^2 \\ &\leq \left\langle \left(T^{\sharp_A}T + S^{\sharp_A}S\right)x, x\right\rangle_A + \left|\left\langle \left(T^{\sharp_A}S + S^{\sharp_A}T\right)x, x\right\rangle_A\right|.\end{aligned}$$

Further, by applying the Cauchy-Schwarz inequality we get

$$\|Tx + Sx\|_A^2 \leq \|T^{\sharp_A}T + S^{\sharp_A}S\|_A + \|T^{\sharp_A}S + S^{\sharp_A}T\|_A.$$

So, by taking the supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the above inequality we get

$$\|T + S\|_A^2 \leq \|T^{\sharp_A}T + S^{\sharp_A}S\|_A + \|T^{\sharp_A}S + S^{\sharp_A}T\|_A.$$

Similarly, we show that

$$\|T - S\|_A^2 \leq \|T^{\sharp_A}T + S^{\sharp_A}S\|_A + \|T^{\sharp_A}S + S^{\sharp_A}T\|_A.$$

Hence, we get the first inequality in (2.10). Moreover, by applying the triangle inequality together with (1.2) and (1.3) we see that

$$\begin{aligned}\|T \pm S\|_A &\leq \sqrt{\|T^{\sharp_A}T + S^{\sharp_A}S\|_A + \|T^{\sharp_A}S + S^{\sharp_A}T\|_A} \\ &\leq \sqrt{\|T\|_A^2 + \|S\|_A^2 + 2\|T\|_A\|S\|_A} \\ &= \sqrt{(\|T\|_A + \|S\|_A)^2} \\ &= \|T\|_A + \|S\|_A.\end{aligned}$$

This immediately proves the required result. □

Remark 1. We note that the inequalities obtained in Theorem 4 cover and refine the recent inequalities due to Bhunia et al. (see [8, Theorem 2.4].

As an application of Theorem 4, we derive another improvement of the first inequality in (1.7).

Corollary 1. *Let $T\in\mathbb{B}_A(\mathcal{H})$. Then*

$$\frac{1}{4}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A\leq\frac{\sqrt{2}}{2}\sqrt{\omega_A^4(T)+\left\|\Im_A^2(T)\Re_A^2(T)\right\|_A}\leq\omega_A^2(T).$$

Proof. Since $T=\Re_A(T)+i\Im_A(T)$, then by using (2.2) we observe that

$$\frac{1}{4}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A=\frac{1}{2}\left\|\left[\Re_A^2(T)\right]^{\sharp_A}+\left[\Im_A^2(T)\right]^{\sharp_A}\right\|_A.$$

Moreover, in view of Lemma 1 (vi), the operators $\Re_A^2(T)$ and $\Im_A^2(T)$ are A-positive and

$$\left[\left[\Re_A^2(T)\right]^{\sharp_A}\right]^{\sharp_A}=\left[\Re_A^2(T)\right]^{\sharp_A}\quad\text{and}\quad\left[\left[\Im_A^2(T)\right]^{\sharp_A}\right]^{\sharp_A}=\left[\Im_A^2(T)\right]^{\sharp_A}.\tag{2.11}$$

So, by using Theorem 4 together with (2.11) we observe that

$$\begin{aligned}&\frac{1}{16}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2\\&\leq\frac{1}{4}\|\left(\left[\Re_A^2(T)\right]^{\sharp_A}\right)^2+\left(\left[\Im_A^2(T)\right]^{\sharp_A}\right)^2\|_A\\&\qquad+\frac{1}{4}\|\left[\Re_A^2(T)\right]^{\sharp_A}\left[\Im_A^2(T)\right]^{\sharp_A}+\left[\Im_A^2(T)\right]^{\sharp_A}\left[\Re_A^2(T)\right]^{\sharp_A}\|_A\\&\leq\frac{1}{4}\|\left(\left[\Re_A^2(T)\right]^{\sharp_A}\right)^2+\left(\left[\Im_A^2(T)\right]^{\sharp_A}\right)^2\|_A\\&\qquad+\frac{1}{4}\|\left[\Re_A^2(T)\right]^{\sharp_A}\left[\Im_A^2(T)\right]^{\sharp_A}\|_A+\frac{1}{4}\|\left[\Im_A^2(T)\right]^{\sharp_A}\left[\Re_A^2(T)\right]^{\sharp_A}\|_A.\end{aligned}$$

On the other hand, since $\|X\|_A=\|X^{\sharp_A}\|_A$ for all $X\in\mathbb{B}_A(\mathcal{H})$, then by (2.11) we have

$$\begin{aligned}\|\left[\Re_A^2(T)\right]^{\sharp_A}\left[\Im_A^2(T)\right]^{\sharp_A}\|_A&=\left\|\left[\left[\Im_A^2(T)\right]^{\sharp_A}\right]^{\sharp_A}\left[\left[\Re_A^2(T)\right]^{\sharp_A}\right]^{\sharp_A}\right\|_A\\&=\|\left[\Im_A^2(T)\right]^{\sharp_A}\left[\Re_A^2(T)\right]^{\sharp_A}\|_A.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\frac{1}{16}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2\\&\leq\frac{1}{4}\|\left(\left[\Re_A^2(T)\right]^{\sharp_A}\right)^2+\left(\left[\Im_A^2(T)\right]^{\sharp_A}\right)^2\|_A+\frac{1}{2}\|\left[\Re_A^2(T)\right]^{\sharp_A}\left[\Im_A^2(T)\right]^{\sharp_A}\|_A\\&\leq\frac{1}{4}\left\|\Re_A^4(T)+\Im_A^4(T)\right\|_A+\frac{1}{2}\left\|\Im_A^2(T)\Re_A^2(T)\right\|_A.\end{aligned}$$

This implies that

$$\begin{aligned}\frac{1}{4}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A&\leq\frac{1}{2}\sqrt{\left\|\Re_A^4(T)+\Im_A^4(T)\right\|_A+2\left\|\Im_A^2(T)\Re_A^2(T)\right\|_A}\\&\leq\frac{1}{2}\sqrt{\left\|\Re_A^4(T)\right\|_A+\left\|\Im_A^4(T)\right\|_A+2\left\|\Im_A^2(T)\Re_A^2(T)\right\|_A}\end{aligned}$$

$$= \frac{1}{2}\sqrt{\left\|\mathfrak{R}_A(T)\right\|_A^4 + \left\|\mathfrak{I}_A(T)\right\|_A^4 + 2\left\|\mathfrak{I}_A^2(T)\mathfrak{R}_A^2(T)\right\|_A},$$

where the last equality follows from Lemma 1 (iii) since $\mathfrak{R}_A(T)$ and $\mathfrak{I}_A(T)$ are A-selfadjoint operators. By using (2.4) and (2.5) we obtain

$$\frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A \le \frac{\sqrt{2}}{2}\sqrt{\omega_A^4(T) + \left\|\mathfrak{I}_A^2(T)\mathfrak{R}_A^2(T)\right\|_A}.$$

This shows the first inequality in Corollary 1. Finally, by using (1.2) and similar arguments as above, we see that

$$\begin{aligned}\frac{\sqrt{2}}{2}\sqrt{\omega_A^4(T) + \left\|\mathfrak{I}_A^2(T)\mathfrak{R}_A^2(T)\right\|_A} &\le \frac{\sqrt{2}}{2}\sqrt{\omega_A^4(T) + \left\|\mathfrak{R}_A(T)\right\|_A^2\left\|\mathfrak{I}_A(T)\right\|_A^2} \\ &\le \frac{\sqrt{2}}{2}\sqrt{\omega_A^4(T) + \omega_A^4(T)} = \omega_A^2(T).\end{aligned}$$

Hence, the proof is complete. □

In order to prove our second main result in this paper, we need the following paper.

Lemma 4. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\|T \pm S\|_A \le \sqrt{\|TT^{\sharp_A} + SS^{\sharp_A}\|_A + 2\omega_A(TS^{\sharp_A})} \le \|T\|_A + \|S\|_A.$$

Proof. It was shown in [15, Lemma 2.14.] that for $T, S \in \mathbb{B}_A(\mathcal{H})$ we have

$$\max\left\{\|T+S\|_A^2, \|T-S\|_A^2\right\} - \|TT^{\sharp_A} + SS^{\sharp_A}\|_A \le 2\omega_A\left(TS^{\sharp_A}\right).$$

This immediately proves the first inequality in Lemma 4. The second inequality in Lemma 4 can be seen easily by remarking that $\omega_A(TS^{\sharp_A}) \le \|TS^{\sharp_A}\|_A$ and then proceeding as in the proof of Theorem 4.

This finishes the proof of the desired result. □

Another improvement of the first inequality in (1.7), that involves $\mathfrak{R}_A(T)$ and $\mathfrak{I}_A(T)$ can be seen as follows.

Theorem 5. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A \le \frac{\sqrt{2}}{2}\sqrt{\omega_A^4(T) + \omega_A\left(\mathfrak{I}_A^2(T)\mathfrak{R}_A^2(T)\right)} \le \omega_A^2(T).$$

Proof. Since $T = \mathfrak{R}_A(T) + i\mathfrak{I}_A(T)$. Then, by using an argument similar to that used in proof of Theorem 1, we get

$$\begin{aligned}\frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A &= \frac{1}{4}\left\|\left(TT^{\sharp_A} + T^{\sharp_A}T\right)^{\sharp_A}\right\|_A \\ &= \frac{1}{2}\left\|\left([\mathfrak{R}_A(T)]^{\sharp_A}\right)^2 + \left([\mathfrak{I}_A(T)]^{\sharp_A}\right)^2\right\|_A\end{aligned}$$

$$= \frac{1}{2}\left\| \left[\Re_A^2(T)\right]^{\sharp_A} + \left[\Im_A^2(T)\right]^{\sharp_A} \right\|_A .$$

So, by applying Lemma 4 together with (2.11) and then using similar arguments as above, we get

$$\begin{aligned}
&\frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A \\
&\leq \frac{1}{2}\sqrt{\left\| \left(\left[\Re_A^2(T)\right]^{\sharp_A}\right)^2 + \left(\left[\Im_A^2(T)\right]^{\sharp_A}\right)^2 \right\|_A + 2\omega_A\left(\left[\Re_A^2(T)\right]^{\sharp_A}\left[\Im_A^2(T)\right]^{\sharp_A}\right)} \\
&\leq \frac{1}{2}\sqrt{\|\Re_A^2(T)\|^2 + \|\Im_A^2(T)\|^2 + 2\omega_A\left(\Im_A^2(T)\Re_A^2(T)\right)} \\
&\leq \frac{1}{2}\sqrt{\|\Re_A(T)\|_A^4 + \|\Im_A(T)\|_A^4 + 2\omega_A\left(\Im_A^2(T)\Re_A^2(T)\right)} \\
&\leq \frac{\sqrt{2}}{2}\sqrt{\omega_A^4(T) + \omega_A\left(\Im_A^2(T)\Re_A^2(T)\right)},
\end{aligned}$$

where the last inequality follows by applying (2.4) together with (2.5). Now, we will prove the second inequality in Theorem 5. By using the second inequality in (1.6), we see that

$$\begin{aligned}
\frac{\sqrt{2}}{2}\sqrt{\omega_A^4(T) + \omega_A\left(\Im_A^2(T)\Re_A^2(T)\right)} &\leq \frac{\sqrt{2}}{2}\sqrt{\omega_A^4(T) + \left\|\Im_A^2(T)\Re_A^2(T)\right\|_A} \\
&\leq \frac{\sqrt{2}}{2}\sqrt{\omega_A^4(T) + \|\Re_A(T)\|_A^2\,\|\Im_A(T)\|_A^2} \\
&\leq \frac{\sqrt{2}}{2}\sqrt{\omega_A^4(T) + \omega_A^4(T)} = \omega_A^2(T).
\end{aligned}$$

Hence, the proof is complete. □

Another refinement of the triangle inequality related to $\|\cdot\|_A$ can be stated as follows.

Theorem 6. *Let $T,S \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\begin{aligned}
\|T+S\|_A &\leq \sqrt{\frac{1}{2}\left(\|T\|_A^2 + \|S\|_A^2 + \sqrt{\left(\|T\|_A^2 - \|S\|_A^2\right)^2 + 4\|TS^{\sharp_A}\|_A^2}\right) + 2\omega_A(S^{\sharp_A}T)} \\
&\leq \|T\|_A + \|S\|_A .
\end{aligned}$$

Proof. Let $x \in \mathcal{H}$ be such that $\|x\|_A = 1$. Then, we see that

$$\begin{aligned}
\|(T+S)x\|_A^2 &= \|Tx\|_A^2 + \|Sx\|_A^2 + 2|\langle Tx, Sx\rangle_A| \\
&= \langle T^{\sharp_A}Tx, x\rangle_A + \langle S^{\sharp_A}Sx, x\rangle_A + 2|\langle S^{\sharp_A}Tx, x\rangle_A| \\
&\leq \langle \left(T^{\sharp_A}T + S^{\sharp_A}S\right)x, x\rangle_A + 2\omega_A(S^{\sharp_A}T)
\end{aligned}$$

$$= \left\|T^{\sharp_A}T + S^{\sharp_A}S\right\|_A + 2\omega_A(S^{\sharp_A}T),$$

where the last equality follows from Lemma 1 (ii) since $T^{\sharp_A}T + S^{\sharp_A}S \geq_A 0$. Hence,

$$\|(T+S)x\|_A^2 \leq \left\|T^{\sharp_A}T + S^{\sharp_A}S\right\|_A + 2\omega_A(S^{\sharp_A}T).$$

So, by taking the supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the last inequality we get

$$\|T+S\|_A^2 \leq \left\|T^{\sharp_A}T + S^{\sharp_A}S\right\|_A + 2\omega_A(S^{\sharp_A}T). \tag{2.12}$$

On the other hand, let $\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. Since $T^{\sharp_A}T + S^{\sharp_A}S \geq_A 0$, then by applying Lemma 1 (ii) we observe that

$$\begin{aligned}
\|T^{\sharp_A}T + S^{\sharp_A}S\|_A &= r_A(T^{\sharp_A}T + S^{\sharp_A}S) \\
&= r_{\mathbb{A}}\left[\begin{pmatrix} T^{\sharp_A}T + S^{\sharp_A}S & 0 \\ 0 & 0 \end{pmatrix}\right] \\
&= r_{\mathbb{A}}\left[\begin{pmatrix} T^{\sharp_A} & S^{\sharp_A} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T & 0 \\ S & 0 \end{pmatrix}\right].
\end{aligned}$$

Moreover, an application of (1.5) gives

$$\begin{aligned}
\|T^{\sharp_A}T + S^{\sharp_A}S\|_A &= r_{\mathbb{A}}\left[\begin{pmatrix} T & 0 \\ S & 0 \end{pmatrix}\begin{pmatrix} T^{\sharp_A} & S^{\sharp_A} \\ 0 & 0 \end{pmatrix}\right] \\
&= r_{\mathbb{A}}\left[\begin{pmatrix} TT^{\sharp_A} & TS^{\sharp_A} \\ ST^{\sharp_A} & SS^{\sharp_A} \end{pmatrix}\right] \\
&\leq r\left[\begin{pmatrix} \|T\|_A^2 & \|TS^{\sharp_A}\|_A \\ \|ST^{\sharp_A}\|_A & \|S\|_A^2 \end{pmatrix}\right],
\end{aligned} \tag{2.13}$$

where the last inequality follows from [17] together with (1.3). In addition, since $\mathcal{R}(T^{\sharp_A}) \subseteq \overline{\mathcal{R}(A)}$ then $P_{\overline{\mathcal{R}(A)}}T^{\sharp_A} = T^{\sharp_A}$. So, by applying (1.1) we see that

$$\begin{aligned}
\|TS^{\sharp_A}\|_A &= \|P_{\overline{\mathcal{R}(A)}}SP_{\overline{\mathcal{R}(A)}}T^{\sharp_A}\|_A \\
&= \|P_{\overline{\mathcal{R}(A)}}ST^{\sharp_A}\|_A \\
&= \sup\left\{|\langle P_{\overline{\mathcal{R}(A)}}ST^{\sharp_A}x, y\rangle_A|\,;\, x,y \in \mathcal{H},\, \|x\|_A = \|y\|_A = 1\right\} = \|ST^{\sharp_A}\|_A,
\end{aligned}$$

where the last equality holds since $AP_{\overline{\mathcal{R}(A)}} = A$. So, it is not difficult to verify that

$$r\left[\begin{pmatrix} \|T\|_A^2 & \|TS^{\sharp_A}\|_A \\ \|TS^{\sharp_A}\|_A & \|S\|_A^2 \end{pmatrix}\right] = \frac{1}{2}\left(\|T\|_A^2 + \|S\|_A^2 + \sqrt{\left(\|T\|_A^2 - \|S\|_A^2\right)^2 + 4\|TS^{\sharp_A}\|_A^2}\right).$$

This implies, through (2.13), that

$$\|T^{\sharp_A}T + S^{\sharp_A}S\|_A \leq \frac{1}{2}\left(\|T\|_A^2 + \|S\|_A^2 + \sqrt{\left(\|T\|_A^2 - \|S\|_A^2\right)^2 + 4\|TS^{\sharp_A}\|_A^2}\right).$$

Therefore, we prove the first inequality in Theorem 6 by combining the last inequality together with (2.12). Now, one observes that

$$\begin{aligned}
&\frac{1}{2}\left(\|T\|_A^2+\|S\|_A^2+\sqrt{\left(\|T\|_A^2-\|S\|_A^2\right)^2+4\|TS^{\sharp_A}\|_A^2}\right)+2\omega_A(S^{\sharp_A}T)\\
&\leq \frac{1}{2}\left(\|T\|_A^2+\|S\|_A^2+\sqrt{\left(\|T\|_A^2-\|S\|_A^2\right)^2+4\|T\|_A^2\|S\|_A^2}\right)+2\|S^{\sharp_A}T\|_A\\
&\leq \frac{1}{2}\left(\|T\|_A^2+\|S\|_A^2+\sqrt{\left(\|T\|_A^2+\|S\|_A^2\right)^2}\right)+2\|T\|_A\|S\|_A\\
&=\|T\|_A^2+\|S\|_A^2+2\|T\|_A\|S\|_A\\
&=\left(\|T\|_A+\|S\|_A\right)^2.
\end{aligned}$$

This immediately proves the second inequality in Theorem 6. Therefore, the proof is finished. □

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