Miskolc Mathematical Notes
Vol. 24 (2023), No. 1, pp. 5–13
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.3769

QUANTUM HERMITE-HADAMARD TYPE INEQUALITIES AND RELATED INEQUALITIES FOR SUBADDITIVE FUNCTIONS

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Received 16 March, 2021

Abstract. In this work, Hermite-Hadamard type inequalities for subadditive functions via quantum integrals are established. Moreover, Hermite-Hadamard type inequalities for the product of two subadditive functions are also obtained. It is worth to mentioning that some existing inequalities of Hermite-Hadamard type for subadditive functions are obtained by considering the limit of the real number $q \in (0,1)$ as $q \to 1^-$ in the key results.

2010 *Mathematics Subject Classification:* 26A09; 26D10; 26D15; 33E20

Keywords: subadditive functions, convex functions, Hermite-Hadamard inequalities, quantum calculus

1. Introduction

The main work on the general theory of subadditive functions is due to Hill and Phillips [9]. This link also contains part of Rosenbaum's work [14] on the subadditive function of several variables. Additivity, subadditivity, and superadditivity are important concepts in measure theory, as well as in various areas of mathematics and mathematical inequality. There are many examples of additive, subadditive, and superadditive functions, especially in various areas of mathematics such as norms, square roots, error functions, growth rates, differential equations, and integral mean. Inequalities, especially the theory of additive functions, is one of the most advanced areas of theoretical and applied mathematics, as well as physics and other applied sciences. Here we refer to the results of [5, 9, 11, 14] and the corresponding references cited therein.

Definition 1. A function $\mathcal{F}$ defined on a set H of real numbers and with the range contained in the set R^+ of all positive real numbers, is subadditive on H if, for all elements $\varkappa$ and γ of H such that $\varkappa+\gamma$ is an element of H

$$\mathcal{F}(\varkappa+\gamma) \leq \mathcal{F}(\varkappa)+\mathcal{F}(\gamma).$$

This work was partially supported by National Natural Science Foundation of China (No. 11971241).

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If the equality holds, $\mathcal{F}$ is called additive; if the inequality is reversed, $\mathcal{F}$ is superadditive.

Remark 1. If $\mathcal{F}$ is convex and subadditive on H and if $\mathcal{F}(0)=0$, then $\mathcal{F}$ is additive on H.

Definition 2. A function $\mathcal{F}:[0,\kappa_2]\to R$, $\kappa_2>0$ is said to be starshaped if for every $\varkappa\in[0,\kappa_2]$ and $\tau\in[0,1]$ we have $\mathcal{F}(\tau\varkappa)\leq\tau\mathcal{F}(\varkappa)$.

According to the above definitions, if a subadditive function $\mathcal{F}:A\subset[0,\infty)\to R$ is also starshaped, then $\mathcal{F}$ is a convex function.

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [7]):

$$\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\leq\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)d\varkappa\leq\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}, \tag{1.1}$$

where $\mathcal{F}:I\subset\mathbb{R}\to\mathbb{R}$ is a convex function on the interval I of real numbers and $\kappa_1,\kappa_2\in I$ with $\kappa_1<\kappa_2$.

Inequality (1.1) has become an important pillar of mathematical analysis and optimization. Given the variety of settings, you can see that this inequality has many uses. Also, many inequalities are obtained in a special way for a given choice of the function $\mathcal{F}$. The Hermite-Hadamard inequality (1.1), for example, is important in its rich geometry, and many studies are being carried out to prove its proofs, improvements, extensions, and new generalizations. You can check [6, 8, 15] and the references included there.

2. Preliminaries of q-Calculus

In this section, we present some required definitions about q-calculus. Throughout the paper, we consider that $0<q<1$.

Definition 3 ([13])**.** Let $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a function. Then, the q_{κ_1}-definite integral on $[\kappa_1,\kappa_2]$ is defined as:

$$\begin{aligned}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\tau)\ {}_{\kappa_1}d_q\tau &= (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\kappa_2+(1-q^n)\kappa_1) \\ &= (\kappa_2-\kappa_1)\int_0^1\mathcal{F}((1-\tau)\kappa_1+\tau\kappa_2)\ d_q\tau\ .\end{aligned} \tag{2.1}$$

Moreover, we will need to use the following lemma in our main results:

Lemma 1. *For $a\in\mathbb{R}\backslash\{-1\}$ we have the equality*

$$\int_{\kappa_1}^{\kappa_2}(\varkappa-\kappa_1)^a\ {}_{\kappa_1}d_q\varkappa=\frac{(\kappa_2-\kappa_1)^{a+1}}{[a+1]_q}.$$

On the other side, Bermudo et al. gave the following new definition:

Definition 4 ([2]). Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a function. Then, the q^{κ_2}-definite integral on $[\kappa_1, \kappa_2]$ is defined as:

$$\begin{aligned}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\tau)\ {}^{\kappa_2}d_q\tau &= (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\kappa_1 + (1-q^n)\kappa_2) \\ &= (\kappa_2-\kappa_1)\int_0^1 \mathcal{F}(\tau\kappa_1 + (1-\tau)\kappa_2)\ d_q\tau\ .\end{aligned}$$

We have to give the following notations which will be used many times in the next sections (see, [10]):

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

Recently, many studies have been conducted in quantum analysis. In 2018, one of these is the following q-Hermite-Hadamard inequality proved by Alp [1]:

Theorem 1 ([1]). *Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a convex differentiable function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then we have*

$$\mathcal{F}\left(\frac{q\kappa_1 + \kappa_2}{[2]_q}\right) \le \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}_{\kappa_1}d_q\varkappa \ \le \frac{q\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{[2]_q},$$

where $[2]_q = 1 + q$.

Theorem 2 ([2]). *If $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a convex differentiable function on $[\kappa_1, \kappa_2]$. Then, we have the following q-Hermite-Hadamard inequalities*

$$\mathcal{F}\left(\frac{\kappa_1 + q\kappa_2}{[2]_q}\right) \le \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa \ \le \frac{\mathcal{F}(\kappa_1) + q\mathcal{F}(\kappa_2)}{[2]_q}.$$

Recently, Alp et al. [1] obtained quantum estimations for q-midpoint and q- type inequalities and for more recent results one can read [3, 4, 17].

The main objective of this research is to offer q-Hermite-Hadamard type inequalities for subadditive functions by using the notions of q_{κ_1} and q^{κ_2}-integrals.

3. Main Results

The $q-$Hermite-Hadamard type inequalities for subadditive functions are given in the following form:

Theorem 3. *We assume that a continuous mapping $\mathcal{F} : I = [0, \infty) \to R$ is subadditive, $\kappa_1, \kappa_2 \in I^\circ$ with $\kappa_1 < \kappa_2$, then the following inequalities hold*

$$\begin{aligned} \mathcal{F}(\kappa_1 + \kappa_2) &\leq \frac{1}{\kappa_2 - \kappa_1}\left[\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa + \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}_{\kappa_1}d_q\varkappa\right] \\ &\leq \frac{1}{\kappa_1}\left[\int_0^{\kappa_1} \mathcal{F}(\varkappa)\ {}_0d_q\varkappa + \int_0^{\kappa_1} \mathcal{F}(\varkappa)\ {}^{\kappa_1}d_q\varkappa\right] \\ &\quad + \frac{1}{\kappa_2}\left[\int_0^{\kappa_2} \mathcal{F}(\varkappa)\ {}_0d_q\varkappa + \int_0^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa\right], \end{aligned} \tag{3.1}$$

where $0 < q < 1$.

Proof. Since $\mathcal{F}$ is a subadditive function, we can write

$$\begin{aligned} \mathcal{F}(\kappa_1 + \kappa_2) &= \mathcal{F}(\tau\kappa_1 + (1-\tau)\kappa_2 + \tau\kappa_2 + (1-\tau)\kappa_1) \\ &\leq \mathcal{F}(\tau\kappa_1 + (1-\tau)\kappa_2) + \mathcal{F}(\tau\kappa_2 + (1-\tau)\kappa_1). \end{aligned} \tag{3.2}$$

q-integrating inequality (3.2) with respect to τ over $[0,1]$, from Definitions 3 and 4, we have

$$\begin{aligned} \mathcal{F}(\kappa_1 + \kappa_2) &\leq \int_0^1 \mathcal{F}(\tau\kappa_1 + (1-\tau)\kappa_2)\ d_q\tau + \int_0^1 \mathcal{F}(\tau\kappa_2 + (1-\tau)\kappa_1)\ d_q\tau \\ &= \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa + \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}_{\kappa_1}d_q\varkappa \end{aligned}$$

and the first inequality in (3.1) is proved. To prove the second inequality in (3.1), we use the subadditivity of $\mathcal{F}$, we have

$$\mathcal{F}(\tau\kappa_1 + (1-\tau)\kappa_2) \leq \mathcal{F}(\tau\kappa_1) + \mathcal{F}((1-\tau)\kappa_2) \tag{3.3}$$

and

$$\mathcal{F}(\tau\kappa_2 + (1-\tau)\kappa_1) \leq \mathcal{F}(\tau\kappa_2) + \mathcal{F}((1-\tau)\kappa_1). \tag{3.4}$$

Adding the inequalities (3.3) and (3.4), then q-integrating the resultant one with respect to τ over $[0,1]$, we have

$$\begin{aligned} &\frac{1}{\kappa_2 - \kappa_1}\left[\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa + \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\ {}_{\kappa_1}d_q\varkappa\right] \\ &\leq \frac{1}{\kappa_1}\left[\int_0^{\kappa_1} \mathcal{F}(\varkappa)\ {}_0d_q\varkappa + \int_0^{\kappa_1} \mathcal{F}(\varkappa)\ {}^{\kappa_1}d_q\varkappa\right] \\ &\quad + \frac{1}{\kappa_2}\left[\int_0^{\kappa_2} \mathcal{F}(\varkappa)\ {}_0d_q\varkappa + \int_0^{\kappa_2} \mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa\right], \end{aligned}$$

which completes the proof. $\square$

Remark 2. In Theorem 3, if we take the limit $q \to 1^-$, then we have the following inequality of Hermite-Hadamard type for subadditive functions (see, [16]):

$$\frac{1}{2}\mathcal{F}(\kappa_1+\kappa_2) \le \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\,d\varkappa \le \frac{1}{\kappa_1}\int_0^{\kappa_1}\mathcal{F}(\varkappa)\,d\varkappa + \frac{1}{\kappa_2}\int_0^{\kappa_2}\mathcal{F}(\varkappa)\,d\varkappa.$$

Corollary 1. *Under the conditions of Theorem 3, if we take $\mathcal{F}(\tau\varkappa) \le \tau\mathcal{F}(\varkappa)$, then we get the following q-Hermite-Hadamard type inequality for convex functions (see, [2]):*

$$\begin{aligned}\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) &\le \frac{1}{2}\mathcal{F}(\kappa_1+\kappa_2) \\ &\le \frac{1}{2(\kappa_2-\kappa_1)}\left[\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\ {}^{\kappa_2}d_q\varkappa + \int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\ {}_{\kappa_1}d_q\varkappa\right] \\ &\le \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}.\end{aligned} \tag{3.5}$$

Remark 3. If we take the limit $q \to 1^-$ in Corollary 1, then we obtain the inequality (1.1) of Hermite-Hadamard for convex functions.

Theorem 4. *We assume that the continuous mappings $\mathcal{F}, \mathcal{G} : I = [0,\infty) \to R$ are subadditive, $\kappa_1, \kappa_2 \in I^\circ$ with $\kappa_1 < \kappa_2$, then the following inequality holds:*

$$\begin{aligned}&\mathcal{F}(\kappa_1+\kappa_2)\,\mathcal{G}(\kappa_1+\kappa_2) \\ &\le \frac{1}{\kappa_2-\kappa_1}\left[\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\,\mathcal{G}(\varkappa)\ {}^{\kappa_2}d_q\varkappa + \int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\,\mathcal{G}(\varkappa)\ {}_{\kappa_1}d_q\varkappa\right] \\ &\quad + \int_0^1 [\mathcal{F}(\tau\kappa_1)+\mathcal{F}((1-\tau)\kappa_2)]\,[\mathcal{G}(\tau\kappa_2)+\mathcal{G}((1-\tau)\kappa_1)]\,d_q\tau \\ &\quad + \int_0^1 [\mathcal{F}(\tau\kappa_2)+\mathcal{F}((1-\tau)\kappa_1)]\,[\mathcal{G}(\tau\kappa_1)+\mathcal{G}((1-\tau)\kappa_2)]\,d_q\tau\end{aligned} \tag{3.6}$$

where $0 < q < 1$.

Proof. By using the subadditivity of the functions $\mathcal{F}$ and $\mathcal{G}$, we can write

$$\begin{aligned}\mathcal{F}(\kappa_1+\kappa_2) &= \mathcal{F}(\kappa_1\tau+(1-\tau)\kappa_2+(1-\tau)\kappa_1+\tau\kappa_2) \\ &\le \mathcal{F}(\kappa_1\tau+(1-\tau)\kappa_2)+\mathcal{F}((1-\tau)\kappa_1+\tau\kappa_2)\end{aligned}$$

and

$$\mathcal{G}(\kappa_1+\kappa_2) \le \mathcal{G}(\kappa_1\tau+(1-\tau)\kappa_2)+\mathcal{G}((1-\tau)\kappa_1+\tau\kappa_2).$$

Multiplying the above inequalities, we have:

$$\begin{aligned}\mathcal{F}(\kappa_1+\kappa_2)\,\mathcal{G}(\kappa_1+\kappa_2) \le &[\mathcal{F}(\kappa_1\tau+(1-\tau)\kappa_2)\,\mathcal{G}(\kappa_1\tau+(1-\tau)\kappa_2) \\ &+\mathcal{F}((1-\tau)\kappa_1+\tau\kappa_2)\,\mathcal{G}((1-\tau)\kappa_1+\tau\kappa_2)] \\ &+[\mathcal{F}(\kappa_1\tau+(1-\tau)\kappa_2)\,\mathcal{G}((1-\tau)\kappa_1+\tau\kappa_2) \\ &+\mathcal{F}((1-\tau)\kappa_1+\tau\kappa_2)\,\mathcal{G}(\kappa_1\tau+(1-\tau)\kappa_2)]\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\leq [\mathcal{F}(\kappa_1\tau+(1-\tau)\kappa_2)\,\mathcal{G}(\kappa_1\tau+(1-\tau)\kappa_2)\\
&\quad+\mathcal{F}((1-\tau)\kappa_1+\tau\kappa_2)\,\mathcal{G}((1-\tau)\kappa_1+\tau\kappa_2)]\\
&\quad+[\mathcal{F}(\kappa_1\tau)+\mathcal{F}((1-\tau)\kappa_2)][\mathcal{G}((1-\tau)\kappa_1)+\mathcal{G}(\tau\kappa_2)]\\
&\quad+[\mathcal{F}((1-\tau)\kappa_1)+\mathcal{F}(\tau\kappa_2)][\mathcal{G}(\kappa_1\tau)+\mathcal{G}((1-\tau)\kappa_2)].
\end{aligned}$$

Taking the $q-$integral of both sides in (3.7) with respect to τ over $[0,1]$ and by Definitions 3 and 4, we get

$$\begin{aligned}
&\mathcal{F}(\kappa_1+\kappa_2)\,\mathcal{G}(\kappa_1+\kappa_2)\\
&\leq \frac{1}{\kappa_2-\kappa_1}\left[\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\,\mathcal{G}(\varkappa)\ {}^{\kappa_2}d_q\varkappa+\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\,\mathcal{G}(\varkappa)\ {}_{\kappa_1}d_q\varkappa\right]\\
&\quad+\int_0^1[\mathcal{F}(\tau\kappa_1)+\mathcal{F}((1-\tau)\kappa_2)][\mathcal{G}(\tau\kappa_2)+\mathcal{G}((1-\tau)\kappa_1)]\,d_q\tau\\
&\quad+\int_0^1[\mathcal{F}(\tau\kappa_2)+\mathcal{F}((1-\tau)\kappa_1)][\mathcal{G}(\tau\kappa_1)+\mathcal{G}((1-\tau)\kappa_2)]\,d_q\tau,
\end{aligned}$$

which proves inequality (3.6). □

Remark 4. In Theorem 4, if we set the limit $q\to 1^-$, then we have the following inequality of Hermite-Hadamard type for the product of two subadditive functions (see, [16]):

$$\begin{aligned}
&\frac{1}{2}\mathcal{F}(\kappa_1+\kappa_2)\,\mathcal{G}(\kappa_1+\kappa_2)\\
&\leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\,\mathcal{G}(\varkappa)\,d\varkappa\\
&\quad+\int_0^1[\mathcal{F}(\kappa_1\tau)\,\mathcal{G}((1-\tau)\kappa_1)+\mathcal{F}((1-\tau)\kappa_2)\,\mathcal{G}(\tau\kappa_2)]\,d\tau\\
&\quad+\int_0^1[\mathcal{F}(\kappa_1\tau)\,\mathcal{G}(\tau\kappa_2)\,d\tau+\mathcal{F}(\tau\kappa_2)\,\mathcal{G}(\tau\kappa_1)]\,d\tau.
\end{aligned}$$

Corollary 2. *Under the conditions of Theorem 4, if we take $\mathcal{F}(\tau\varkappa)\leq\tau\mathcal{F}(\varkappa)$, then we get the following $q-$Hermite-Hadamard type inequality for the product of two convex functions:*

$$\begin{aligned}
\mathcal{F}(\kappa_1+\kappa_2)\,\mathcal{G}(\kappa_1+\kappa_2)&\leq\frac{1}{\kappa_2-\kappa_1}\left[\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\,\mathcal{G}(\varkappa)\ {}^{\kappa_2}d_q\varkappa+\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\,\mathcal{G}(\varkappa)\ {}_{\kappa_1}d_q\varkappa\right]\\
&\quad+\frac{2q^2}{[2]_q[3]_q}M(\kappa_1,\kappa_2)+\left(\frac{1}{[3]_q}+\frac{q(1+q^2)}{[2]_q[3]_q}\right)N(\kappa_1,\kappa_2),
\end{aligned}$$

where

$$\begin{aligned}
M(\kappa_1,\kappa_2)&=\mathcal{F}(\kappa_1)\,\mathcal{G}(\kappa_1)+\mathcal{F}(\kappa_2)\,\mathcal{G}(\kappa_2) \qquad (3.8)\\
N(\kappa_1,\kappa_2)&=\mathcal{F}(\kappa_1)\,\mathcal{G}(\kappa_2)+\mathcal{F}(\kappa_2)\,\mathcal{G}(\kappa_1).
\end{aligned}$$

Remark 5. In Corollary 2, if we take the limit $q \to 1^{-}$, then we obtain the inequality (2) of [12, Theorem 1].

Theorem 5. *We assume that the continuous mappings $\mathcal{F}, \mathcal{G} : I = [0,\infty) \to R$ are subadditive, $\kappa_1, \kappa_2 \in I^{\circ}$ with $\kappa_1 < \kappa_2$, then the following inequality holds:*

$$\begin{aligned}
&\frac{1}{\kappa_2-\kappa_1}\left[\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\mathcal{G}(\varkappa)\ {}^{\kappa_2}d_q\varkappa + \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\mathcal{G}(\varkappa)\ {}_{\kappa_1}d_q\varkappa\right] \qquad (3.9)\\
&\leq \frac{1}{\kappa_1}\left[\int_{0}^{\kappa_1} \mathcal{F}(\varkappa)\mathcal{G}(\varkappa)\ {}_{0}d_q\varkappa + \int_{0}^{\kappa_1} \mathcal{F}(\varkappa)\mathcal{G}(\varkappa)\ {}^{\kappa_1}d_q\varkappa\right]\\
&\quad + \frac{1}{\kappa_2}\left[\int_{0}^{\kappa_2} \mathcal{F}(\varkappa)\mathcal{G}(\varkappa)\ {}^{\kappa_2}d_q\varkappa + \int_{0}^{\kappa_2} \mathcal{F}(\varkappa)\mathcal{G}(\varkappa)\ {}_{0}d_q\varkappa\right]\\
&\quad + \int_0^1 \left[\mathcal{F}(\tau\kappa_1)\mathcal{G}((1-\tau)\kappa_2) + \mathcal{F}((1-\tau)\kappa_2)\mathcal{G}(\tau\kappa_1)\right] d_q\tau\\
&\quad + \int_0^1 \left[\mathcal{F}((1-\tau)\kappa_1)\mathcal{G}(\tau\kappa_2) + \mathcal{F}(\tau\kappa_2)\mathcal{G}((1-\tau)\kappa_1)\right] d_q\tau,
\end{aligned}$$

where $0 < q < 1$.

Proof. Since $\mathcal{F}$ and $\mathcal{G}$ are subadditive functions, we can write

$$\begin{aligned}
\mathcal{F}(\kappa_1\tau + (1-\tau)\kappa_2) &\leq \mathcal{F}(\kappa_1\tau) + \mathcal{F}((1-\tau)\kappa_2)\\
\mathcal{G}(\kappa_1\tau + (1-\tau)\kappa_2) &\leq \mathcal{G}(\kappa_1\tau) + \mathcal{G}((1-\tau)\kappa_2),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}((1-\tau)\kappa_1 + \tau\kappa_2) &\leq \mathcal{F}((1-\tau)\kappa_1) + \mathcal{F}(\tau\kappa_2)\\
\mathcal{G}((1-\tau)\kappa_1 + \tau\kappa_2) &\leq \mathcal{G}((1-\tau)\kappa_1) + \mathcal{G}(\tau\kappa_2).
\end{aligned}$$

From the above inequalities, we get

$$\begin{aligned}
&\mathcal{F}(\kappa_1\tau + (1-\tau)\kappa_2)\mathcal{G}(\kappa_1\tau + (1-\tau)\kappa_2) + \mathcal{F}((1-\tau)\kappa_1 + \tau\kappa_2)\mathcal{G}((1-\tau)\kappa_1 + \tau\kappa_2)\\
&\leq \mathcal{F}(\kappa_1\tau)\mathcal{G}(\kappa_1\tau) + \mathcal{F}(\kappa_1\tau)\mathcal{G}((1-\tau)\kappa_2)\\
&\quad + \mathcal{F}((1-\tau)\kappa_2)\mathcal{G}(\kappa_1\tau) + \mathcal{F}((1-\tau)\kappa_2)\mathcal{G}((1-\tau)\kappa_2)\\
&\quad + \mathcal{F}((1-\tau)\kappa_1)\mathcal{G}((1-\tau)\kappa_1) + \mathcal{F}((1-\tau)\kappa_1)\mathcal{G}(\tau\kappa_2)\\
&\quad + \mathcal{F}(\tau\kappa_2)\mathcal{G}((1-\tau)\kappa_1) + \mathcal{F}(\tau\kappa_2)\mathcal{G}(\tau\kappa_2)
\end{aligned}$$

and taking the $q-$integral with respect to τ on $[0,1]$, then using Definitions 3 and 4, we obtain inequality (3.9). □

Remark 6. In Theorem 5, if we set the limit $q \to 1^{-}$, then we obtain the following inequality of Hermite-Hadamard type for the product of two subadditive functions (see, [16]):

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa)\mathcal{G}(\varkappa)d\varkappa \leq \frac{1}{\kappa_1}\int_{0}^{\kappa_1} \mathcal{F}(\varkappa)\mathcal{G}(\varkappa)d\varkappa + \frac{1}{\kappa_2}\int_{0}^{\kappa_2} \mathcal{F}(\varkappa)\mathcal{G}(\varkappa)d\varkappa$$

$$+\int_0^1 \mathcal{F}(\kappa_1\tau)\,\mathcal{G}((1-\tau)\kappa_2)\,d\tau+\int_0^1 \mathcal{F}(\tau\kappa_2)\,\mathcal{G}((1-\tau)\kappa_1)\,d\tau.$$

Corollary 3. *In Theorem 5, if we assume that $\mathcal{F}(\tau\varkappa)\leq\tau\mathcal{F}(\varkappa)$, then we have the following inequality of $q-$Hermite-Hadamard type for the product of two convex functions:*

$$\frac{1}{\kappa_2-\kappa_1}\left[\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\,\mathcal{G}(\varkappa)\ {}^{\kappa_2}d_q\varkappa+\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)\,\mathcal{G}(\varkappa)\ {}_{\kappa_1}d_q\varkappa\right]$$
$$\leq\frac{2q^2}{[2]_q\,[3]_q}N(\kappa_1,\kappa_2)+\left(\frac{1}{[3]_q}+\frac{q\left(1+q^2\right)}{[2]_q\,[3]_q}\right)M(\kappa_1,\kappa_2),$$

where $M(\kappa_1,\kappa_2)$ and $N(\kappa_1,\kappa_2)$ are defined by (3.8).

Remark 7. In Corollary 3, if we take the limit $q\to 1^-$, then we obtain the inequality (1) of [12, Theorem 1].

4. Conclusions

In this investigation, via the q_{κ_1} and q^{κ_2}-quantum integrals, we established q-Hermite-Hadamard type inequalities and some related inequalities for subadditive functions. It is also revealed that the findings shown here are a clear generalisation of those that have already been published. It is an important and new problem for the next researcher to be able to give similar inequalities in their future work for different integral operators.

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