Miskolc Mathematical Notes
Vol. 23 (2022), No. 2, pp. 811–845

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2022.3766

A CLASS OF CONFORMABLE BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS

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Received 14 March, 2021

Abstract. In this paper, we study conformable backward stochastic differential equations driven by a Brownian motion and a compensated random measure. We derive the conformable Itô's formula with jumps and a priori estimates and we obtain the existence and uniqueness of solutions under some assumptions in the framework of the conformable derivative. In addition we get a predictable representation of the solution. Comparison theorems for the operator g under different conditions are given. We also establish the inverse comparison theorem for the operator g under a Lipschitz condition.

2010 *Mathematics Subject Classification:* 35J61; 35B09; 35B33; 35B40; 35Q55

Keywords: backward stochastic differential equations, conformable derivative, existence, predictable representation, comparison theorem, inverse comparison theorem

1. Introduction

In stock transaction, the holder always expects that the option pricing problem can be calculated and analyzed more accurately in order to avoid the risk and to achieve the expected return and backward stochastic differential equations is a good tool to deal with this kind of problem. The linear backward stochastic differential equation was proposed by Bismut [4] in 1978. In 1990, Pardoux and Peng [19] obtained an existence and uniqueness theorem for non-linear backward stochastic differential equations under a Lipschitz condition; for applications in mathematical finance, stochastic control, partial differential equations, stochastic differential games and economics we refer the reader to [5, 9, 11].

Mandelbrot [14] proposed in 1963 that the distribution of returns in the financial market has a typical thick-tailed characteristic which does not obey the general normal distribution, and has the characteristics of self similarity and long-term correlation. This has led to a large number of pricing models driven by Brownian motion

This work is partially supported by the National Natural Science Foundation of China (12161015), Training Object of High Level and Innovative Talents of Guizhou Province ((2016)4006), Guizhou Data Driven Modeling Learning and Optimization Innovation Team ([2020]5016), and Major Project of Guizhou Postgraduate Education and Teaching Reform (YJSJGKT[2021]041).

that do not conform to the real market (see [7, 10, 17, 18]). The fractional backward stochastic differential equation can give a reasonable explanation to this kind of problem. Fractional Brownian motion is neither a semi martingale nor a Markov process [15], so it can describe phenomena that cannot be described by semi martingales and Markov processes, and it plays an important role in finance, hydrology, meteorology, transportation, etc. It has self similarity and long-term memory [6], so it is practical to describe asset price changes by fractional backward stochastic differential equations. However, fractional derivatives have unique non-local characteristics (essentially different from the local properties of classical derivatives), so there are difficulties in the theoretical analysis and application [13]. Khalil et al. [12] defined a derivative called the conformable fractional derivative and when $\rho = 1$ its definition is consistent with the classical derivative. The conformable derivative has not only some properties of fractional derivative, but also some properties of integer order derivatives [2, 12]. In this paper we consider conformable backward stochastic differential equations.

This article is mainly divided into six parts. First, we give some important lemmas that are applicable to conformable backward stochastic differential equation. We give the Itô's formula with jumps, state a priori estimates, and prove the existence of a unique solution. Then, the predictable representations of different types of solutions are studied. Next, we establish comparison theorems for the operator g under different constraints, and give a class of inverse comparison theorems. Finally, some examples are given to illustrate the theoretical results.

2. Preliminaries

Let $B(\cdot)$ be a standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, $\mathfrak{B}(\mathbb{R})$ denotes the Borel sets of $\mathbb{R}$ and $\mathbb{E}$ denotes the expected value. Next we give some definitions of spaces.

$L^2_Q(\mathbb{R})$ is a space of measurable function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying $\int_{\mathbb{R}} |\varphi(s)|^2 Q(ds) < \infty$, where Q is a ϖ-finite measure; $\mathbb{L}^2(\mathbb{R})$ is a space of random variable $\zeta : \Omega \to \mathbb{R}$ satisfying $\mathbb{E}[|\zeta|^2] < \infty$; $\mathbb{H}^2(\mathbb{R})$ is a space of predictable process $Y : \Omega \times [a,T] \to \mathbb{R}$ satisfying $\mathbb{E}[\int_a^T |Y(t)|^2 dt] < \infty$ with $\|Y\|^2_{\mathbb{H}^2} = \mathbb{E}[\int_a^T e^{\sigma t}|Y(t)|^2 dt]$, σ is a constant; $\mathbb{H}^2_N(\mathbb{R})$ is a space of predictable process $Z : \Omega \times [a,T] \times \mathbb{R} \to \mathbb{R}$ satisfying

$$\mathbb{E}[\int_a^T \int_{\mathbb{R}} |Z(t,s)|^2 Q(t,ds)\eta(t)dt] < \infty$$

with $\|Z\|^2_{\mathbb{H}^2_N} = \mathbb{E}[\int_a^T \int_{\mathbb{R}} e^{\sigma t}|Z(t,s)|^2 Q(t,ds)\eta(t)dt]$; $\mathbb{S}^2(\mathbb{R})$ is a space of adapted, càdlàg process $X : \Omega \times [a,T] \to \mathbb{R}$ satisfying $\mathbb{E}[\sup_{t\in[a,T]} |X(t)|^2] < \infty$ with $\|X\|^2_{\mathbb{S}^2} = \mathbb{E}[\sup_{t\in[a,T]} e^{\sigma t}|X(t)|^2]$.

Definition 1 ([2, Definition 2.1]). The conformable derivative with low index ρ of a function $f:[a,\infty)\to\mathbb{R}$ is defined as

$$D^a_\rho f(x)=\lim_{\varepsilon\to 0}\frac{f(x+\varepsilon(x-a)^{1-\rho})-f(x)}{\varepsilon},\ x>a,\ 0<\rho\le 1,$$

while $D^a_\rho f(a)=\lim_{x\to a^+}D^a_\rho f(x)$.

Remark 1 ([1, Theorem 2]). Fix $0<\rho\le 1$ and $x>a$. A function $f:[a,\infty)\to\mathbb{R}$ has a conformable derivative $D^a_\rho f(x)$ if and only if it is differentiable at x and $D^a_\rho f(x)=(x-a)^{1-\rho}f'(x)$ holds.

Definition 2 ([2, Definition 3.1]). The conformable integral with low index ρ of a function $f:[a,\infty)\to\mathbb{R}$ is defined as

$$I^a_\rho f(x)=\int_a^x f(t)d\frac{(t-a)^\rho}{\rho}=\int_a^x f(t)(t-a)^{\rho-1}dt,\ x>a,\ 0<\rho\le 1.$$

Definition 3 ([8, p.15]). Let ϑ be compensator of the random measure N. For any $(t,s)\in[a,T]\times\mathbb{R}$, the compensated random measure $\tilde{N}$ is defined as

$$\tilde{N}(\omega,dt,ds)=N(\omega,dt,ds)-\vartheta(\omega,dt,ds).$$

Lemma 1 ([8, p.17]). *If $\eta:\Omega\times[a,T]\to[a,\infty)$ is a predictable process, Q is a kernel from $(\Omega\times[a,T],\mathfrak{P})$ into $(\mathbb{R},\mathfrak{B}(\mathbb{R}))$ satisfying $\int_a^T\int_{\mathbb{R}}z^2Q(t,ds)\eta(t)dt<\infty$, then*

$$\vartheta(dt,ds)=Q(t,ds)\eta(t)dt.$$

Lemma 2 ([16, Theorem 7.3]). *Let $f\in\mathbb{L}^2(\mathbb{R})$. For any $a\le t\le T$, define $x(t)=\int_a^t f(s)dB(s)$ and $A(t)=\int_a^t|f(s)|^2ds$. Then for any $p>0$, there exist universal positive constants c_p, C_p(depending only on p), such that*

$$c_p\mathbb{E}[|A(t)|^{\frac{p}{2}}]\le\mathbb{E}[\sup_{a\le t\le T}|x(s)|^p]\le C_p\mathbb{E}[|A(t)|^{\frac{p}{2}}].$$

Lemma 3 ([8, Theorem 2.3.1]). *(i) Let $V:\Omega\times[a,T]\to\mathbb{R}$ be a predictable process satisfying $\int_a^T|V(s)|^2dt<\infty$. Then $\int_a^t V(s)dB(s)$ is a continuous local martingale with the quadratic variation process*

$$\left[\int_a^\cdot V(\tau)dB(\tau),\int_a^\cdot V(\tau)dB(\tau)\right](t)=\int_a^t|V(\tau)|^2d\tau,\ a\le t\le T.$$

(ii) Let $V:\Omega\times[a,T]\times\mathbb{R}\to\mathbb{R}$ be a predictable process satisfying

$$\int_a^T\int_{\mathbb{R}}|V(t,s)|^2Q(t,ds)\eta(t)dt<\infty.$$

Then $\int_a^t\int_{\mathbb{R}}V(\tau,s)\tilde{N}(d\tau,ds)$ is a càdlàg local martingale with the quadratic variation process

$$\left[\int_a^\cdot\int_{\mathbb{R}}V(\tau,s)\tilde{N}(d\tau,ds),\int_a^\cdot\int_{\mathbb{R}}V(\tau,s)\tilde{N}(d\tau,ds)\right](t)$$

$$= \int_a^t \int_{\mathbb{R}} |V(\tau,s)|^2 N(d\tau,ds),\ a \le t \le T.$$

Lemma 4 ([8, Theorem 2.3.2]). *Let $V : \Omega \times [a,T] \times \mathbb{R} \to \mathbb{R}$ be a predictable process satisfying*

$$\int_a^T \int_{\mathbb{R}} |V(t,ds)| Q(t,ds) \eta(t) dt < \infty.$$

Then

$$(\int_a^T \int_{\mathbb{R}} V(t,ds) \tilde{N}(dt,ds), a \le t \le T)$$

is a càdlàg local martingale and

$$(\int_a^T \int_{\mathbb{R}} V(t,ds) N(dt,ds), a \le t \le T)$$

is a càdlàg process.

Lemma 5 ([8, p.23]). *If V is a non-negative predictable process satisfying*

$$\mathbb{E}\left[\int_a^T \int_{\mathbb{R}} V(t,s) Q(t,ds) \eta(t) dt\right] < \infty,$$

then

$$\mathbb{E}\left[\int_a^T \int_{\mathbb{R}} V(t,s) N(dt,ds)\right] = \mathbb{E}\left[\int_a^T \int_{\mathbb{R}} V(t,s) Q(t,ds) \eta(t) dt\right].$$

Lemma 6 ([8, p.42]). *Let $\gamma > 0$ and $x_1, x_2 \in \mathbb{R}$. Then*

$$2|x_1 x_2| \le \frac{1}{\gamma}|x_1|^2 + \gamma |x_2|^2. \tag{2.1}$$

Lemma 7 ([8, Theorem 2.5.1]). *Let B be a $(\mathbb{P}, \mathcal{F})$-Brownian motion, N be a $(\mathbb{P}, \mathcal{F})$-random measure with compensator $\vartheta(d\tau, ds) = Q(\tau, ds)\eta(\tau)d\tau$. Assume an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$. Then*

$$B^{\mathbb{Q}}(t) = B(t) - \int_a^t \phi(\tau) d\tau,\ a \le t \le T,$$

$$\tilde{N}^{\mathbb{Q}}(t,A) = N(t,A) - \int_a^t \int_{\mathbb{R}} (1 + \kappa(\tau,s)) Q(\tau,ds) \eta(\tau) d\tau,\ a \le t \le T,\ \ A \in \mathfrak{B}(\mathbb{R}),$$

are $(\mathbb{Q}, \mathcal{F})$-Brownian motion and a $(\mathbb{Q}, \mathcal{F})$-random measure.

Lemma 8 ([8, Proposition 2.5.1]). *Let M be the stochastic exponential defined by*

$$\frac{dM(t)}{M(t)} = \phi(t) dB(t) + \int_{\mathbb{R}} \kappa(t,s) \tilde{N}(dt,ds),\ M(a) = 1,$$

where $\phi(\cdot)$ and $\kappa(\cdot,s)$ are predictable processes satisfying

$$|\phi(t)| \le K,\ \int_{\mathbb{R}} |\kappa(t,s)|^2 Q(t,ds) \eta(t) \le K,\ a \le t \le T,\ \ \kappa(t,s) > -1,\ a \le t \le T,\ s \in \mathbb{R}.$$

The process M is a square integrable, positive martingale.

3. Existence and uniqueness results

In this part, we consider the following conformable backward stochastic differential equation with jumps

$$\begin{cases} D^a_\rho X(t) = -g(t,X(t),Y(t),Z(t,\cdot)) + Y(t)\dfrac{dB(t)}{dt} \\ \qquad\qquad + \dfrac{\int_{\mathbb{R}} Z(t,s)\tilde{N}(dt,ds)}{dt},\ 0<\rho\leq 1,\ t\in[a,T], \\ X(T) = \zeta \in \mathbb{L}^2(\mathbb{R}), \end{cases} \tag{3.1}$$

where D^a_ρ is the conformable derivative, X is an adapted process, Y and Z are given control processes, $g : \Omega\times[a,T]\times\mathbb{R}\times\mathbb{R}\times L^2_Q(\mathbb{R}) \to \mathbb{R}$ is predictable, $B(\cdot)$ is a given Brownian motion and $\tilde{N}$ is a compensated random measure.

From now on, in order to simplify the notation, we think of ω as an implicit function in function $g(\omega,t,X(t),Y(t),Z(t,s))$, so as not to cause confusion.

Using Definition 2, (3.1) has an equivalent form

$$\begin{cases} dX(t) = -(t-a)^{\rho-1}g(t,X(t),Y(t),Z(t,\cdot))dt + (t-a)^{\rho-1}Y(t)dB(t) \\ \qquad\qquad + (t-a)^{\rho-1}\displaystyle\int_R Z(t,s)\tilde{N}(dt,ds), \\ X(T) = \zeta. \end{cases} \tag{3.2}$$

Definition 4. A triple $(X,Y,Z) \in \mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_N(\mathbb{R})$ is called the solution of (3.1) if (X,Y,Z) satisfies

$$\begin{aligned} X(t) = \zeta &+ \int_t^T (\tau-a)^{\rho-1}g(\tau,X(\tau),Y(\tau),Z(\tau,\cdot))d\tau - \int_t^T (\tau-a)^{\rho-1}Y(\tau)dB(\tau) \\ &- \int_t^T (\tau-a)^{\rho-1}\int_{\mathbb{R}} Z(\tau,s)\tilde{N}(d\tau,ds),\ 0<\rho\leq 1,\ a\leq t\leq T. \end{aligned} \tag{3.3}$$

Next we give some important assumptions:

$(S1)$ The terminal value $\zeta\in\mathbb{L}^2(\mathbb{R})$.

$(S2)$ For all (x,y,z), $(x',y',z') \in \mathbb{R}\times\mathbb{R}\times L^2_Q(\mathbb{R})$, g is Lipschitz continuous and

$$|g(t,x,y,z)-g(t,x',y',z')|^2 \leq K\left(|x-x'|^2+|y-y'|^2+\int_{\mathbb{R}}|z(s)-z'(s)|^2Q(t,ds)\eta(t)\right),$$

a.s., a.e. $t\in[a,T]$, $K\in\mathbb{R}$.

$(S3)$ $\mathbb{E}[\int_a^T |g(t,0,0,0)|^2dt] < \infty$.

Lemma 9. *Suppose that* (X,Y,Z), $(X',Y',Z') \in \mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_N(\mathbb{R})$, *and* (ζ,g), (ζ',g') *satisfy* $(S1)$, $(S2)$ *and* $(S3)$. *For any* $a\leq t\leq T$, *if*

$$e^{\sigma t}|X(t)-X'(t)|^2 + \sigma\int_t^T e^{\sigma\tau}|X(\tau)-X'(\tau)|^2d\tau$$

$$
\begin{aligned}
&+\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Y(\tau)-Y'(\tau)|^2 d\tau \\
&+\int_t^T \int_{\mathbb{R}} e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Z(\tau,s)-Z'(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau \\
\leq{}& e^{\sigma T}|\zeta-\zeta'|^2+2\int_t^T e^{\sigma\tau}(\tau-a)^{\rho-1}|X(\tau)-X'(\tau)| \\
&\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|d\tau,
\end{aligned}
$$

then

$$
\begin{aligned}
&\mathbb{E}[e^{\sigma t}|X(t)-X'(t)|^2]+\mathbb{E}\left[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Y(\tau)-Y'(\tau)|^2 d\tau\right] \\
&\quad+\mathbb{E}\left[\int_t^T \int_{\mathbb{R}} e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Z(\tau,s)-Z'(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau\right] \\
&\leq \mathbb{E}[e^{\sigma T}|\zeta-\zeta'|^2]+\frac{1}{\sigma}\mathbb{E}\bigg[\int_t^T e^{\sigma t}(\tau-a)^{2(\rho-1)} \\
&\quad\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|^2 d\tau\bigg].
\end{aligned}
$$

Proof. From (2.1), one can see that

$$
\begin{aligned}
&2\int_t^T e^{\sigma\tau}(\tau-a)^{\rho-1}|X(\tau)-X'(\tau)| \\
&\quad\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|d\tau \\
&\leq \frac{1}{\sigma}\int_t^T e^{\sigma\tau}|X(\tau)-X'(\tau)|^2 d\tau+\sigma\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)} \\
&\quad\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|^2 d\tau,\ a\leq t\leq T.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&e^{\sigma t}|X(t)-X'(t)|^2+\sigma\int_t^T e^{\sigma\tau}|X(\tau)-X'(\tau)|^2 d\tau \\
&\quad+\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Y(\tau)-Y'(\tau)|^2 d\tau \\
&\quad+\int_t^T \int_{\mathbb{R}} e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Z(\tau,s)-Z'(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau \\
&\leq e^{\sigma T}|\zeta-\zeta'|^2+\frac{1}{\sigma}\int_t^T e^{\sigma\tau}|X(\tau)-X'(\tau)|^2 d\tau+\sigma\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)} \\
&\quad\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|^2 d\tau,\ a\leq t\leq T.
\end{aligned}
$$

Using Lemma 3 and Theorem I.51 in [20], we note the integrals in the above inequality are uniformly integrable martingales. Thus

$$\begin{aligned}
&\mathbb{E}[e^{\sigma t}|X(t)-X'(t)|^2]+\sigma\mathbb{E}\left[\int_t^T e^{\sigma\tau}|X(\tau)-X'(\tau)|^2 d\tau\right]\\
&\quad+\mathbb{E}\left[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Y(\tau)-Y'(\tau)|^2 d\tau\right]\\
&\quad+\mathbb{E}\left[\int_t^T\int_{\mathbb{R}} e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Z(\tau,s)-Z'(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau\right]\\
&\leq \mathbb{E}[e^{\sigma T}|\zeta-\zeta'|^2]+\sigma\mathbb{E}\left[\int_t^T e^{\sigma\tau}|X(\tau)-X'(\tau)|^2 d\tau\right]\\
&\quad+\frac{1}{\sigma}\mathbb{E}\left[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|g(\tau,X(\tau),Y(\tau),Z(\tau))\right.\\
&\quad\left.-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|^2 d\tau\right],\ a\leq t\leq T.
\end{aligned}$$

Let $\gamma=\sigma$, and we have

$$\begin{aligned}
&\mathbb{E}[e^{\sigma t}|X(t)-X'(t)|^2]+\mathbb{E}\left[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Y(\tau)-Y'(\tau)|^2 d\tau\right]\\
&\quad+\mathbb{E}\left[\int_t^T\int_{\mathbb{R}} e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Z(\tau,s)-Z'(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau\right]\\
&\leq \mathbb{E}[e^{\sigma T}|\zeta-\zeta'|^2]+\frac{1}{\sigma}\mathbb{E}\left[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|g(\tau,X(\tau),Y(\tau),Z(\tau))\right.\\
&\quad\left.-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|^2 d\tau\right],\ a\leq t\leq T,
\end{aligned}$$

as required. The proof is complete. □

Lemma 10. *Assume that* $(S2)$ *is satisfied and let* $0<\rho\leq 1$. *Then*

$$\int_a^T e^{\sigma\tau}|Y(\tau)-Y'(\tau)|^2 d\tau\leq (T-a)^{2(1-\rho)}\int_a^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Y(\tau)-Y'(\tau)|^2 d\tau, \tag{3.4}$$

and

$$\begin{aligned}
&\int_a^T\int_{\mathbb{R}} e^{\sigma\tau}|Z(\tau,s)-Z'(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau\\
&\leq (T-a)^{2(1-\rho)}\int_a^T\int_{\mathbb{R}} e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Z(\tau,s)-Z'(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau.
\end{aligned}\tag{3.5}$$

Proof. We have

$$\int_a^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Y(\tau)-Y'(\tau)|^2 d\tau\geq\int_a^T e^{\sigma\tau}(T-a)^{2(\rho-1)}|Y(\tau)-Y'(\tau)|^2 d\tau$$

$$= (T-a)^{2(\rho-1)} \int_a^T e^{\sigma\tau}|Y(\tau)-Y'(\tau)|^2 d\tau.$$

Thus

$$\int_a^T e^{\sigma\tau}|Y(\tau)-Y'(\tau)|^2 d\tau \le (T-a)^{2(1-\rho)} \int_a^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Y(\tau)-Y'(\tau)|^2 d\tau,$$

where the function $(\cdot - a)^{2(\rho-1)}$ monotonicity has been used. The inequality in (3.5) can be obtained in a similar manner to that in (3.4). □

Lemma 11. *For any $a \le t \le T$, we have*

$$\mathbb{E}\left[\int_t^T e^{\sigma\tau}|X(\tau)-X'(\tau)|^2 d\tau\right] \le T\mathbb{E}[\sup_{t\le\tau\le T} e^{\sigma\tau}|X(\tau)-X'(\tau)|^2]. \tag{3.6}$$

Proof. From Lemma 2 and the Hölder inequality, we have

$$\begin{aligned}
\mathbb{E}\left[\int_t^T e^{\sigma\tau}|X(\tau)-X'(\tau)|^2 d\tau\right] &\le \sqrt{T-t}\,\mathbb{E}\left[\left(\int_t^T e^{2\sigma\tau}|X(\tau)-X'(\tau)|^4 d\tau\right)^{\frac{1}{2}}\right] \\
&\le (T-t)\mathbb{E}[\sup_{t\le\tau\le T} e^{\sigma\tau}|X(\tau)-X'(\tau)|^2] \\
&\le T\mathbb{E}[\sup_{t\le\tau\le T} e^{\sigma\tau}|X(\tau)-X'(\tau)|^2],\ a \le t \le T.
\end{aligned}$$

The proof is complete. □

We introduce the conformable Itô's formula with jumps here.

Theorem 1. *Suppose $U(\cdot) = U(X(\cdot),\cdot) \in C^{2,1}(\mathbb{R}\times\mathbb{R}_+,\mathbb{R})$. Then, for any $a \le t \le T$, we have*

$$\begin{aligned}
D_\rho^a U(t) d\frac{(t-a)^\rho}{\rho} &= \left(\frac{\partial u}{\partial t} - (t-a)^{\rho-1} g(t,X(t),Y(t),Z(t,\cdot))\frac{\partial u}{\partial x}\right. \\
&\quad \left. + \frac{1}{2}(t-a)^{2(\rho-1)}Y^2(t)\frac{\partial^2 u}{\partial x^2}\right)dt + (t-a)^{\rho-1}Y(t)\frac{\partial u}{\partial x}dB(t) \\
&\quad + (t-a)^{\rho-1}\frac{\partial u}{\partial x}\int_{\mathbb{R}} Z(t,s)\tilde{N}(dt,ds) \\
&\quad + \frac{1}{2}(t-a)^{2(\rho-1)}\frac{\partial^2 u}{\partial x^2}\int_{\mathbb{R}} Z^2(t,s)N(dt,ds),\ 0<\rho\le 1.
\end{aligned} \tag{3.7}$$

Proof. For any $a \le t \le T$, by Remark 1 and Lemma 3, we have

$$\begin{aligned}
D_\rho^a U(t) d\frac{(t-a)^\rho}{\rho} &= dU(t) \\
&= \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dX(t) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(dX(t))^2 \\
&= \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dX(t) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t-a)^{2(\rho-1)}Y^2(t)dt
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}(t-a)^{2(\rho-1)}\frac{\partial^2 u}{\partial x^2}\int_{\mathbb{R}} Z^2(t,s)N(dt,ds)\\
=&\frac{\partial u}{\partial t}dt-(t-a)^{\rho-1}\frac{\partial u}{\partial x}g(t,X(t),Y(t),Z(t,\cdot))dt\\
&+(t-a)^{\rho-1}\frac{\partial u}{\partial x}Y(t)dB(t)+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t-a)^{2(\rho-1)}Y^2(t)dt\\
&+(t-a)^{\rho-1}\frac{\partial u}{\partial x}\int_{\mathbb{R}} Z(t,s)\tilde{N}(dt,ds)\\
&+\frac{1}{2}(t-a)^{2(\rho-1)}\frac{\partial^2 u}{\partial x^2}\int_{\mathbb{R}} Z^2(t,s)N(dt,ds),\ 0<\rho\leq 1.
\end{aligned}
$$

The proof is complete. □

Theorem 2. *Suppose (S1), (S2) and (S3) hold and let* $(X,Y,Z),(X',Y',Z')\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_N(\mathbb{R})$ *be solutions to (3.1). Then*

$$
\begin{aligned}
&\|Y-Y'\|^2_{\mathbb{H}^2}+\|Z-Z'\|^2_{\mathbb{H}^2_N}\\
&\leq (T-a)^{2(1-\rho)}\bigg(\mathbb{E}[e^{\sigma T}|\zeta-\zeta'|^2]+\frac{1}{\sigma}\mathbb{E}\bigg[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}\\
&\quad\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|^2d\tau\bigg]\bigg),
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
&\|Y-Y'\|^2_{\mathbb{H}^2}+\|Z-Z'\|^2_{\mathbb{H}^2_N}\\
&\leq \hat{K}\bigg(\mathbb{E}[e^{\sigma T}|\zeta-\zeta'|^2]+\mathbb{E}\bigg[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}\\
&\quad\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X(\tau),Y(\tau),Z(\tau))|^2d\tau\bigg]\bigg),
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
\|X-X'\|^2_{\mathbb{S}^2}\leq&\Big(1+\sqrt{\sigma}(T-a)^{4(1-\rho)}\tilde{K}\Big)\mathbb{E}[e^{\sigma T}|\zeta-\zeta'|^2]\\
&+\sqrt{\sigma}\bigg(1+\frac{(T-a)^{4(1-\rho)}\tilde{K}}{\sigma}\bigg)\mathbb{E}\bigg[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}\\
&\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|^2d\tau\bigg],
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
\|X-X'\|^2_{\mathbb{S}^2}\leq&\hat{K}\bigg(\mathbb{E}[e^{\sigma T}|\zeta-\zeta'|^2]+\mathbb{E}\bigg[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}\\
&\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X(\tau),Y(\tau),Z(\tau))|^2d\tau\bigg]\bigg),
\end{aligned}
\tag{3.11}
$$

where $\tilde{K}$ depends on K_1, K_2, and $\hat{K}$ depends on K, T, σ, a, and $a \le t \le T$.

Proof. Case 1. Applying Theorem 1 to $e^{\sigma t}|X(t)-X'(t)|^2$, then

$$\begin{aligned}
&e^{\sigma t}|X(t)-X'(t)|^2+\sigma\int_t^T e^{\sigma\tau}|X(\tau)-X'(\tau)|^2 d\tau\\
&\quad+\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Y(\tau)-Y'(\tau)|^2 d\tau\\
&\quad+\int_t^T\int_{\mathbb{R}} e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Z(\tau,s)-Z'(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau\\
&=e^{\sigma T}|\zeta-\zeta'|^2+2\int_t^T e^{\sigma\tau}(\tau-a)^{\rho-1}|X(\tau)-X'(\tau)|\\
&\quad\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|d\tau\\
&\quad-2\int_t^T e^{\sigma\tau}(\tau-a)^{\rho-1}|X(\tau)-X'(\tau)||Y(\tau)-Y'(\tau)|dB(\tau)\\
&\quad-2\int_t^T e^{\sigma\tau}(\tau-a)^{\rho-1}|X(\tau)-X'(\tau)|\int_{\mathbb{R}}|Z(\tau,s)-Z'(\tau,s)|\tilde{N}(d\tau,ds)\\
&\quad-\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}\int_{\mathbb{R}}|Z(\tau,s)-Z'(\tau,s)|^2\tilde{N}(d\tau,ds),\ a\le t\le T. \qquad (3.12)
\end{aligned}$$

Thus

$$\begin{aligned}
&e^{\sigma t}|X(t)-X'(t)|^2+\sigma\int_t^T e^{\sigma\tau}|X(\tau)-X'(\tau)|^2 d\tau\\
&\quad+\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Y(\tau)-Y'(\tau)|^2 d\tau\\
&\quad+\int_t^T\int_{\mathbb{R}} e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Z(\tau,s)-Z'(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau\\
&\le e^{\sigma T}|\zeta-\zeta'|^2+2\int_t^T e^{\sigma\tau}(\tau-a)^{\rho-1}|X(\tau)-X'(\tau)|\\
&\quad\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|d\tau, a\le t\le T.
\end{aligned}$$

From Lemma 9, we have

$$\begin{aligned}
&\mathbb{E}\left[e^{\sigma t}|X(t)-X'(t)|^2\right]+\mathbb{E}\left[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Y(\tau)-Y'(\tau)|^2 d\tau\right]\\
&\quad+\mathbb{E}\left[\int_t^T\int_{\mathbb{R}} e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Z(\tau,s)-Z'(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau\right]\\
&\le\mathbb{E}[e^{\sigma T}|\zeta-\zeta'|^2]+\frac{1}{\sigma}\mathbb{E}\left[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|g(\tau,X(\tau),Y(\tau),Z(\tau))\right.\\
&\quad\left.-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|^2 d\tau\right], a\le t\le T.
\end{aligned}$$

Next from Lemma 10, estimate (3.8) holds.
Case 2. By the Lipschitz condition of g' and (2.1), we get

$$\begin{aligned}
&2(t-a)^{\rho-1}|X(t)-X'(t)|\times|g(t,X(t),Y(t),Z(t))-g'(t,X'(t),Y'(t),Z'(t))|\\
&\le 2\gamma|X(t)-X'(t)|^2+\frac{1}{\gamma}(t-a)^{2(\rho-1)}\\
&\quad\times|g(t,X(t),Y(t),Z(t))-g'(t,X(t),Y(t),Z(t))|^2\\
&\quad+\frac{K}{\gamma}(t-a)^{2(\rho-1)}\Big(|X(t)-X'(t)|^2+|Y(t)-Y'(t)|^2\\
&\quad+\int_{\mathbb{R}}|Z(t,s)-Z'(t,s)|^2Q(t,ds)\eta(t)\Big),\ a\le t\le T.
\end{aligned}$$

Similar to the proof of the estimate (3.8), but here we take $\gamma=\frac{\sigma+\sqrt{\sigma^2-8K}}{4},\sigma\ge 2\sqrt{2K}$ instead of $\gamma=\sigma$ in Lemma 9, then the proof of estimate (3.9) is complete.

Furthermore, we can also establish the following estimates in the same way:

$$\begin{aligned}
\mathbb{E}[e^{\sigma t}|X(t)-X'(t)|^2]&\le\mathbb{E}[e^{\sigma T}|\zeta-\zeta'|^2]+\frac{1}{\sigma}\mathbb{E}\Big[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}\\
&\quad\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau)|^2d\tau\Big],\ a\le t\le T,
\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}
\mathbb{E}\Big[e^{\sigma t}|X(t)-X'(t)|^2\Big]&\le\hat{K}\Big(\mathbb{E}[e^{\sigma T}|\zeta-\zeta'|^2]+\mathbb{E}\Big[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}\\
&\quad\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X(\tau),Y(\tau),Z(\tau)|^2d\tau\Big]\Big),\ a\le t\le T.
\end{aligned}\tag{3.14}$$

Case 3. By (3.12) and Definition 3 we have

$$\begin{aligned}
&e^{\sigma t}|X(t)-X'(t)|^2+\sigma\int_t^T e^{\sigma\tau}|X(\tau)-X'(\tau)|^2d\tau\\
&\quad+\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Y(\tau)-Y'(\tau)|^2d\tau\\
&\quad+\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}\int_{\mathbb{R}}|Z(\tau,s)-Z'(\tau,s)|^2N(d\tau,ds)\\
&=e^{\sigma T}|\zeta-\zeta'|^2+2\int_t^T e^{\sigma\tau}(\tau-a)^{\rho-1}|X(\tau)-X'(\tau)|\\
&\quad\times|(g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|d\tau\\
&\quad-2\int_t^T e^{\sigma\tau}(\tau-a)^{\rho-1}|X(\tau)-X'(\tau)||Y(\tau)-Y'(\tau)|dB(\tau)\\
&\quad-2\int_t^T e^{\sigma\tau}(\tau-a)^{\rho-1}|X(\tau)-X'(\tau)|\int_{\mathbb{R}}|Z(\tau,s)-Z'(\tau,s)|\tilde{N}(d\tau,ds),\ a\le t\le T.
\end{aligned}$$

Note

$$
\begin{aligned}
&\sup_{t\in[a,T]} e^{\sigma t}|X(t)-X'(t)|^2\\
&\le e^{\sigma T}|\zeta-\zeta'|^2+2\int_t^T e^{\sigma\tau}(\tau-a)^{\rho-1}|X(\tau)-X'(\tau)|\\
&\quad\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|d\tau\\
&\quad+4\sup_{t\in[a,T]}\left|\int_t^T e^{\sigma\tau}(\tau-a)^{\rho-1}|X(\tau)-X'(\tau)||Y(\tau)-Y'(\tau)|dB(\tau)\right|\\
&\quad+4\sup_{t\in[a,T]}\left|\int_t^T e^{\sigma\tau}(\tau-a)^{\rho-1}|X(\tau)-X'(\tau)|\int_{\mathbb{R}}|Z(\tau,s)-Z'(\tau,s)|\tilde{N}(d\tau,ds)\right|.
\end{aligned}
$$

According to Lemma 2, Lemma 5 and (2.1), we get

$$
\begin{aligned}
&\mathbb{E}[\sup_{t\in[a,T]} e^{\sigma t}|X(t)-X'(t)|^2]\\
&\le \mathbb{E}[e^{\sigma T}|\zeta-\zeta'|^2]+2\mathbb{E}\left[\int_t^T e^{\sigma\tau}(\tau-a)^{\rho-1}|X(\tau)-X'(\tau)|\right.\\
&\quad\left.\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|d\tau\right]\\
&\quad+\frac{K_1+K_2}{\gamma}\mathbb{E}[\sup_{t\in[a,T]} e^{\sigma t}|X(t)-X'(t)|^2]\\
&\quad+K_1\sigma\mathbb{E}\left[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Y(\tau)-Y'(\tau)|^2d\tau\right]\\
&\quad+K_2\sigma\mathbb{E}\left[\int_t^T\int_{\mathbb{R}} e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Z(\tau,s)-Z'(\tau,s)|^2Q(\tau,ds)\eta(\tau)d\tau\right],
\end{aligned}
$$

where $a\le t\le T,\ K_1,K_2\in\mathbb{R}$ and $0<\rho\le 1$.
From Lemma 9, we have

$$
\begin{aligned}
\|X-X'\|^2_{\mathbb{S}^2}\le{}&\mathbb{E}[e^{\sigma T}|\zeta-\zeta'|^2]+\frac{1}{\gamma}\mathbb{E}\left[\int_t^T e^{\sigma\tau}|X(\tau)-X'(\tau)|^2d\tau\right]\\
&+\gamma\mathbb{E}\left[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|g(\tau,X(\tau),Y(\tau),Z(\tau))\right.\\
&\left.-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|^2d\tau\right]\\
&+\frac{K_1+K_2}{\gamma}\mathbb{E}[\sup_{t\in[a,T]} e^{\sigma t}|X(t)-X'(t)|^2]\\
&+K_1\gamma\mathbb{E}\left[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Y(\tau)-Y'(\tau)|^2d\tau\right]
\end{aligned}
$$

$$+K_2\gamma\mathbb{E}\left[\int_t^T\int_{\mathbb{R}} e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|Z(\tau,s)-Z'(\tau,s)|^2Q(\tau,ds)\eta(\tau)d\tau\right].$$

Next by (3.8) and (3.6), one has

$$\begin{aligned}\frac{\gamma-K_1-K_2-T}{\gamma}\|X-X'\|^2 \le &\left(1+\gamma(T-a)^{4(1-\rho)}\tilde{K}\right)\mathbb{E}[e^{\sigma T}|\zeta-\zeta'|^2]\\ &+\gamma\left(1+\frac{(T-a)^{4(1-\rho)}\tilde{K}}{\sigma}\right)\mathbb{E}\left[\int_t^T e^{\sigma\tau}\times(\tau-a)^{2(\rho-1)}\right.\\ &\left.\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|^2d\tau\right],\end{aligned}$$

where $\tilde{K}$ depends on K_1 and K_2, $a \le t \le T$. Let $\gamma > K_1+K_2+T$, and then

$$\begin{aligned}\|X-X'\|^2_{\mathbb{S}^2} \le &\left(1+\gamma(T-a)^{4(1-\rho)}\tilde{K}\right)\mathbb{E}[e^{\sigma T}|\zeta-\zeta'|^2]\\ &+\gamma\left(1+\frac{(T-a)^{4(1-\rho)}\tilde{K}}{\sigma}\right)\mathbb{E}\left[\int_t^T e^{\sigma\tau}\times(\tau-a)^{2(\rho-1)}\right.\\ &\left.\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|^2d\tau\right].\end{aligned}$$

Choose $\gamma=\sqrt{\sigma}$, and we have

$$\begin{aligned}\|X-X'\|^2_{\mathbb{S}^2} \le &\left(1+\sqrt{\sigma}(T-a)^{4(1-\rho)}\tilde{K}\right)\mathbb{E}[e^{\sigma T}|\zeta-\zeta'|^2]\\ &+\sqrt{\sigma}\left(1+\frac{(T-a)^{4(1-\rho)}\tilde{K}}{\sigma}\right)\mathbb{E}\left[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)},\right.\\ &\left.\times|g(\tau,X(\tau),Y(\tau),Z(\tau))-g'(\tau,X'(\tau),Y'(\tau),Z'(\tau))|^2d\tau\right],\end{aligned}$$

which proves estimate (3.10).
Case 4. By the Lipschitz property of g' and some classical inequalities, then similar to the proof process of estimate (3.10), we can directly get estimate (3.11). □

Remark 2. When $\rho=1$, the above estimates are the same as in [8, Lemma 3.1.1].

We now prove the existence of a unique solution to the conformable backward stochastic differential equation (3.1). The essence of this proof is to transform (3.1) into an equivalent integral equation (3.3), and then prove the existence and uniqueness of the solution of (3.3). Its proof idea is similar to the proof idea of the existence and uniqueness of the solution to ordinary differential equations.

The following theorem is the core in this paper.

Theorem 3. *Assume that* $(S1)$, $(S2)$ *and* $(S3)$ *are satisfied. The conformable backward stochastic differential equation* (3.1) *has a unique solution* $(X,Y,Z) \in \mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_N(\mathbb{R})$, *provided* $\frac{1}{2}<\rho\le 1$.

Proof. We first prove the existence of the solution. Suppose $X^0(T)=\zeta$, $X^0(t)=Y^0(t)=Z^0(t,s)=0$, $(t,s)\in[a,T]\times\mathbb{R}$ and $(X^n,Y^n,Z^n)\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_N(\mathbb{R})$. Let $a\le t\le T$, and then let

$$\begin{aligned}X^{n+1}(t)=\zeta&+\int_t^T(\tau-a)^{\rho-1}g(\tau,X^n(\tau),Y^n(\tau),Z^n(\tau,\cdot))d\tau\\&-\int_t^T(\tau-a)^{\rho-1}Y^{n+1}(\tau)dB(\tau)\\&-\int_t^T(\tau-a)^{\rho-1}\int_{\mathbb{R}}Z^{n+1}(\tau,s)\tilde{N}(d\tau,ds).\end{aligned}\tag{3.15}$$

For any $m,n\in\mathbb{R}$, $|m+n|^2\le 2(m^2+n^2)$. From $(S2)$, we get

$$\begin{aligned}&|g(t,X^n(t),Y^n(t),Z^n(t))|^2\\&\le|g(t,X^n(t),Y^n(t),Z^n(t))+g(t,0,0,0)|^2\\&\le 2|g(t,X^n(t),Y^n(t),Z^n(t))|^2+2|g(t,0,0,0)|^2\\&\le 2|g(t,0,0,0)|^2+2K\left(|X^n(t)|^2+|Y^n(t)|^2+\int_{\mathbb{R}}|Z^n(t)|^2Q(t,ds)\eta(t)\right),\end{aligned}$$

where $a\le t\le T$, $K\in\mathbb{R}$ and g is independent of $(X^{n+1},Y^{n+1},Z^{n+1})$.

According to (3.6), we have

$$\begin{aligned}\mathbb{E}\left[\int_a^T|g(t,X^n(t),Y^n(t),Z^n(t))|^2dt\right]\le 2\mathbb{E}\left[\int_a^T|g(t,0,0,0)|^2dt\right]+2K(T\parallel X^n\|^2_{\mathbb{S}^2}\\+\|Y^n\|^2_{\mathbb{H}^2}+\|Z^n\|^2_{\mathbb{H}^2_N})<\infty,\ a\le t\le T.\end{aligned}$$

For any sequence of stopping times $(t_n)_{n\ge1}$, one has

$$\begin{aligned}\mathbb{E}[[X^{n+1},X^{n+1}](t_n)]=\zeta^2&+\mathbb{E}\left[\left(\int_{t_n}^T(\tau-a)^{\rho-1}g(\tau,X^n(\tau),Y^n(\tau),Z^n(\tau))d\tau\right)^2\right]\\&+\mathbb{E}\left[\int_{t_n}^T(\tau-a)^{2(\rho-1)}|Y^{n+1}(\tau)|^2d\tau\right]\\&+\mathbb{E}\left[\int_{t_n}^T\int_{\mathbb{R}}(\tau-a)^{2(\rho-1)}|Z^{n+1}(\tau,s)|^2Q(\tau,ds)\eta(\tau)d\tau\right].\end{aligned}$$

By the Cauchy-Schwarz inequality and the properties of integrals, we get

$$\begin{aligned}&\mathbb{E}\left[\left(\int_{t_n}^T(\tau-a)^{\rho-1}g(\tau,X^n(\tau),Y^n(\tau),Z^n(\tau))d\tau\right)^2\right]\\&\le\mathbb{E}\left[\int_{t_n}^T(\tau-a)^{2(\rho-1)}d\tau\int_{t_n}^T|g(\tau,X^n(\tau),Y^n(\tau),Z^n(\tau)|^2)d\tau\right]\\&\le\frac{1}{2\rho-1}(T-a)^{2\rho-1}\mathbb{E}\left[\int_{t_n}^T|g(\tau,X^n(\tau),Y^n(\tau),Z^n(\tau)|^2)d\tau\right],\end{aligned}$$

$$\mathbb{E}\left[\int_{t_n}^{T}(\tau-a)^{2(\rho-1)}|Y^{n+1}(\tau)|^2 d\tau\right] \le \frac{1}{2\rho-1}(T-a)^{2\rho-1}\mathbb{E}\left[\int_{t_n}^{T}|Y^{n+1}(\tau)|^2 d\tau\right],$$

and

$$\begin{aligned}&\mathbb{E}\left[\int_{t_n}^{T}\int_{\mathbb{R}}(\tau-a)^{2(\rho-1)}|Z^{n+1}(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau\right]\\ &\le \frac{1}{2\rho-1}(T-a)^{2\rho-1}\mathbb{E}\left[\int_{t_n}^{T}\int_{\mathbb{R}}|Z^{n+1}(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau\right],\end{aligned}$$

where $\frac{1}{2} < \rho \le 1$. Thus for any $a \le t \le T$, we have $\mathbb{E}[[X^{n+1},X^{n+1}](t)] < \infty$ and $X^{n+1}(t)$ is a uniformly integrable martingale. Taking the conditional expected values on (3.15), we get

$$\mathbb{E}[X^{n+1}(t)] = \mathbb{E}\left[\zeta + \int_t^T (\tau-a)^{\rho-1} g(\tau,X^n(\tau),Y^n(\tau),Z^n(\tau,\cdot))d\tau|\mathcal{F}_t\right].$$

Also the triple $(X^{n+1},Y^{n+1},Z^{n+1})$ must satisfy the equation

$$\begin{aligned}&\zeta + \int_a^T (\tau-a)^{\rho-1} g(\tau,X^n(\tau),Y^n(\tau),Z^n(\tau,\cdot))d\tau\\ &= \mathbb{E}\left[\zeta + \int_a^T (\tau-a)^{\rho-1} g(\tau,X^n(\tau),Y^n(\tau),Z^n(\tau,\cdot))d\tau\middle|\mathcal{F}_t\right]\\ &\quad + \int_a^T (\tau-a)^{\rho-1} Y^{n+1}(\tau)dB(\tau) + \int_a^T (\tau-a)^{\rho-1}\int_{\mathbb{R}} Z^{n+1}(\tau,s)\tilde{N}(d\tau,ds),\end{aligned}$$

then Y^{n+1} and Z^{n+1} can be determined by ζ and

$$\mathbb{E}[\zeta + \int_a^T (\tau-a)^{\rho-1} g(\tau,X^n(\tau),Y^n(\tau),Z^n(\tau,\cdot))d\tau|\mathcal{F}_t].$$

By Fatou's Lemma and Doob's inequality we can conclude that there exists a triple $(X^{n+1},Y^{n+1},Z^{n+1}) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$.

For any $a \le t \le T$, from Lemma 9, Lemma 10, (3.10) and (3.15) we have

$$\begin{aligned}&\|X^{n+1}-X^n\|^2_{\mathbb{S}^2} + \|Y^{n+1}-Y^n\|^2_{\mathbb{H}^2} + \|Z^{n+1}-Z^n\|^2_{\mathbb{H}^2_N}\\ &\le K_3\left[\int_t^T e^{\sigma\tau}(\tau-a)^{2(\rho-1)}|g(\tau,X^n(\tau),Y^n(\tau),Z^n(\tau))\right.\\ &\quad \left. - g(\tau,X^{n-1}(\tau),Y^{n-1}(\tau),Z^{n-1}(\tau))|^2 d\tau\right]\\ &\le 2KK_3(T-a)^{2(1-\rho)}(T\|X^n-X^{n-1}\|^2_{\mathbb{S}^2}\\ &\quad + \|Y^n-Y^{n-1}\|^2_{\mathbb{H}^2} + \|Z^n-Z^{n-1}\|^2_{\mathbb{H}^2_N})\\ &\le 2KK_3(T+1)(T-a)^{2(1-\rho)}(\|X^n-X^{n-1}\|^2_{\mathbb{S}^2}\\ &\quad + \|Y^n-Y^{n-1}\|^2_{\mathbb{H}^2} + \|Z^n-Z^{n-1}\|^2_{\mathbb{H}^2_N}),\end{aligned} \tag{3.16}$$

where $K_3 = \sqrt{\sigma} + \frac{(T-a)^{4(1-\rho)}\tilde{K}}{\sqrt{\sigma}} + \frac{(T-a)^{2(1-\rho)}}{\sigma}$. Thus there exists a limit $(X,Y,Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$ of the converging sequence $(X^{n+1}, Y^{n+1}, Z^{n+1})_{n\in\mathbb{N}}$, and (X,Y,Z) satisfies (3.3).

Next, we prove the uniqueness of the solution. Suppose (X,Y,Z) and (X',Y',Z') are solutions of (3.1). For any $a \le t \le T$, we have

$$\begin{aligned} X(t) - X'(t) = &\int_t^T (\tau - a)^{\rho-1}[g(\tau, X(\tau), Y(\tau), Z(\tau,\cdot)) \\ &- g(\tau, X'(\tau), Y'(\tau), Z'(\tau,\cdot))]d\tau \\ &- \int_t^T (\tau-a)^{\rho-1}(Y(\tau) - Y'(\tau))dB(\tau) \\ &- \int_t^T (\tau-a)^{\rho-1}\int_{\mathbb{R}}(Z(\tau,s) - Z'(\tau,s))\tilde{N}(d\tau, ds). \end{aligned}$$

Then

$$\begin{aligned} &\mathbb{E}\left[\sup_{a\le t\le T} e^{\sigma t}|X(t) - X'(t)|^2\right] \\ &= \mathbb{E}\left[\sup_{a\le t\le T} e^{\sigma t}\left|\int_t^T (\tau-a)^{\rho-1}[g(\tau, X(\tau), Y(\tau), Z(\tau,\cdot))\right.\right. \\ &\left.\left.- g(\tau, X'(\tau), Y'(\tau), Z'(\tau,\cdot))]d\tau\right|^2\right] \\ &+ \mathbb{E}\left[\sup_{a\le t\le T} e^{\sigma t}\int_t^T (\tau-a)^{2(\rho-1)}|Y(\tau) - Y'(\tau)|^2 d\tau\right] \\ &+ \mathbb{E}\left[\sup_{a\le t\le T} e^{\sigma t}\int_t^T \int_{\mathbb{R}}(\tau-a)^{2(\rho-1)}|Z(\tau,s) - Z'(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau\right]. \end{aligned}$$

From Lemma 10, (3.9) and (3.11), we can get

$$\|X - X'\|^2_{\mathbb{S}^2} + \|Y - Y'\|^2_{\mathbb{H}^2} + \|Z - Z'\|^2_{\mathbb{H}^2_N} = 0,$$

which proves the uniqueness of the solution (X,Y,Z). The proof is complete. □

4. Predictable representation of solution

We introduce two types of predictable representations of conformable backward stochastic differential equations. Assume that the terminal condition ζ satisfies $(S1)$, and a generator $g : [a,T] \to \mathbb{R}$ is a predictable process.

Proposition 1. *Let* $a \le t \le T$, $\frac{1}{2} < \rho \le 1$, *and we consider the backward stochastic differential equation with zero generator*

$$\begin{cases} dX(t) = (t-a)^{\rho-1} Y(t) dB(t) + (t-a)^{\rho-1} \int_{\mathbb{R}} Z(t,s) \tilde{N}(dt,ds), \\ X(T) = \zeta. \end{cases}$$

Then there exists a unique solution (X,Y,Z) *given by*

$$X(t) = \zeta - \int_t^T (\tau-a)^{\rho-1} Y(\tau) dB(\tau) - \int_t^T (\tau-a)^{\rho-1} \int_{\mathbb{R}} Z(\tau,s) \tilde{N}(d\tau,ds). \tag{4.1}$$

Moreover, we have

$$X(t) = \mathbb{E}[\zeta|\mathcal{F}_t],$$

and

$$\zeta = \mathbb{E}[\zeta] + \int_t^T (\tau-a)^{\rho-1} Y(\tau) dB(\tau) + \int_t^T (\tau-a)^{\rho-1} \int_{\mathbb{R}} Z(\tau,s) \tilde{N}(d\tau,ds).$$

Proof. Any solution (X,Y,Z) of the backward stochastic differential equation (4.1) satisfies

$$\zeta = X(a) + \int_a^T (\tau-a)^{\rho-1} Y(\tau) dB(\tau) + \int_a^T (\tau-a)^{\rho-1} \int_{\mathbb{R}} Z(\tau,s) \tilde{N}(d\tau,ds).$$

From Lemmas 3 and Lemma 5, for any sequence of stopping times $(t_n)_{n\ge 1}$, one has

$$\begin{aligned} \mathbb{E}[[X,X](t_n)] &= \zeta^2 + \mathbb{E}\left[\int_{t_n}^T (\tau-a)^{2(\rho-1)} |Y(\tau)|^2 d\tau\right] \\ &\quad + \mathbb{E}\left[\int_{t_n}^T \int_{\mathbb{R}} (\tau-a)^{2(\rho-1)} |Z(\tau,s)|^2 Q(\tau,ds)\eta(\tau) d\tau\right] \\ &\le \zeta^2 + \frac{1}{2\rho-1}(T-a)^{2\rho-1} \mathbb{E}\left[\int_{t_n}^T |Y(\tau)|^2 d\tau\right] \\ &\quad + \frac{1}{2\rho-1}(T-a)^{2\rho-1} \mathbb{E}\left[\int_{t_n}^T \int_{\mathbb{R}} |Z(\tau,s)|^2 Q(\tau,ds)\eta(\tau) d\tau\right], \end{aligned} \tag{4.2}$$

where $\frac{1}{2} < \rho \le 1$. The stochastic integrals in (4.2) are uniformly integrable martingales, then X is a square integral martingale and $\mathbb{E}[[X,X](a)] < \infty$. Furthermore, we derive $Y \in \mathbb{H}^2(\mathbb{R})$ and $Z \in \mathbb{H}^2_N(\mathbb{R})$. By Doob's inequality and Theorem I.9 in [20], we can conclude that $X \in \mathbb{S}^2(\mathbb{R})$. Taking the conditional expected value on both sides of (4.1), we get $X(t) = \mathbb{E}[\zeta|\mathcal{F}_t]$, $a \le t \le T$. Next taking the martingale $X(t) = \mathbb{E}[\zeta|\mathcal{F}_t]$ on both sides of (4.1), we have

$$\begin{aligned} \zeta = \mathbb{E}[\zeta] &+ \int_t^T (\tau-a)^{\rho-1} Y(\tau) dB(\tau) \\ &+ \int_t^T (\tau-a)^{\rho-1} \int_{\mathbb{R}} Z(\tau,s) \tilde{N}(d\tau,ds), \ a \le t \le T. \end{aligned}$$

In addition, the processes (Y,Z) can be uniquely expressed by ζ and $\mathbb{E}[\zeta]$. □

Proposition 2. *Let $a \leq t \leq T$, $\frac{1}{2} < \rho \leq 1$, and we consider the backward stochastic differential equation with generator independent of (X,Y,Z)*

$$\begin{cases} dX(t) = -(t-a)^{\rho-1}g(t)dt + (t-a)^{\rho-1}Y(t)dB(t) \\ \qquad\qquad + (t-a)^{\rho-1}\int_{\mathbb{R}} Z(t,s)\tilde{N}(dt,ds), \\ X(T) = \zeta, \end{cases}$$

where g satisfies $\mathbb{E}[\int_a^T |g(\tau)|^2 d\tau] < \infty$. Then there exists a unique solution $(X,Y,Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$ satisfying

$$\begin{aligned} X(t) = \zeta + \int_t^T (\tau-a)^{\rho-1}g(\tau)d\tau - \int_t^T (\tau-a)^{\rho-1}Y(\tau)dB(\tau) \\ - \int_t^T (\tau-a)^{\rho-1}\int_{\mathbb{R}} Z(\tau,s)\tilde{N}(d\tau,ds). \end{aligned} \tag{4.3}$$

Moreover we have

$$X(t) = \mathbb{E}\left[\zeta + \int_t^T (\tau-a)^{\rho-1}g(\tau)d\tau \middle| \mathcal{F}_t\right],$$

and

$$\begin{aligned} \zeta + \int_a^T (\tau-a)^{\rho-1}g(\tau)d\tau = \mathbb{E}\left[\zeta + \int_a^T (\tau-a)^{\rho-1}g(\tau)d\tau\right] + \int_a^T (\tau-a)^{\rho-1}Y(\tau)dB(\tau) \\ + \int_a^T (\tau-a)^{\rho-1}\int_{\mathbb{R}} Z(\tau,s)\tilde{N}(d\tau,ds). \end{aligned}$$

Proof. For any sequence of stopping times $(t_n)_{n\geq 1}$, from Lemma 3 and Lemma 5, one has

$$\begin{aligned} \mathbb{E}[[X,X](t_n)] = \zeta^2 + \mathbb{E}\left[\left(\int_{t_n}^T (\tau-a)^{\rho-1}g(\tau)d\tau\right)^2\right] \\ + \mathbb{E}\left[\int_{t_n}^T (\tau-a)^{2(\rho-1)}|Y(\tau)|^2 d\tau\right] \\ + \mathbb{E}\left[\int_{t_n}^T \int_{\mathbb{R}} (\tau-a)^{2(\rho-1)}|Z(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau\right]. \end{aligned} \tag{4.4}$$

Next, from the Schwarz inequality we have

$$\begin{aligned} \mathbb{E}[[X,X](t_n)] \leq \zeta^2 + \frac{1}{2\rho-1}(T-a)^{2\rho-1}\mathbb{E}\left[\int_{t_n}^T |g(\tau)|^2 d\tau\right] \\ + \frac{1}{2\rho-1}(T-a)^{2\rho-1}\mathbb{E}\left[\int_{t_n}^T e^{\sigma\tau}|Y(\tau)|^2 d\tau\right] \end{aligned}$$

$$+\frac{1}{2\rho-1}(T-a)^{2\rho-1}\mathbb{E}\left[\int_{t_n}^{T}\int_{\mathbb{R}}e^{\sigma\tau}|Z(\tau,s)|^2Q(\tau,ds)\eta(\tau)d\tau\right], \quad (4.5)$$

where $\frac{1}{2}<\rho\le 1$.

Notice that $\int_{t_n}^{T}e^{\sigma\tau}|Y(\tau)|^2d\tau$ and $\int_{t_n}^{T}\int_{\mathbb{R}}e^{\sigma\tau}|Z(\tau,s)|^2Q(\tau,ds)\eta(\tau)d\tau$ are uniformly integrable martingales. Since $\mathbb{E}[\int_a^T|g(\tau)|^2d\tau]<\infty$, we can get $\mathbb{E}[[X,X](a)]<\infty$. Furthermore, we also get $Y\in\mathbb{H}^2(\mathbb{R})$ and $Z\in\mathbb{H}_N^2(\mathbb{R})$. By Doob's inequality and Theorem I.9 in [20], we see that $X\in\mathbb{S}^2(\mathbb{R})$. Since $t\to\int_a^T(\tau-a)^{\rho-1}g(\tau)d\tau$ is a.s. continuous, see Sect. 4.3 in [3], then taking the conditional expected value in (5.4), we get $X(t)=\mathbb{E}[\zeta+\int_t^T(\tau-a)^{\rho-1}g(\tau)d\tau|\mathcal{F}_t]$, $a\le t\le T$. Next taking the martingale $X(t)=\mathbb{E}[\zeta+\int_t^T(\tau-a)^{\rho-1}g(\tau)d\tau|\mathcal{F}_t]$ on both sides of (5.4), we have

$$\zeta+\int_a^T(\tau-a)^{\rho-1}g(\tau)d\tau=\mathbb{E}\left[\zeta+\int_a^T(\tau-a)^{\rho-1}g(\tau)d\tau\right]+\int_a^T(\tau-a)^{\rho-1}Y(\tau)dB(\tau)$$
$$+\int_a^T(\tau-a)^{\rho-1}\int_{\mathbb{R}}Z(\tau,s)\tilde{N}(d\tau,ds).$$

This finishes the proof. □

5. Comparison theorem and converse comparison theorem

For any $(\omega,t,X(t),Y(t),Z(t))\in\Omega\times[a,T]\times\mathbb{R}\times\mathbb{R}\times L_Q^2(\mathbb{R})$, let

$$\begin{aligned}g(\omega,t,X(t),Y(t),Z(t,s))&=h\Big(\omega,t,X(t),Y(t),\int_{\mathbb{R}}Z(t,s)\delta(t,s)Q(t,ds)\eta(t)\Big)\\&=h(\omega,t,X(t),Y(t),U(t,s)),\end{aligned}$$

where the process $\delta:\Omega\times[a,T]\times\mathbb{R}\to\mathbb{R}$ is predictable and non-negative.

Hence equation (3.2) can be written in the following form

$$\begin{cases}dX(t)=-(t-a)^{\rho-1}h\Big(t,X(t),Y(t),\int_{\mathbb{R}}Z(t,s)\delta(t,s)Q(t,ds)\eta(t)\Big)dt\\ \qquad\qquad +(t-a)^{\rho-1}Y(t)dB(t)+(t-a)^{\rho-1}\int_{R}Z(t,s)\tilde{N}(dt,ds),\\ X(T)=\zeta.\end{cases} \quad (5.1)$$

Definition 5. A triple $(X(t),Y(t),Z(t))\in\mathbb{R}\times\mathbb{R}\times L_Q^2(\mathbb{R})$ is called the solution of equation (5.1) if $(X(t),Y(t),Z(t))$ satisfies

$$X(t)=\zeta+\int_t^T(\tau-a)^{\rho-1}h\Big(\tau,X(\tau),Y(\tau),\int_{\mathbb{R}}Z(\tau,s)\delta(\tau,s)Q(\tau,ds)\eta(\tau)\Big)d\tau$$
$$-\int_t^T(\tau-a)^{\rho-1}Y(\tau)dB(\tau)-\int_t^T(\tau-a)^{\rho-1}\int_{\mathbb{R}}Z(\tau,s)\tilde{N}(d\tau,ds),$$

where $a\le t\le T$.

From now on, in order to simplify the notation, we think of ω as an implicit function in the function $h(\omega,t,X(t),Y(t),\int_{\mathbb{R}} Z(t,s)\delta(t,s)Q(t,ds)\eta(t))$, so as not to cause confusion.

Next we introduce the following hypotheses, denoted by H_{cs}.

(i) the generator $h:\Omega\times[a,T]\times\mathbb{R}\times\mathbb{R}\times L_Q^2(\mathbb{R})\to\mathbb{R}$ is predictable, continuous and satisfies the Lipschitz condition

$$|h(\omega,t,x,y,u)-h(\omega,t,x',y',u')|\le K(|x-x'|+|y-y'|+|u-u'|),$$

a.s., a.e. $(\omega,t)\in\Omega\times[a,T]$, for any (x,y,u), $(x',y',u')\in\mathbb{R}\times\mathbb{R}\times L_Q^2(\mathbb{R})$,

(ii) for any $(t,x,y)\in\Omega\times\mathbb{R}\times\mathbb{R}$, the mapping $u\longmapsto h(t,x,y,u)$ is non-decreasing,

(iii) the mapping $t\longmapsto\int_{\mathbb{R}}|\delta(t,s)|^2Q(t,ds)\eta(t)$ is bounded,

(iv) $\mathbb{E}[\int_a^T|h(t,0,0,0)|^2dt]<\infty$.

Theorem 4. *Consider the conformable backward stochastic differential equation*

$$\begin{aligned}X(t)=\zeta+&\int_t^T(\tau-a)^{\rho-1}h(\tau,X(\tau),Y(\tau),\int_{\mathbb{R}}Z(\tau,s)\delta(\tau,s)Q(\tau,ds)\eta(\tau))d\tau\\&-\int_t^T(\tau-a)^{\rho-1}Y(\tau)dB(\tau)-\int_t^T(\tau-a)^{\rho-1}\int_{\mathbb{R}}Z(\tau,s)\tilde{N}(d\tau,ds),\end{aligned}$$

where $a\le t\le T$. Assume that the terminal values ζ, $\zeta'\in\mathbb{L}^2(\mathbb{R})$ and the generators h, h' satisfy the hypotheses H_{cs}. Let (X,Y,Z), $(X',Y',Z')\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}_N^2(\mathbb{R})$ be the unique solutions with (ζ,h) and (ζ',h'). For any $(t,x,y,z)\in[a,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, if $\zeta\ge\zeta'$ and $h(t,x,y,u)\ge h'(t,x,y,u)$, then $X(t)\ge X'(t)$, where $u=\int_{\mathbb{R}}z(t,s)\delta(t,s)Q(t,ds)\eta(t)$. In addition, for any $a\le t_0\le T$, if $X(t_0)=X'(t_0)$, then $X(t)=X'(t),t_0\le t\le T$.

Proof. The existence and uniqueness of solutions (X,Y,Z), (X',Y',Z') follows from Proposition 2. Define $\tilde{X}(t)=X(t)-X'(t)$, $\tilde{Y}(t)=Y(t)-Y'(t)$, $\tilde{Z}(t)=Z(t)-Z'(t)$ and $U(t,s)=\int_{\mathbb{R}}Z(t,s)\delta(t,s)Q(t,ds)\eta(t)$. The generator h' is Lipschitz continuous in x, y and z, and we can define the following processes

$$\alpha(t)=\begin{cases}\frac{h(t,X(t),Y(t),U(t,s))-h(t,X'(t),Y(t),U(t,s))}{\tilde{X}(t)} & if\ X(t)\ne X'(t),\\ 0 & otherwise,\end{cases}$$

$$\beta(t)=\begin{cases}\frac{h(t,X'(t),Y(t),U(t,s))-h(t,X'(t),Y'(t),U(t,s))}{\tilde{Y}(t)} & if\ Y(t)\ne Y'(t),\\ 0 & otherwise,\end{cases}$$

and

$$\psi(t)=\begin{cases}\frac{h(t,X'(t),Y'(t),U(t,s))-h(t,X'(t),Y'(t),U'(t,s))}{\int_{\mathbb{R}}\tilde{Z}(t,s)\delta(t,s)Q(t,ds)\eta(t)} & if\ Z(t)\ne Z'(t),\\ 0 & otherwise.\end{cases}$$

Thus

$$\begin{aligned} d\tilde{X}(t) = & -(t-a)^{\rho-1}\Big[h(t,X(t),Y(t),\int_{\mathbb{R}} Z(t,s)\delta(t,s)Q(t,ds)\eta(t)) \\ & - h'(t,X'(t),Y'(t),\int_{\mathbb{R}} Z'(t,s)\delta(t,s)Q(t,ds)\eta(t))\Big]dt \\ & + (t-a)^{\rho-1}\tilde{Y}(t)dB(t) + (t-a)^{\rho-1}\int_{\mathbb{R}} \tilde{Z}(t,s)\tilde{N}(dt,ds) \\ = & -(t-a)^{\rho-1}\Big[\tilde{h}(t,X'(t),Y'(t),\int_{\mathbb{R}} Z'(t,s)\delta(t,s)Q(t,ds)\eta(t)) + \alpha(t)\tilde{X}(t) \\ & + \beta(t)\tilde{Y}(t) + \psi(t)\int_{\mathbb{R}} \tilde{Z}(t,s)\delta(t,s)Q(t,ds)\eta(t)\Big]dt + (t-a)^{\rho-1}\tilde{Y}(t)dB(t) \\ & + (t-a)^{\rho-1}\int_{\mathbb{R}} \tilde{Z}(t,s)\tilde{N}(dt,ds), \end{aligned}$$

where

$$\begin{aligned} & \tilde{h}(t,X'(t),Y'(t),\int_{\mathbb{R}} Z'(t,s)\delta(t,s)Q(t,ds)\eta(t)) \\ & = h(t,X'(t),Y'(t),\int_{\mathbb{R}} Z'(t,s)\delta(t,s)Q(t,ds)\eta(t)) \\ & \quad - h'(t,X'(t),Y'(t),\int_{\mathbb{R}} Z'(t,s)\delta(t,s)Q(t,ds)\eta(t)). \end{aligned}$$

Applying Lemma 1 to $\tilde{X}(t)\exp(\int_a^t (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon)$, and we have

$$\begin{aligned} & d\tilde{X}(t)\exp\left(\int_a^t (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) \\ & = (t-a)^{\rho-1}\alpha(t)\tilde{X}(t)\exp\left(\int_a^t (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)dt \\ & \quad + \exp\left(\int_a^t (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)d\tilde{X}(t) \\ & = -(t-a)^{\rho-1}\exp\left(\int_a^t (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) \\ & \quad \times \tilde{h}\left(t,X'(t),Y'(t),\int_{\mathbb{R}} Z'(t,s)\delta(t,s)Q(t,ds)\eta(t)\right)dt \\ & \quad - (t-a)^{\rho-1}\beta(t)\tilde{Y}(t)\exp\left(\int_a^t (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)dt - (t-a)^{\rho-1}\psi(t) \\ & \quad \times \exp\left(\int_a^t (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\int_{\mathbb{R}} \tilde{Z}(t,s)\delta(t,s)Q(t,ds)\eta(t)dt + (t-a)^{\rho-1} \\ & \quad \times \exp\left(\int_a^t (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(t)dB(t) + (t-a)^{\rho-1} \end{aligned}$$

$$\times \exp\left(\int_a^t (\upsilon - a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\int_{\mathbb{R}} \tilde{Z}(t,s)\tilde{N}(dt,ds)$$

By integrating both sides of the equation from t to T, one has

$$\begin{aligned}
&\tilde{X}(t)\exp\left(\int_a^t (\upsilon - a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\\
&= \tilde{\zeta}\exp\left(\int_a^T (\upsilon - a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) + \int_t^T (\tau - a)^{\rho-1}\\
&\quad \times \exp\left(\int_a^\tau (\upsilon - a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\\
&\quad \times \tilde{h}\left(\tau, X'(\tau), Y'(\tau), \int_{\mathbb{R}} Z'(\tau,s)\delta(\tau,s)Q(\tau,ds)\eta(\tau)\right)d\tau\\
&\quad + \int_t^T (\tau - a)^{\rho-1}\beta(\tau)\exp\left(\int_a^\tau (\upsilon - a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)d\tau\\
&\quad + \int_t^T (\tau - a)^{\rho-1}\psi(\tau)\exp\left(\int_a^\tau (\upsilon - a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\\
&\quad \times \int_{\mathbb{R}} \tilde{Z}(\tau,s)\delta(\tau,s)Q(\tau,ds)\eta(\tau)d\tau\\
&\quad - \int_t^T (\tau - a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon - a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)dB(\tau)\\
&\quad - \int_t^T (\tau - a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon - a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\int_{\mathbb{R}} \tilde{Z}(\tau,s)\tilde{N}(d\tau,ds),\ a \le t \le T.
\end{aligned}$$

From Lemma 7, we can introduce an equivalent probability measure

$$\begin{aligned}
&\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = M(t),\ M(a) = 1\\
&\frac{dM(t)}{M(t)} = \beta(t)dB(t) + \int_{\mathbb{R}} \psi(t)\delta(t,s)\tilde{N}(dt,ds),\ a \le t \le T.
\end{aligned}$$

We know that $|\beta(t)| \le K$, $\int_{\mathbb{R}} |\psi(t)\delta(t,s)|^2 Q(t,ds)\eta(t) \le K$ and $\psi(t)\delta(t,s) > 1$. According to Proposition 8, the process M is a square integrable martingale under the equivalent probability measure $\mathbb{Q}$. Then changing the probability measure, we have

$$\begin{aligned}
&\tilde{X}(t)\exp\left(\int_a^t (\upsilon - a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\\
&= \tilde{\zeta}\exp\left(\int_a^T (\upsilon - a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) + \int_t^T (\tau - a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon - a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\\
&\quad \times \tilde{h}\left(\tau, X'(\tau), Y'(\tau), \int_{\mathbb{R}} Z'(\tau,s)\delta(\tau,s)Q(\tau,ds)\eta(\tau)\right)d\tau
\end{aligned}$$

$$
\begin{aligned}
&-\int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)dB^{\mathbb{Q}}(\tau)\\
&-\int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\int_{\mathbb{R}}\tilde{Z}(\tau,s)\tilde{N}^{\mathbb{Q}}(d\tau,ds), \qquad (5.2)
\end{aligned}
$$

where $a \le t \le T$. By Lemma 2, Schwarz inequality, the definition of expectation and the boundedness of the functions, we have

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\sup\int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)dB^{\mathbb{Q}}(\tau)\right]\\
&\le K\mathbb{E}^{\mathbb{Q}}\left[\sqrt{\int_t^T \left|(\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)\right|^2 d\tau}\right]\\
&\le K\mathbb{E}\left[\left|\frac{d\mathbb{Q}}{d\mathbb{P}}\right|^2\right]\mathbb{E}\left[\int_t^T \left|(\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)\right|^2 d\tau\right]\\
&\le K\sqrt{\frac{(T-a)^{4\rho-3}}{4\rho-3}}\mathbb{E}\left[\left|\frac{d\mathbb{Q}}{d\mathbb{P}}\right|^2\right]\mathbb{E}\left[\sqrt{\int_t^T \left|\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)\right|^4 d\tau}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\sup\int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\int_{\mathbb{R}}\tilde{Z}(\tau,s)\tilde{N}^{\mathbb{Q}}(d\tau,ds)\right]\\
&\le K\mathbb{E}^{\mathbb{Q}}\Bigg[\Bigg(\int_t^T\int_{\mathbb{R}}\Big|(\tau-a)^{\rho-1}\\
&\qquad\times\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Z}(\tau,s)\Big|^2 Q(\tau,ds)\eta(\tau)d\tau\Bigg)\Bigg]\\
&\le K\mathbb{E}\left[\left|\frac{d\mathbb{Q}}{d\mathbb{P}}\right|^2\right]\mathbb{E}\Bigg[\int_t^T\int_{\mathbb{R}}\Big|(\tau-a)^{\rho-1}\\
&\qquad\times\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Z}(\tau,s)\Big|^2 Q(\tau,ds)\eta(\tau)d\tau\Bigg]\\
&\le K\sqrt{\frac{(T-a)^{4\rho-3}}{4\rho-3}}\mathbb{E}\left[\left|\frac{d\mathbb{Q}}{d\mathbb{P}}\right|^2\right]\mathbb{E}\Bigg[\Bigg(\int_t^T\int_{\mathbb{R}}\\
&\qquad\Big|\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Z}(\tau,s)\Big|^4 Q(\tau,ds)\eta(\tau)d\tau\Bigg)^{\frac{1}{2}}\Bigg].
\end{aligned}
$$

Taking the conditional expectation on both sides of equation (5.2), we get

$$
\tilde{X}(t)\exp\left(\int_a^t (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)
$$

$$
\begin{aligned}
&= \mathbb{E}^{\mathbb{Q}}\Bigg[\tilde{\zeta}\exp\left(\int_a^T (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) \\
&\quad + \int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^{\tau}(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) \\
&\quad \times \tilde{h}\left(\tau, X'(\tau), Y'(\tau), \int_{\mathbb{R}} Z'(\tau,s)\delta(\tau,s)Q(\tau,ds)\eta(\tau)\right)d\tau\Bigg|\mathcal{F}_t\Bigg]. \qquad (5.3)
\end{aligned}
$$

Hence for any $a \le t \le T$, we have $X(t) \ge X'(t)$.

For the last part note we add a hypothesis $X(t_0) = X'(t_0)$, that is, $\tilde{X}(t_0) = 0$, where $a \le t_0 \le T$. Notice that $\tilde{\zeta} \ge 0$ and $\tilde{h}(t, X'(t), Y'(t), \int_{\mathbb{R}} Z'(t,s)\delta(t,s)Q(t,ds)\eta(t)) \ge 0$, then we can conclude that $\tilde{\zeta} = 0$ and $\tilde{h}(t, X'(t), Y'(t), \int_{\mathbb{R}} Z'(t,s)\delta(t,s)Q(t,ds)\eta(t)) = 0$, where $t_0 \le t \le T$. By (5.2), one has

$$
\begin{aligned}
&\tilde{X}(t)\exp\left(\int_a^t (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) \\
&= -\int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^{\tau}(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)dB^{\mathbb{Q}}(\tau) \\
&\quad - \int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^{\tau}(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) \\
&\quad \times \int_{\mathbb{R}} \tilde{Z}(\tau,s)\tilde{N}^{\mathbb{Q}}(d\tau,ds), \ t_0 \le t \le T. \qquad (5.4)
\end{aligned}
$$

Taking the conditional expectation in (5.4), the proof of the second conclusion is complete. □

On the basis of Theorem 4, we propose an inverse principle of the comparison principle.

Theorem 5. *Consider the conformable backward stochastic differential equation*

$$
\begin{aligned}
X(t) &= \zeta + \int_t^T (\tau-a)^{\rho-1} h(\tau, X(\tau), Y(\tau), \int_{\mathbb{R}} Z(\tau,s)\delta(\tau,s)Q(\tau,ds)\eta(\tau))d\tau \\
&\quad - \int_t^T (\tau-a)^{\rho-1} Y(\tau)dB(\tau) - \int_t^T (\tau-a)^{\rho-1}\int_{\mathbb{R}} Z(\tau,s)\tilde{N}(d\tau,ds),
\end{aligned}
$$

where $a \le t \le T$. Assume that the generators h, h' satisfy the hypotheses H_{cs}. Let (X,Y,Z), $(X',Y',Z') \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}_N^2(\mathbb{R})$ be the unique solutions with h and h'. For any $(t,x,y,u) \in [a,T] \times \mathbb{R} \times \mathbb{R} \times L_Q^2(\mathbb{R})$, if $\zeta = \zeta' \in \mathbb{L}^2(\mathbb{R})$ and $X(t) \ge X'(t)$, then $h(t,x,y,u) \ge h'(t,x,y,u)$, where $u(t) = \int_{\mathbb{R}} z(t,s)\delta(t,s)Q(t,ds)\eta(t)$.

Proof. According to the proof of Theorem 4, we can still get formula (5.3),

$$
\tilde{X}(t)\exp\left(\int_a^t (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)
$$

$$
\begin{aligned}
&= \mathbb{E}^{\mathbb{Q}}\bigg[\tilde{\zeta}\exp\left(\int_a^T (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) + \int_t^T (\tau-a)^{\rho-1} \\
&\quad \times \exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) \\
&\quad \times \tilde{h}\left(\tau, X'(\tau), Y'(\tau), \int_{\mathbb{R}} Z'(\tau,s)\delta(\tau,s)Q(\tau,ds)\eta(\tau)\right)d\tau \bigg| \mathcal{F}_t\bigg].
\end{aligned}
$$

Since $\zeta = \zeta'$, the above formula can be rewritten as

$$
\begin{aligned}
&\tilde{X}(t)\exp\left(\int_a^t (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) \\
&= \mathbb{E}^{\mathbb{Q}}\bigg[\int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) \\
&\quad \times \tilde{h}\left(\tau, X'(\tau), Y'(\tau), \int_{\mathbb{R}} Z'(\tau,s)\delta(\tau,s)Q(\tau,ds)\eta(\tau)\right)d\tau \bigg| \mathcal{F}_t\bigg].
\end{aligned}
$$

Hence for any $X(t) \geq X'(t)$, we have $h(t,x,y,u) \geq h'(t,x,y,u)$. The proof is complete. □

Theorem 6. *Consider the following conformable backward stochastic differential equation*

$$
\begin{aligned}
X(t) = \zeta &+ \int_t^T (\tau-a)^{\rho-1} g(\tau, X(\tau), Y(\tau), Z(\tau,\cdot))d\tau \\
&- \int_t^T (\tau-a)^{\rho-1} Y(\tau)dB(\tau) - \int_t^T (\tau-a)^{\rho-1} \\
&\times \int_{\mathbb{R}} Z(\tau,s)\tilde{N}(d\tau,ds),\ 0<\rho\leq 1,\ a\leq t\leq T.
\end{aligned}
$$

Assume that

(i) *the terminal value* $\zeta \in \mathbb{L}^2(\mathbb{R})$,

(ii) *the generator* $g : \Omega\times[a,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is predictable and Lipschitz in x and y*

$$|g(t,x,y,z) - g(t,x',y',z)| \leq K(|x-x'| + |y-y'|),$$

In addition, for any (x,y,z), $(x,y,z') \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$, *there exists constants* $-1 < C_1 \leq 0$ *and* $C_2 \geq 0$ *such that*

$$g(t,x,y,z) - g(t,x,y,z') \leq \int_{\mathbb{R}} (z(t,s) - z'(t,s))\delta^{x,y,z,z'}(t,s)Q(t,ds)\eta(t),$$

where $\delta^{x,y,z,z'}(t,s) : \Omega\times[a,T]\times\mathbb{R}\to\mathbb{R}$ *is predictable and satisfies*

$$C_1(1\wedge|s|) \leq \delta^{x,y,z,z'}(t,s) \leq C_2(1\wedge|s|),$$

(iii) $\mathbb{E}[\int_a^T |g(t,0,0,0)|^2 dt] < \infty$.

Let $(X,Y,Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$ and $(X',Y',Z') \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$ be the unique solutions to the above equation with (ζ,g) and (ζ',g'). For any $(t,x,y,z) \in [a,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, if $\zeta \geq \zeta'$ and $g(t,x,y,z) \geq g'(t,x,y,z)$, then $X(t) \geq X'(t)$. In addition, for any $a \leq t_0 \leq T$, if $X(t_0) = X'(t_0)$, then $X(t) = X'(t), t_0 \leq t \leq T$.

Proof. Notice that $\delta^{x,y,z,z'}(t,s)$ depends on x,y,z,z' and when it is greater than $C_1(1 \wedge |s|)$, it can also be negative. According to hypothesis (ii), we can conclude that the generator g is also Lipschitz in z such that

$$|g(t,x,y,z) - g(t,x,y,z')| \leq C_3 \int_{\mathbb{R}} |z(t,s) - z'(t,s)|(1 \wedge |s|)Q(t,ds)\eta(t),$$

where $C_3 \geq \max(-C_1, C_2)$. In other words, there exists a predictable process $\bar{\delta}^{x,y,z,z'}$: $\Omega \times [a,T] \times \mathbb{R} \to \mathbb{R}$ such that

$$g(t,x,y,z) - g(t,x,y,z') \geq \int_{\mathbb{R}} (z(t,s) - z'(t,s))\bar{\delta}^{x,y,z,z'}(t,s)Q(t,ds)\eta(t),$$

where $-C_3(1 \wedge |s|) \leq \bar{\delta}^{x,y,z,z'}(t,s) \leq C_3(1 \wedge |s|)$. Hence we can get

$$\begin{aligned}|g(t,x,y,z) - g(t,x',y',z')| \leq K(|x-x'| + |y-y'| + \int_{\mathbb{R}} |z(t,s) - z'(t,s)| \\ \times \max(|\delta^{x,y,z,z'}(t,s)|, |\bar{\delta}^{x,y,z,z'}(t,s)|)Q(t,ds)\eta(t)).\end{aligned}$$

The existence and uniqueness of the solutions (X,Y,Z) and (X',Y',Z') follow from Theorem 3.

Let

$$\tilde{X}(t) = X(t) - X'(t),\ \tilde{Y}(t) = Y(t) - Y'(t),\ \tilde{Z}(t) = Z(t) - Z'(t), \text{ and}$$

$$U(t,s) = \int_{\mathbb{R}} Z(t,s)\delta(t,s)Q(t,ds)\eta(t).$$

Since

$$\begin{aligned}g(\omega,t,X(t),Y(t),Z(t,s)) &= h(\omega,t,X(t),Y(t),\int_{\mathbb{R}} Z(t,s)\delta(t,s)Q(t,ds)\eta(t)) \\ &= h(\omega,t,X(t),Y(t),U(t,s)),\end{aligned}$$

we also have

$$\alpha(t) = \begin{cases} \frac{g(t,X(t),Y(t),Z(t,s)) - g(t,X'(t),Y(t),Z(t,s))}{\tilde{X}(t)} & if\ X(t) \neq X'(t), \\ 0 & otherwise, \end{cases}$$

and

$$\beta(t) = \begin{cases} \frac{g(t,X'(t),Y(t),Z(t,s)) - g(t,X'(t),Y'(t),Z(t,s))}{\tilde{Y}(t)} & if\ Y(t) \neq Y'(t), \\ 0 & otherwise. \end{cases}$$

Then

$$d\tilde{X}(t) = -(t-a)^{\rho-1}[\alpha(t)\tilde{X}(t) + \beta(t)\tilde{Y}(t) + g(t,X',Y',Z)$$

$$-g(t,X',Y',Z')+\tilde{g}(t,X',Y',Z')]dt+(t-a)^{\rho-1}\tilde{Y}(t)dB(t)$$
$$+(t-a)^{\rho-1}\int_{\mathbb{R}}\tilde{Z}(t,s)\tilde{N}(t,s).$$

Applying Lemma 1 to $\tilde{X}(t)\exp(\int_a^t(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon)$, we have

$$\begin{aligned}
&d\tilde{X}(t)\exp\left(\int_a^t(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\\
&=-(t-a)^{\rho-1}\beta(t)\tilde{Y}(t)\exp\left(\int_a^t(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)dt\\
&\quad-(t-a)^{\rho-1}\exp\left(\int_a^t(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)[g(t,X',Y',Z)-g(t,X',Y',Z')]dt\\
&\quad-(t-a)^{\rho-1}\exp\left(\int_a^t(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{g}\left(t,X'(t),Y'(t),Z'(t,s)\right)dt\\
&\quad+(t-a)^{\rho-1}\exp\left(\int_a^t(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(t)dB(t)\\
&\quad+(t-a)^{\rho-1}\exp\left(\int_a^t(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\int_{\mathbb{R}}\tilde{Z}(t,s)\tilde{N}(dt,ds)
\end{aligned}$$

By integrating both sides of the equation from t to T, one has

$$\begin{aligned}
&\tilde{X}(t)\exp\left(\int_a^t(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\\
&=\tilde{\zeta}\exp\left(\int_a^T(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\\
&\quad+\int_t^T(\tau-a)^{\rho-1}\beta(\tau)\exp\left(\int_a^\tau(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)d\tau\\
&\quad+\int_t^T(\tau-a)^{\rho-1}\exp\left(\int_a^\tau(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\\
&\quad\times[g(\tau,X'(\tau),Y'(\tau),Z(\tau))-g(\tau,X'(\tau),Y'(\tau),Z'(\tau))]d\tau\\
&\quad+\int_t^T(\tau-a)^{\rho-1}\exp\left(\int_a^\tau(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{g}\left(\tau,X'(\tau),Y'(\tau),Z'(\tau,s)\right)d\tau\\
&\quad-\int_t^T(\tau-a)^{\rho-1}\exp\left(\int_a^\tau(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)dB(\tau)\\
&\quad-\int_t^T(\tau-a)^{\rho-1}\exp\left(\int_a^\tau(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\int_{\mathbb{R}}\tilde{Z}(\tau,s)\tilde{N}(d\tau,ds),\ a\le t\le T.
\end{aligned}$$

Notice that $g(t,x,y,z)-g(t,x,y,z')\ge\int_{\mathbb{R}}\tilde{z}(t,s)\bar{\delta}^{x,y,z,z'}(t,s)Q(t,ds)\eta(t)$, then

$$\tilde{X}(t)\exp\left(\int_a^t(\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)$$

$$
\begin{aligned}
&\geq \tilde{\zeta} \exp\left(\int_a^T (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) + \int_t^T (\tau-a)^{\rho-1}\beta(\tau) \\
&\quad \times \exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)d\tau + \int_t^T (\tau-a)^{\rho-1} \\
&\quad \times \exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\int_{\mathbb{R}} \tilde{Z}(\tau,s)\bar{\delta}^{x,y,z,z'}(\tau,s)Q(\tau,ds)\eta(\tau)d\tau \\
&\quad + \int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{g}\Big(\tau, X'(\tau), Y'(\tau), Z'(\tau,s)\Big)d\tau \\
&\quad - \int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)dB(\tau) \\
&\quad - \int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\int_{\mathbb{R}} \tilde{Z}(\tau,s)\tilde{N}(d\tau,ds).
\end{aligned}
$$

Define

$$
\begin{aligned}
&\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = M(t),\ M(a) = 1 \\
&\frac{dM(t)}{M(t)} = \beta(t)dB(t) + \int_{\mathbb{R}} \bar{\delta}^{x,y,z,z'}(t,s)\tilde{N}(dt,ds),\ a \leq t \leq T.
\end{aligned}
$$

According to Lemma 7, we get

$$
\begin{aligned}
&\tilde{X}(t)\exp\left(\int_a^t (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) \\
&\geq \tilde{\zeta} \exp\left(\int_a^T (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) \\
&\quad + \int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{g}\Big(\tau, X'(\tau), Y'(\tau), Z'(\tau,s)\Big)d\tau \\
&\quad - \int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)dB^{\mathbb{Q}}(\tau) \\
&\quad - \int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\int_{\mathbb{R}} \tilde{Z}(\tau,s)\tilde{N}^{\mathbb{Q}}(d\tau,ds).
\end{aligned}
$$

Taking the conditional expectation on both sides, we have

$$
\begin{aligned}
&\tilde{X}(t)\exp\left(\int_a^t (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) \\
&\geq \mathbb{E}^{\mathbb{Q}}\Bigg[\tilde{\zeta} \exp\left(\int_a^T (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) + \int_t^T (\tau-a)^{\rho-1} \\
&\quad \times \exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{g}\Big(\tau, X'(\tau), Y'(\tau), Z'(\tau,s)\Bigg|\mathcal{F}_t\Bigg].
\end{aligned}
$$

Since $\tilde{\zeta} \geq 0$ and $\tilde{g}(t,X',Y',Z') \geq 0$, we can conclude that $\tilde{X}(t) \geq 0$. The proof of the first assertion is complete.

According to $X(t_0) = X'(t_0)$, $a \leq t_0 \leq T$, we have $\tilde{\zeta} = 0$ and $\tilde{g}(t,X',Y',Z') = 0$. Then

$$\begin{aligned}
&\tilde{X}(t)\exp\left(\int_a^t (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\\
&\geq -\int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)dB^{\mathbb{Q}}(\tau)\\
&\quad -\int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\int_{\mathbb{R}}\tilde{Z}(\tau,s)\tilde{N}^{\mathbb{Q}}(d\tau,ds).
\end{aligned}$$

Taking the conditional expectation on both sides of the above inequality, we can get $\tilde{X}(t) \geq 0$.

In the same way, since

$$g(t,x,y,z)-g(t,x,y,z') \leq \int_{\mathbb{R}}(z(t,s)-z'(t,s))\delta^{x,y,z,z'}(t,s)Q(t,ds)\eta(t),$$

one has

$$\begin{aligned}
&\tilde{X}(t)\exp\left(\int_a^t (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\\
&\leq \tilde{\zeta}\exp\left(\int_a^T (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)+\int_t^T (\tau-a)^{\rho-1}\beta(\tau)\\
&\quad\times\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)d\tau+\int_t^T (\tau-a)^{\rho-1}\\
&\quad\times\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\int_{\mathbb{R}}\tilde{Z}(\tau,s)\delta^{x,y,z,z'}(\tau,s)Q(\tau,ds)\eta(\tau)d\tau\\
&\quad+\int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{g}\Big(\tau,X'(\tau),Y'(\tau),Z'(\tau,s)\Big)d\tau\\
&\quad-\int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)dB(\tau)\\
&\quad-\int_t^T (\tau-a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon-a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\int_{\mathbb{R}}\tilde{Z}(\tau,s)\tilde{N}(d\tau,ds).
\end{aligned}$$

Define

$$\begin{aligned}
&\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = M(t),\ M(a)=1\\
&\frac{dM(t)}{M(t)} = \beta(t)dB(t)+\int_{\mathbb{R}}\delta^{x,y,z,z'}(t,s)\tilde{N}(dt,ds),\ a\leq t\leq T.
\end{aligned}$$

According to Lemma 7, $\tilde{\zeta} = 0$ and $\tilde{g}(t, X', Y', Z') = 0$, we get

$$\begin{aligned}
&\tilde{X}(t)\exp\left(\int_a^t (\upsilon - a)^{\rho-1}\alpha(\upsilon)d\upsilon\right) \\
&\le -\int_t^T (\tau - a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon - a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\tilde{Y}(\tau)dB^{\mathbb{Q}}(\tau) \\
&\quad -\int_t^T (\tau - a)^{\rho-1}\exp\left(\int_a^\tau (\upsilon - a)^{\rho-1}\alpha(\upsilon)d\upsilon\right)\int_{\mathbb{R}} \tilde{Z}(\tau, s)\tilde{N}^{\mathbb{Q}}(d\tau, ds).
\end{aligned}$$

Taking the conditional expectation on both sides, we can conclude that $\tilde{X}(t) \le 0$. Combined with the inequality $\tilde{X}(t) \ge 0$, we get $\tilde{X}(t) = 0,\ t_0 \le t \le T$. The proof is complete. □

Notice that the comparison theorem does not hold when the generator g only satisfies conditions $(S1)$, $S(2)$ and $(S3)$. In the next section, we will give a counter example.

Remark 3. When $\rho = 1$, the above comparison theorems are the same as in [21, Theorem 2.5] and [21, Theorem 2.6].

6. EXAMPLES

Example 1. Consider the following linear conformable backward stochastic differential equation

$$\begin{aligned}
X(t) = \zeta + \int_t^T (\tau - a)^{\rho-1} b(\tau)X(\tau)d\tau + \int_t^T (\tau - a)^{\rho-1} \\
\times c(\tau)Y(\tau)d\tau - \int_t^T (\tau - a)^{\rho-1}Y(\tau)dB(\tau),\ \ a < t \le T.
\end{aligned}$$

Let $\zeta \in \mathbb{L}^2(\mathbb{R})$, and let $b(\cdot)$ and $c(\cdot)$ be $\mathcal{F}$-predictable, bounded processes. From Theorem 3, there exists a unique solution $(X, Y) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$.

Define

$$\begin{aligned}
\hat{X}(t) = X(t)\exp\left(\int_a^t (\nu - a)^{\rho-1} b(\nu)d\nu\right),\ \ a < t \le T \\
\hat{Y}(t) = Y(t)\exp\left(\int_a^t (\nu - a)^{\rho-1} b(\nu)d\nu\right),\ \ a < t \le T.
\end{aligned}$$

Then we apply the Itô's formula to $\hat{X}(t)$, and one has

$$\begin{aligned}
d\hat{X}(t) = (t - a)^{\rho-1} b(t)\exp\left(\int_a^t (\nu - a)^{\rho-1} b(\nu)d\nu\right)dt \\
+\exp\left(\int_a^t (\nu - a)^{\rho-1} b(\nu)d\nu\right)dX(t)
\end{aligned}$$

$$\begin{aligned}
&= (t-a)^{\rho-1}b(t)\exp\left(\int_a^t (\nu-a)^{\rho-1}b(\nu)d\nu\right)dt \\
&\quad - (t-a)^{\rho-1}b(t)\exp\left(\int_a^t (\nu-a)^{\rho-1}b(\nu)d\nu\right)dt \\
&\quad - (t-a)^{\rho-1}c(t)\exp\left(\int_a^t (\nu-a)^{\rho-1}b(\nu)d\nu\right)Y(t)dt \\
&\quad + (t-a)^{\rho-1}\exp\left(\int_a^t (\nu-a)^{\rho-1}b(\nu)d\nu\right)Y(t)dB(t) \\
&= -(t-a)^{\rho-1}c(t)\exp\left(\int_a^t (\nu-a)^{\rho-1}b(\nu)d\nu\right)Y(t)dt \\
&\quad + (t-a)^{\rho-1}\exp\left(\int_a^t (\nu-a)^{\rho-1}b(\nu)d\nu\right)Y(t)dB(t).
\end{aligned}$$

By integrating both sides of the equation from t to T, we get

$$\hat{X}(t) = \hat{X}(T) + \int_t^T (\tau-a)^{\rho-1}c(\tau)\hat{Y}(\tau)d\tau - \int_t^T (\tau-a)^{\rho-1}\hat{Y}(\tau)dB(\tau).$$

Let

$$\begin{aligned}
&\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|\mathcal{F}_t = M(t),\ M(a) = 1, \\
&\frac{dM(t)}{M(t)} = c(t)dB(t).
\end{aligned}$$

According to Lemma 7, we have

$$\hat{X}(t) = \hat{X}(T) - \int_t^T (\tau-a)^{\rho-1}\hat{Y}(\tau)dB^{\mathbb{Q}}(\tau).$$

Since the process $\hat{Y}$ is a.s. square integrable, we can conclude that $\hat{X}$ is a $\mathbb{Q}$-local martingale, and by Lemma 2 and some important inequality we have $\hat{X}(t) = \mathbb{E}^{\mathbb{Q}}[\hat{X}(T)|\mathcal{F}_t]$. In addition, we can get

$$\begin{aligned}
\zeta\exp\left(\int_a^T (\nu-a)^{\rho-1}b(\nu)d\nu\right) &= \mathbb{E}^{\mathbb{Q}}\left[\zeta\exp\left(\int_a^T (\nu-a)^{\rho-1}b(\nu)d\nu\right)\middle|\mathcal{F}_t\right] \\
&\quad - \int_t^T (\tau-a)^{\rho-1}\hat{Y}(\tau)dB^{\mathbb{Q}}(\tau).
\end{aligned}$$

The above results satisfy Proposition 1.

Example 2. Let $a \le t \le T$ and $0 < \rho \le 1$, and we consider the following conformable backward stochastic differential equation

$$\begin{cases} D_\rho^a X(t) = -rX(t) + \beta(t-a)^{1-\rho}X(t)\dfrac{dB(t)}{dt} + \dfrac{\int_{\mathbb{R}} X(t)\tilde{N}(dt,ds)}{dt}, \\ X(T) = \zeta, \end{cases} \tag{6.1}$$

where $r, \beta \in \mathbb{R}$. From Remark 1, equation (6.1) has the equivalent form

$$\begin{cases} dX(t) = -r(t-a)^{\rho-1}X(t)dt + \beta X(t)dB(t) \\ \qquad\qquad + (t-a)^{\rho-1}\int_{\mathbb{R}} X(t)\tilde{N}(dt,ds), \\ X(T) = \zeta. \end{cases}$$

Suppose $U(t,X(t)) = \ln X(t)$. From Theorem 1, we have

$$\begin{aligned} d\ln X(t) &= \frac{1}{X(t)}dX(t) - \frac{1}{2X^2(t)}(dX(t))^2 \\ &= -r(t-a)^{\rho-1}dt + \beta dB(t) + \frac{1}{X(t)}(t-a)^{\rho-1}\int_{\mathbb{R}} X(t)\tilde{N}(dt,ds) \\ &\quad - \frac{1}{2}\beta^2 dt - \frac{1}{2X^2(t)}(t-a)^{2(\rho-1)}\int_{\mathbb{R}} X^2(t)N(dt,ds) \\ &= -\left[r(t-a)^{\rho-1} + \frac{1}{2}\beta^2\right]dt + \beta dB(t) \\ &\quad + \frac{1}{X(t)}(t-a)^{\rho-1}\int_{\mathbb{R}} X(t)\tilde{N}(dt,ds) \\ &\quad - \frac{1}{2X^2(t)}(t-a)^{2(\rho-1)}\int_{\mathbb{R}} X^2(t)N(dt,ds). \end{aligned}$$

Then

$$\begin{aligned} \ln X(t) = \ln\zeta &+ \int_t^T \left[r(t-a)^{\rho-1} + \frac{1}{2}\beta^2\right]dt \\ &- \int_t^T \beta dB(t) + \int_t^T \frac{1}{X(t)}(t-a)^{\rho-1}\int_{\mathbb{R}} X(t)\tilde{N}(dt,ds) \\ &- \int_t^T \frac{1}{2X^2(t)}(t-a)^{2(\rho-1)}\int_{\mathbb{R}} X^2(t)N(dt,ds), \end{aligned}$$

and the solution of (6.1) is expressed as

$$\begin{aligned} X(t) = \zeta\exp\Bigg(&\frac{r}{\rho}[(T-a)^{\rho} - (t-a)^{\rho}] + \frac{1}{2}\beta^2(T-t) - \beta(B(T)-B(t)) \\ &- \int_t^T \beta dB(t) + \int_t^T \frac{1}{X(t)}(t-a)^{\rho-1}\int_{\mathbb{R}} X(t)\tilde{N}(dt,ds) \\ &- \int_t^T \frac{1}{2X^2(t)}(t-a)^{2(\rho-1)}\int_{\mathbb{R}} X^2(t)N(dt,ds)\Bigg). \end{aligned}$$

Taking the conditional expected value on both sides of the above equation, one has

$$X(t) = \mathbb{E}[\zeta\exp\{\frac{r}{\rho}[(T-a)^{\rho} - (t-a)^{\rho}] + \frac{1}{2}\beta^2(T-t)\}|\mathcal{F}_t],$$

which satisfies Proposition 2.

Example 3. Consider the conformable backward stochastic differential equations

$$\begin{aligned} X_1(t) = \zeta_1 + \int_t^T (\tau - a)^{\rho-1}[g_1(\tau, X_1(\tau), Y_1(\tau), Z_1(\tau,\cdot)) + \nu_1(\tau)]d\tau \\ - \int_t^T (\tau - a)^{\rho-1} Y_1(\tau)dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z_1(\tau,s)\tilde{N}(d\tau,ds), \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} X_2(t) = \zeta_2 + \int_t^T (\tau - a)^{\rho-1}[g_2(\tau, X_2(\tau), Y_2(\tau), Z_2(\tau,\cdot)) + \nu_2(\tau)]d\tau \\ - \int_t^T (\tau - a)^{\rho-1} Y_2(\tau)dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z_2(\tau,s)\tilde{N}(d\tau,ds), \end{aligned} \tag{6.3}$$

where $0 < \rho \leq 1$, $a \leq t \leq T$ and $\nu_1(\cdot), \nu_2(\cdot) \in \mathbb{S}^2(\mathbb{R})$. Assume that the generator g_1 and g_2 satisfy condition H_{cs}. At this time, for any $a \leq t \leq T$, let $\zeta_1 = 0$, $\zeta_2 = \frac{(t-a)^\rho}{\rho}$ and $g_1(t, X_1(t), Y_1(t), Z_1(t)) + \nu_1(t) = 0$, $g_2(t, X_2(t), Y_2(t), Z_2(t)) + \nu_2(t) = 1$, then $(X_1, Y_1, Z_1) = (-K(B(T) - B(t)), K(t-a)^{1-\rho}, 0)$ is the unique solution of equation (6.2) and $(X_2, Y_2, Z_2) = (\frac{(T-a)^\rho}{\rho} - K(B(T) - B(t)), K(t-a)^{1-\rho}, 0)$ is the unique solution of equation (6.3). Obviously, $\zeta_2 \geq \zeta_1$ and $g_2 + \nu_2 \geq g_1 + \nu_1$, and we know that $X_2(t) \geq X_1(t)$, which satisfies Theorem 4.

In the financial market, such as the Merton model, $\nu(\cdot)$ represents the investor's consumption rate, $X(t)$ represents his property at time t, $Y(t)$ represents the portfolio strategy at time t under the condition of small fluctuation, and $Z(t,s)$ represents the portfolio strategy at time t affected by large fluctuation. At this time, the comparison principle has the following explanation, that is, if an investor wants to achieve a higher fiscal revenue at a future moment, then the investor must either invest more money now, or reduce consumption before time T.

Example 4. Consider the following price model of a stock

$$\begin{aligned} X_i(t) = \zeta_i + m\int_t^T (\tau - a)^{\rho-1} g_i(\tau)d\tau - n\int_t^T (\tau - a)^{\rho-1} Y_i(\tau)dB(\tau) \\ - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z_i(\tau)\tilde{N}(d\tau,ds), \ a \leq t \leq T, \end{aligned} \tag{6.4}$$

where $i = 1,2$ and $m,n \in \mathbb{R}^+$. Let generator g_i satisfy the condition H_{cs}. Choose $Y_1(t) = T(t-a)^{1-\rho}$, $Y_2(t) = 0$ and $Z_1(t) = Z_2(t) = 0$. For any $\zeta_1 = \zeta_2 = \zeta$, if $X_1(t) = t$ and $X_2(t) = t - 1$, we can get

$$t = \zeta + m\int_t^T (\tau - a)^{\rho-1} g_1(\tau)d\tau - nT(B(T) - B(t)), \ a \leq t \leq T,$$

and

$$t-1=\zeta+m\int_t^T(\tau-a)^{\rho-1}g_1(\tau)d\tau,\ a\leq t\leq T.$$

Then $g_1(t)=\frac{-(t-a)^{1-\rho}}{m}[1-nTB'(t)]$, $g_2(t)=\frac{-(t-a)^{1-\rho}}{m}$ and $\zeta=T-1$. Clearly, $g_1(t)\geq g_2(t)$ which satisfies Theorem 5.

Example 5. Consider the following conformable backward stochastic differential equation

$$\begin{aligned}X(t)=\zeta+&\int_t^T(\tau-a)^{\rho-1}g(\tau,X(\tau),Y(\tau),Z(\tau))d\tau\\&-\int_t^T(\tau-a)^{\rho-1}Y(\tau)dB(\tau)-\int_t^T(\tau-a)^{\rho-1}\int_{\mathbb{R}}Z(\tau,s)\tilde{N}(d\tau,ds),\end{aligned}\tag{6.5}$$

where the intensity of the Poisson process is 1. Let $g(t,X,Y,Z)=-2Y(t)$ and g only satisfies the conditions $(S1)$, $(S2)$ and $(S3)$. Then

$$\int_a^t\int_{\mathbb{R}}N(d\tau,ds),\ a\leq t\leq T,$$

is a standard Poisson process. If we choose $\zeta=0$ and $\zeta'=\int_a^T\int_{\mathbb{R}}N(d\tau,ds)$, then $(X(t),Y(t),Z(t))=(0,0,0)$ and $(X'(t),Y'(t),Z'(t))=(\int_a^t\int_{\mathbb{R}}N(d\tau,ds)-(T-t),0,1)$ are the unique solutions of the equation (6.5) with (ζ,g) and (ζ',g). The existence and uniqueness of the solutions $(X(t),Y(t),Z(t))$ and $(X'(t),Y'(t),Z'(t))$ follow from Theorem 3. It is clear that $\zeta\leq\zeta'$, but $X(t)\geq X'(t)$.

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