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THE YANG-BAXTER MATRIX EQUATION FOR INVOLUTIONS

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Abstract. In this paper we find all involutive solutions $X \in \mathbb{C}^{n\times n}$ of the Yang-Baxter matrix equation $AXA = XAX$, where $A \in \mathbb{C}^{n\times n}$ is a given involutory matrix. The construction is algorithmic. It is based on the concept of quadratic matrices. Algorithms for generating concrete involutive solutions of the Yang-Baxter matrix equation $AXA = XAX$ are also presented along with several examples.

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1. INTRODUCTION

We recall that $A \in \mathbb{C}^{n\times n}$ is an involutory matrix (involution) if $A^2 = I_n$, where I_n is the $n\times n$ identity matrix. Let $\mathcal{K}_n$ denote the set of all involutory matrices of size n. The purpose of this paper then is to find all explicit involutive solutions X of the Yang-Baxter matrix equation $AXA = XAX$, where A is involution. Our research is inspired by the papers [5] and [7], where the general analysis of finding all explicit solutions of $AXA = XAX$ for given involutory A has been given. In this paper we restrict our attention to involutory matrices X. This problem is related to the quantum Yang-Baxter equation (QYBE). We say that $Z \in \mathbb{R}^{\mathrm{m}^2\times \mathrm{m}^2}$ satisfies the QYBE if

$$(I_m \otimes Z)(Z\otimes I_m)(I_m \otimes Z) = (Z\otimes I_m)(I_m \otimes Z)(Z\otimes I_m) \tag{1.1}$$

where $B\otimes C$ denotes the Kronecker product (tensor product) of the matrices B and C: $B\otimes C = (b_{i,j}C)$. That is, the Kronecker product $B\otimes C$ is a block matrix whose (i, j) blocks are $b_{i,j}C$. Notice that if Z is involution and satisfies (1.1), then $A = I_m \otimes Z$ and $X = Z\otimes I_m$ are involutions as well, and we have $AXA = XAX$.

The Yang-Baxter equation has many applications of modern physics, computer science and mathematics ([6, 8, 9]), for example, in statistical mechanics, integrable

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quantum field theory, condensed matter physics, quantum integrable models, quantum computation, to name a few. Many techniques for the construction of involutive solutions of the Yang-Baxter equation have been developed, see [2–4, 8].

For given $A \in \mathcal{K}_n$ we define the set $\mathcal{S}_n(A)$ of all involutive solutions of the Yang-Baxter matrix equation thus:

$$\mathcal{S}_n(A) = \{X \in \mathcal{K}_n : AXA = XAX\}.$$

Using the concept of quadratic matrices, see Theorem 2, we are able to find the explicit expressions of such solutions for all the case, see Theorems 3–6. This enables the development of methods, easy to implement, for generating concrete involutive solutions of the Yang-Baxter matrix equation, see Section 8.

First we present easily-checkable basic properties of the set $\mathcal{S}_n(A)$.

Lemma 1. *Let $A \in \mathcal{K}_n$. Then we have*

(a): *$A \in \mathcal{S}_n(A)$,*
(b): *if $X \in \mathcal{S}_n(A)$ then $-X \in \mathcal{S}_n(-A)$,*
(c): *if $X \in \mathcal{S}_n(A)$ then A and X are similar; we have $X = (AX)A(AX)^{-1}$,*
(d): *if $P \in \mathbb{C}^{n\times n}$ is nonsingular, then $X \in \mathcal{S}_n(A)$ if and only if $P^{-1}XP \in \mathcal{S}_n(P^{-1}AP)$,*
(e): *if $A = \pm I_n$ then $\mathcal{S}_n(A) = \{A\}$.*

Assume that $A \in \mathbb{C}^{n\times n}$ is a given involution. Then A is diagonalizable. According to Lemma 1 (b), there is no loss of generality in assuming that A is not equal to $\pm I_n$, and there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that

$$A = PDP^{-1}, \quad D = \operatorname{diag}(I_p, -I_{n-p}), \quad 1 \le p < n, \quad n \le 2p. \tag{1.2}$$

Then we have $\mathcal{S}_n(A) = \{PYP^{-1} : Y \in \mathcal{S}_n(D)\}$. Natural questions to ask about $\mathcal{S}_n(D)$ then are: How many $Y \in \mathcal{S}_n(D)$ are there? and, how to find them? In general, the Yang-Baxter matrix equation has infinitely many solutions, see Example 1.

2. Identities for involutive solutions of $DYD = YDY$

We would like to find $Y \in \mathcal{S}_n(D)$, where D is defined by (1.2). Partition Y conformally with D as

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}, \quad Y_1(p \times p). \tag{2.1}$$

From Lemma 1 it follows that D and Y are similar, so $trY = trD$, where trY denotes the trace of Y. This together with (1.2)-(2.1) gives

$$trY_1 + trY_4 = 2p - n \ge 0. \tag{2.2}$$

Lemma 2. *Let D be given by (1.2) and Y be partitioned as in (2.1). Then $DYD = YDY$ if and only if*

$$Y_1^2 - Y_1 = Y_2Y_3, \tag{2.3}$$

$$Y_4^2 + Y_4 = Y_3Y_2, \tag{2.4}$$

$$(Y_1+I_p)Y_2 = Y_2Y_4, \tag{2.5}$$
$$Y_3(Y_1+I_p) = Y_4Y_3. \tag{2.6}$$

Proof. Compare the blocks of the following matrices:

$$DYD = \begin{bmatrix} Y_1 & -Y_2 \\ -Y_3 & Y_4 \end{bmatrix}, \quad YDY = \begin{bmatrix} Y_1^2 - Y_2Y_3 & Y_1Y_2 - Y_2Y_4 \\ Y_3Y_1 - Y_4Y_3 & Y_3Y_2 - Y_4^2 \end{bmatrix}.$$

□

Lemma 3. *Let Y be partitioned as in (2.1). Then Y is an involution if and only if*

$$I_p - Y_1^2 = Y_2Y_3, \tag{2.7}$$
$$I_{n-p} - Y_4^2 = Y_3Y_2, \tag{2.8}$$
$$Y_1Y_2 = -Y_2Y_4, \tag{2.9}$$
$$Y_3Y_1 = -Y_4Y_3. \tag{2.10}$$

Proof. It is evident that $Y^2 = I_n$ holds if and only if

$$Y^2 = \begin{bmatrix} Y_1^2 + Y_2Y_3 & Y_1Y_2 + Y_2Y_4 \\ Y_3Y_1 + Y_4Y_3 & Y_3Y_2 + Y_4^2 \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_{n-p} \end{bmatrix}.$$

□

Lemma 4. *Let (1.2)-(2.2) hold. Assume that $Y \in \mathcal{S}_n(D)$. Then Y_1 and Y_4 (called the quadratic matrices) satisfy the equations*

$$(2Y_1+I_p)(Y_1-I_p) = 0, \quad (2Y_4-I_{n-p})(Y_4+I_{n-p}) = 0. \tag{2.11}$$

Moreover, we have

$$(2Y_1+I_p)Y_2 = 0, \qquad Y_3(2Y_1+I_p) = 0, \tag{2.12}$$
$$I_p - Y_1 = 2Y_2Y_3, \qquad I_{n-p} + Y_4 = 2Y_3Y_2. \tag{2.13}$$

Proof. From (2.3) and (2.7) we get $Y_1^2 - Y_1 = I_p - Y_1^2$. Similarly, from (2.4) and (2.8) we obtain $Y_4^2 + Y_4 = I_{n-p} - Y_4^2$. The above equations are equivalent to (2.11). From (2.5) and (2.9), and from (2.6) and (2.10) we get (2.12). Note that (2.3) together with (2.7), and (2.4) together with (2.8), gives (2.13). □

Theorem 1. *Under the hypotheses of Lemma 4, $Y \in \mathcal{S}_n(D)$ if and only if Y satisfies (2.11)-(2.13).*

Proof. Standard calculations show that the conditions (2.3)-(2.10) are equivalent to (2.11)-(2.13). □

To find concrete solutions of the equation $DYD = YDY$ the following Lemma 5 is useful.

Lemma 5. *Let D be defined by (1.2), and $W = diag(W_1, W_2) \in \mathbb{C}^{n\times n}$ be arbitrary nonsingular matrix, where $W_1(p \times p)$. Assume that $Y \in \mathcal{S}_n(D)$ and define $\hat{Y} = W^{-1}YW$. The we have*

(i): $\hat{Y} \in \mathcal{S}_n(D)$,

(ii): *if we partition $\hat{Y}$ conformally with Y as follows*

$$\hat{Y} = \begin{bmatrix} \hat{Y}_1 & \hat{Y}_2 \\ \hat{Y}_3 & \hat{Y}_4 \end{bmatrix} = \begin{bmatrix} W_1^{-1}Y_1W_1 & W_1^{-1}Y_2W_2 \\ W_2^{-1}Y_3W_1 & W_2^{-1}Y_4W_2 \end{bmatrix}, \tag{2.14}$$

then we get the identities

$$(2\hat{Y}_1 + I_p)\hat{Y}_2 = 0, \quad \hat{Y}_3(2\hat{Y}_1 + I_p) = 0, \quad I_p - \hat{Y}_1 = 2\hat{Y}_2\hat{Y}_3 \tag{2.15}$$

and

$$I_{n-p} + \hat{Y}_4 = 2\hat{Y}_3\hat{Y}_2. \tag{2.16}$$

Proof. From Lemma 1 it follows that $Y \in \mathcal{S}_n(D)$ if and only if $\hat{Y} = W^{-1}YW \in \mathcal{S}_n(W^{-1}DW)$. However, since W is a block diagonal matrix, we obtain the identity $W^{-1}DW = D$, so $\hat{Y} \in \mathcal{S}_n(D)$.

To prove part (ii) of Lemma 5, we apply Lemma 4 to $\hat{Y}$ using the fact that $\hat{Y} \in \mathcal{S}_n(D)$. This completes the proof. $\square$

Remark 1. It is obvious that if W is an arbitrary nonsingular matrix then $W^{-1}DW = D$ holds only for block diagonal matrix $W = \mathrm{diag}(W_1, W_2)$, where $W_1(p \times p)$.

Note that if there are two matrices P and Q such that $A = PDP^{-1}$ and $A = QDQ^{-1}$, where D is defined by (1.2), then $W = P^{-1}Q$ is a block diagonal matrix. This together with Lemma 5 leads to

$$\mathcal{S}_n(A) = \{PYP^{-1} : Y \in \mathcal{S}_n(D)\} = \{QYQ^{-1} : Y \in \mathcal{S}_n(D)\}.$$

Remark 2. Notice that Y_1 and Y_4 satisfying (2.11) are nonsingular because each eigenvalue of Y_1 is either 1 or $-\frac{1}{2}$, and each eigenvalue of Y_4 is either -1 or $\frac{1}{2}$. Trivial solutions of (2.11) are: $Y_1 = I_p$, $Y_1 = -\frac{1}{2}I_p$, and $Y_4 = -I_{n-p}$, $Y_4 = \frac{1}{2}I_{n-p}$. In order to characterize other matrices Y_1 and Y_4 satisfying (2.11), we need some properties of quadratic matrices.

3. Quadratic matrices

We recall that $A \in \mathbb{C}^{n\times n}$ is a quadratic matrix, if there exist $\alpha, \beta \in \mathbb{C}$ such that $(A - \alpha I_n)(A - \beta I_n) = 0$. For the convenience of the reader we repeat the relevant material from [1].

Theorem 2 ([1, Theorem 1.2]). *Let $(A - \alpha I_n)(A - \beta I_n) = 0$, where $\alpha, \beta \in \mathbb{C}$. Assume that $A \neq \alpha I_n, \beta I_n$.*

(1) *Then there exist a unitary matrix $U \in \mathbb{C}^{n\times n}$ and triangular $R \in \mathbb{C}^{n\times n}$ such that $A = URU^*$ (the Schur form), where*

$$R = \begin{bmatrix} \alpha I_k & G \\ 0 & \beta I_{n-k} \end{bmatrix}, \quad 1 \le k < n. \tag{3.1}$$

(2) *Each eigenvalue of A is either α or β.*

(3) *If $\alpha \neq \beta$ then A is diagonalizable, and can be written as $A = PDP^{-1}$, where $P = UW$, and W is involution containing the eigenvectors of R, i.e. $RW = WD$, where*

$$W = \begin{bmatrix} I_k & \frac{G}{\alpha-\beta} \\ 0 & -I_{n-k} \end{bmatrix}, \quad D = \begin{bmatrix} \alpha I_k & 0 \\ 0 & \beta I_{n-k} \end{bmatrix}. \tag{3.2}$$

Lemma 6. *Let the quadratic matrices Y_1 and Y_4 satisfy (2.11), where $n \le 2p$. Assume that $Y_1 \neq I_p, -\frac{1}{2}I_p$ and $Y_4 \neq -I_{n-p}, \frac{1}{2}I_{n-p}$. Then there exist nonsingular matrices P_1 and P_4 such that*

$$Y_1 = P_1 D_1 P_1^{-1}, \quad D_1 = diag(-\tfrac{1}{2}I_r, I_{p-r}), \quad 1 \le r < p \tag{3.3}$$

and

$$Y_4 = P_4 D_4 P_4^{-1}, \quad D_4 = diag(\tfrac{1}{2}I_s, -I_{n-p-s}), \quad 1 \le s < n-p \le p. \tag{3.4}$$

Proof. It follows from Lemma 4 and Theorem 2. □

Remark 3. We see that all possible cases for Y_1 and Y_4 are: $Y_1 = I_p$ or $Y_1 = -\frac{1}{2}I_p$, or Y_1 satisfies (3.3). Similarly, Y_4 is equal to $-I_{n-p}$ or $\frac{1}{2}I_{n-p}$, or Y_4 satisfies (3.4). We consider all these cases.

4. Solutions of $DYD = YDY$ for $Y_1 = I_p$

Theorem 3. *Let (1.2)-(2.2) hold. If $Y_1 = I_p$ or $Y_4 = -I_{n-p}$ then $Y = D$ is the only solution of the quadratic equation $YDY = DYD$.*

Proof. Let $Y_1 = I_p$. Then from (2.12) it follows that $Y_2 = 0$ and $Y_3 = 0$. This together with (2.13) leads to $Y_4 = -I_{n-p}$, hence $Y = D$.

Now assume that $Y_4 = -I_{n-p}$. Then from (2.13) we have $Y_3Y_2 = 0$. Since $tr(Y_3Y_2) = 0$ and $tr(2Y_3Y_2) = tr(2Y_2Y_3) = tr(I_p - Y_1) = p - trY_1$, we get $trY_1 = p$, so $Y_1 = I_p$. This together with (2.12) leads to $Y_2 = 0$ and $Y_3 = 0$, hence $Y = D$. □

5. Solutions of $DYD = YDY$ for $Y_1 = -\frac{1}{2}I_p$

Theorem 4. *Let (1.2)-(2.2) hold and $Y \in \mathcal{S}_n(D)$. Assume that $Y_1 = -\frac{1}{2}I_p$. Then $n = 2p$, $Y_4 = \frac{1}{2}I_p$, and solutions of $YDY = DYD$ are*

$$Y = \begin{bmatrix} -\frac{1}{2}I_p & Y_2 \\ \frac{3}{4}Y_2^{-1} & \frac{1}{2}I_p \end{bmatrix}, \tag{5.1}$$

where $Y_2(p \times p)$ is an arbitrary nonsingular matrix.

Proof. From Theorem 3 it follows that $Y_4 \neq -I_{n-p}$. Assume now that $Y_4 \neq \frac{1}{2}I_{n-p}$. By Remark 3, Y_4 should satisfy (3.4). From (3.4) it follows that $trY_4 = \frac{s}{2} - (n-p-s)$, where $1 \leq s < n-p$. This together with (1.2) gives $s < p$. Therefore, $trY_1 + trY_4 = \frac{s-p}{2} - (n-p-s)$, so from (2.2) it follows that $trY_1 + trY_4 = 2p-n$ if and only if $s = p$, a contradiction. We conclude that $Y_4 = \frac{1}{2}I_{n-p}$. Then $trY_1 + trY_4 = -\frac{p}{2} + \frac{n-p}{2} = \frac{n-2p}{2}$. This together with (2.2) gives $n = 2p$. From (2.13) we get $Y_2Y_3 = \frac{3}{4}I_p$. Therefore, Y_2 and Y_3 are nonsingular, so $Y_3 = \frac{3}{4}Y_2^{-1}$. Clearly, Y has a form (5.1). It is easy to check that such Y satisfies the equation $YDY = YDY$. □

6. Solutions of $DYD = YDY$ for Y_1 satisfying (3.3)

In this section we study two remaining cases: $Y_4 = \frac{1}{2}I_{n-p}$, or Y_4 satisfies (3.4). Through this section we assume that Y_1 satisfies (3.3). We apply Lemma 5 to $W = \text{diag}(P_1, W_2)$, where W_2 is a nonsingular matrix. Then from (2.14) and (3.3) we get that $\hat{Y}_1 = D_1 = \text{diag}(-\frac{1}{2}I_r, I_{p-r})$, and we have

$$2D_1 + I_p = 3\text{diag}(0, I_{p-r}), \qquad I_p - D_1 = \tfrac{3}{2}\text{diag}(I_r, 0).$$

Now we partition $\hat{Y}_2$ and $\hat{Y}_3$ as follows

$$\hat{Y}_2 = \begin{bmatrix} B_2 \\ C_2 \end{bmatrix}, \quad B_2(r \times (n-p)), \quad \hat{Y}_3 = (B_3, C_3), \quad B_3((n-p) \times r). \tag{6.1}$$

Then from (2.15) it follows that

$$C_2 = 0, \quad C_3 = 0, \quad B_2B_3 = \tfrac{3}{4}I_r. \tag{6.2}$$

Theorem 5. *Let (1.2)-(2.2) hold. Let Y_1 satisfy (3.3) and $Y_4 = \frac{1}{2}I_{n-p}$. Then $r = n-p$, where $1 \leq r < p$. Moreover, $Y \in \mathcal{S}_n(D)$ if and only if $Y = W\hat{Y}W^{-1}$, where $W = diag(P_1, I_{n-p})$ and*

$$\hat{Y} = \left[\begin{array}{cc|c} -\frac{1}{2}I_{n-p} & 0 & B_2 \\ 0 & I_{2p-n} & 0 \\ \hline \frac{3}{4}B_2^{-1} & 0 & \frac{1}{2}I_{n-p} \end{array}\right], \tag{6.3}$$

where $B_2(n-p) \times (n-p))$ is an arbitrary nonsingular matrix.

Proof. Here $\hat{Y}_1 = D_1$ and $\hat{Y}_4 = Y_4$, where D_1 is defined by (3.3). Then $trY_1 + trY_4 = tr\hat{Y}_1 + tr\hat{Y}_4 = -\frac{r}{2} + (p-r) + \frac{n-p}{2}$, so by (2.2) we get $r = n-p$. From this and (6.1)-(6.2) it follows that B_2 and B_3 are nonsingular matrices. By (6.2), we get $B_3 = B_2^{-1}$. Clearly, (2.16) also holds. □

Theorem 6. *Let (1.2)-(2.2) hold. Let Y_1 satisfy (3.3) and Y_4 satisfy (3.4). Then $s = r$, where $1 \leq r < n-p \leq p$. Moreover, $Y \in \mathcal{S}_n(D)$ if and only if $Y = W\hat{Y}W^{-1}$,* *where $W = diag(P_1, P_4)$ and*

$$\hat{Y} = \left[\begin{array}{cc|cc} -\frac{1}{2}I_r & 0 & F_1 & 0 \\ 0 & I_{p-r} & 0 & 0 \\ \hline \frac{3}{4}F_1^{-1} & 0 & \frac{1}{2}I_r & 0 \\ 0 & 0 & 0 & -I_{n-p-r} \end{array}\right],$$

where $F_1(r \times r)$ is an arbitrary nonsingular matrix.

Proof. Here $\hat{Y}_1 = D_1$ and $\hat{Y}_4 = D_4$, where D_1 is defined by (3.3), and D_4 is given in (3.4). Then $trY_1 + trY_4 = tr\hat{Y}_1 + tr\hat{Y}_4 = -\frac{r}{2} + (p-r) + \frac{s}{2} - (n-p-s)$, so by (2.2) we get $s = r$.

We use (6.1) and partition B_2 and B_3 as follows

$$B_3 = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \quad F_1(r \times r), \quad B_2 = (G_1, G_2), \quad G_1(r \times r).$$

Then using (6.2) and (2.16) we conclude that $G_1 = \frac{3}{4}F_1^{-1}$, $G_2 = 0$ and $F_2 = 0$. □

7. ALGORITHMS

For given involutory matrix $A \in \mathbb{C}^{n \times n}$ and matrices P, D defined in (1.2), we can compute the concrete involutory solutions X of the equation $AXA = XAX$ as $X = PYP^{-1}$, where Y is a solution of the equation $DYD = YDY$.

In order to help readers implement our methods, we include algorithms for finding such solutions $Y \in \mathcal{S}_n(D)$. The proposed algorithms (Algorithms 1–3) construct **all solutions** $Y \neq D$ (we omit the trivial case where $X = A$, i.e. $Y = D$).

Algorithm 1. *Construction of $Y \in \mathcal{S}_{2p}(D)$, using Theorem 4.*
Take any natural number p and arbitrary nonsingular matrix $Y_2 \in \mathbb{C}^{p \times p}$.
The algorithm is determined now by two steps:

- compute Y_2^{-1},
- form $Y \in \mathbb{C}^{2p \times 2p}$ as follows:

$$Y = \begin{bmatrix} -\frac{1}{2}I_p & Y_2 \\ \frac{3}{4}Y_2^{-1} & \frac{1}{2}I_p \end{bmatrix}.$$

Algorithm 2. *Construction of $Y \in \mathcal{S}_n(D)$, using Theorem 5*
Take any natural numbers n and p such that $1 \leq n - p < p$, and arbitrary nonsingular matrices $P_1 \in \mathbb{C}^{p \times p}$, $B_2 \in \mathbb{C}^{(n-p)\times(n-p)}$.
The algorithm splits into the following steps:

- compute P_1^{-1} and B_2^{-1},
- $Y_1 = P_1 \mathrm{diag}(-\frac{1}{2}I_{n-p}, I_{2p-n})P_1^{-1}$,
- $Y_2 = P_1 \begin{bmatrix} B_2 \\ 0 \end{bmatrix}$,
- $Y_3 = (\frac{3}{4}B_2^{-1}, 0)P_1^{-1}$,

- form $Y \in \mathbb{C}^{n\times n}$ as follows

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & \frac{1}{2}I_{n-p} \end{bmatrix}.$$

Algorithm 3. *Construction of $Y \in \mathcal{S}_n(D)$, using Theorem 6*

Take any natural numbers n, p, r such that $1 \le r < n-p \le p$, and arbitrary nonsingular matrices $P_1 \in \mathbb{C}^{p\times p}$, $P_4 \in \mathbb{C}^{(n-p)\times(n-p)}$, and $F_1 \in \mathbb{C}^{r\times r}$.

The algorithm consists with the following steps:

- *compute P_1^{-1}, P_4^{-1} and F_1^{-1},*
- $Y_1 = P_1 diag(-\frac{1}{2}I_r, I_{p-r})P_1^{-1}$,
- $Y_2 = P_1 diag(F_1, 0)P_4^{-1}$,
- $Y_3 = P_4 diag(\frac{3}{4}F_1^{-1}, 0)P_1^{-1}$,
- $Y_4 = P_4 diag(\frac{1}{2}I_r, -I_{n-p-r})P_4^{-1}$,
- *form $Y \in \mathbb{C}^{n\times n}$ as follows:*

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}.$$

8. Concrete examples

This section contains a few examples to illustrate our theoretical results. All tests are performed in MATLAB using Symbolic Math Toolbox version 8.5 (R2020a).

Example 1. Let $D = \mathrm{diag}(1, -1)$. Then from Theorems 3 and 4 it follows that $\mathcal{S}_2(A) = \{D\} \cup \mathcal{K}$, where

$$\mathcal{K} = \left\{ \begin{bmatrix} -\frac{1}{2} & t \\ \frac{3}{4t} & \frac{1}{2} \end{bmatrix} : 0 \neq t \in \mathbb{C} \right\}.$$

Note that the set K is uniquely determined by parameter t. We have $\{D\} \cap K = \varnothing$.

Example 2. Let $D = \mathrm{diag}(1, 1, -1)$. Here $n = 3$ and $p = 2$. We prove that $\mathcal{S}_3(D) = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3$, where $\mathcal{K}_1 = \{D\}$ and $\mathcal{K}_i \cap \mathcal{K}_j = \varnothing$ for $i \neq j$.

It is obvious that $Y \in \mathcal{S}_3(D)$ iff $Y = D$ or $Y = W\hat{Y}W^{-1}$, where $W = \mathrm{diag}(P_1, 1)$, and $P_1 (2\times 2)$ is an arbitrary nonsingular matrix. By (6.3), we get

$$\hat{Y} = \begin{bmatrix} -\frac{1}{2} & 0 & t \\ 0 & 1 & 0 \\ \frac{3}{4t} & 0 & \frac{1}{2} \end{bmatrix}, \quad 0 \neq t \in \mathbb{C}.$$

Without loss of generality we can assume that $detP_1 = 1$, i.e. $P_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $ab - cd = 1$.

Notice that for such $W = \mathrm{diag}(P_1, 1)$ we get

$$Y = W\hat{Y}W^{-1} = \begin{bmatrix} -\frac{(ad+2bc)}{2} & \frac{3ab}{2} & at \\ -\frac{3cd}{2} & \frac{(bc+2ad)}{2} & ct \\ \frac{3d}{4t} & -\frac{3b}{4t} & \frac{1}{2} \end{bmatrix}.$$

Case (i): $a = 0$. Then $detP_1 = -bc = 1$, hence $b \neq 0$ and $c \neq 0$. Let $u = ct$ and $t_2 = ct$. Then $Y \in \mathcal{K}_2$ iff

$$Y = Y(u, t_2) = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3u}{2} & -\frac{1}{2} & t_2 \\ \frac{3u}{4t_2} & \frac{3}{4t_2} & \frac{1}{2} \end{bmatrix}, \quad u \neq 0, t_2 \neq 0.$$

We see that the set $\mathcal{K}_2$ is uniquely determined by parameters u and t_2.

Case (ii): $a \neq 0$. This together with $ad - bc = 1$ leads to $ad = 1 + bc$. Define $t_3 = at, b_2 = ab, c_2 = c/a$. Then $Y \in \mathcal{K}_3$ iff

$$Y = Y(b_2, c_2, t_3) = \begin{bmatrix} -\frac{1}{2} - \frac{3b_2c_2}{2} & \frac{3b_2}{2} & t_3 \\ -\frac{3c_2(1+b_2c_2)}{2} & 1 + \frac{3b_2c_2}{2} & c_2t_3 \\ \frac{3(1+b_2c_2)}{4t_3} & -\frac{3b_2}{4t_3} & \frac{1}{2} \end{bmatrix}, \quad t_3 \neq 0.$$

Notice that the set $\mathcal{K}_3$ is uniquely determined by parameters b_2, c_2, t_3.

Example 3. Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 6 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

By Theorem 2, the matrices A and P are involutions and we have the spectral decomposition $A = PDP^{-1}$, where $D = \mathrm{diag}(1, 1, -1, -1)$. We would like to find concrete solutions $Y \in \mathcal{S}_4(D)$ and $X = PYP \in \mathcal{S}_4(A)$. We see that here we can use only Algorithm 1 and Algorithm 3.

First, we apply Algorithm 1 for the matrix $Y_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Then we get

$$Y = \begin{bmatrix} -\frac{1}{2} & 0 & 1 & 2 \\ 0 & -\frac{1}{2} & 3 & 4 \\ -\frac{3}{2} & \frac{3}{4} & \frac{1}{2} & 0 \\ \frac{9}{8} & -\frac{3}{8} & 0 & \frac{1}{2} \end{bmatrix}, \quad X = \begin{bmatrix} \frac{11}{8} & -\frac{3}{8} & \frac{5}{8} & \frac{1}{4} \\ \frac{33}{8} & -\frac{19}{8} & \frac{55}{8} & \frac{11}{2} \\ \frac{3}{2} & -\frac{3}{4} & \frac{7}{2} & \frac{15}{4} \\ -\frac{9}{8} & \frac{3}{8} & -\frac{15}{8} & -\frac{5}{2} \end{bmatrix}.$$

Next, we apply Algorithm 3, taking $F_1 = 4$ and

$$P_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then we obtain the following solutions $Y \in \mathcal{S}_4(D)$ and $X = PYP \in \mathcal{S}_4(A)$:

$$Y = \begin{bmatrix} -2 & \frac{3}{2} & 2 & -2 \\ -3 & \frac{5}{2} & 2 & -2 \\ \frac{3}{8} & -\frac{3}{16} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{3}{8} & \frac{3}{16} & -\frac{3}{4} & -\frac{1}{4} \end{bmatrix}, \quad X = \begin{bmatrix} -\frac{11}{4} & \frac{15}{8} & -6 & -\frac{23}{8} \\ -\frac{33}{8} & \frac{49}{16} & -12 & -\frac{137}{16} \\ -\frac{3}{8} & \frac{3}{16} & -1 & -\frac{27}{16} \\ \frac{3}{8} & -\frac{3}{16} & 0 & \frac{11}{16} \end{bmatrix}.$$

9. Conclusions

- *We characterized all explicit involutive solutions of the Yang-Baxter matrix equation $AXA = XAX$.*
- *There are infinitely many such solutions for $A \neq \pm I_n$.*
- *Constructing involutive solutions of $AXA = XAX$ can be done easily by direct implementation of Algoritms 1–3.*

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