Miskolc Mathematical Notes
Vol. 24 (2023), No. 1, pp. 505–513

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.3754

ON QUATERNION-GAUSSIAN FIBONACCI POLYNOMIALS

TÜLAY YAĞMUR

Received 03 March, 2021

Abstract. In this paper, we define Gaussian Fibonacci quaternion polynomials and Gaussian Lucas quaternion polynomials. We also investigate some properties of these quaternion polynomials.

2010 *Mathematics Subject Classification:* 11B37; 11B83; 11R52; 20G20

Keywords: Gaussian numbers, Fibonacci numbers, Gaussian Fibonacci polynomials, quaternions

1. INTRODUCTION

Gaussian number, investigated by Gauss in 1832, is a complex number with integer coefficients. Horadam introduced the concept of complex Fibonacci numbers (Gaussian Fibonacci numbers) in 1963. Then, Gaussian Fibonacci numbers and Gaussian Lucas numbers are studied by many authors. Some example of these studies can be found in [1, 7, 8, 10, 12], among others.

The nth Gaussian Fibonacci number is defined by the relation

$$GF_n = GF_{n-1} + GF_{n-2}, \quad n \geq 2,$$

with initial conditions $GF_0 = i$ and $GF_1 = 1$.

It is easy to see that $GF_n = F_n + iF_{n-1}$, where F_n is the nth Fibonacci number defined recursively by $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$, $F_1 = 1$.

Similarly, the nth Gaussian Lucas number is defined by the relation

$$GL_n = GL_{n-1} + GL_{n-2}, \quad n \geq 2,$$

with initial conditions $GL_0 = 2 - i$ and $GL_1 = 1 + 2i$.

It is clear that $GL_n = L_n + iL_{n-1}$, where L_n is the nth Lucas number defined recursively by $L_n = L_{n-1} + L_{n-2}$ with $L_0 = 2$, $L_1 = 1$.

The Fibonacci polynomials studied by Catalan in 1883 are defined by the relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad n \geq 2,$$

with initial conditions $F_0(x) = 0$ and $F_1(x) = 1$.

The Lucas polynomials studied by Bicknell in 1970 are defined by the relation

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \quad n \geq 2,$$

with initial conditions $L_0(x) = 2$ and $L_1(x) = x$.

In [11], Özkan and Taştan studied Gaussian Fibonacci polynomials, Gaussian Lucas polynomials and also their applications. They gave the definitions of the Gaussian Fibonacci and Lucas polynomials, and Binet's formulas for these polynomials as follows:

The Gaussian Fibonacci polynomials are defined by the relation

$$GF_n(x) = xGF_{n-1}(x) + GF_{n-2}(x), \quad n \geq 2, \tag{1.1}$$

with initial conditions $GF_0(x) = i$ and $GF_1(x) = 1$.

Moreover, it is easy to see that $GF_n(x) = F_n(x) + iF_{n-1}(x)$. Setting $x = 1$ in the eq. (1.1), the Gaussian Fibonacci number GF_n can be obtained.

The Binet's formulas for the Gaussian Fibonacci polynomials are given by

$$GF_n(x) = \frac{\alpha^{n-1}(x)(\alpha(x) + i) - \beta^{n-1}(x)(\beta(x) + i)}{\alpha(x) - \beta(x)}, \tag{1.2}$$

where $\alpha(x) = \frac{x+\sqrt{x^2+4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2+4}}{2}$ are the roots of the equation $t^2 - xt - 1 = 0$.

The Gaussian Lucas polynomials are defined by the relation

$$GL_n(x) = xGL_{n-1}(x) + GL_{n-2}(x), \quad n \geq 2, \tag{1.3}$$

with initial conditions $GL_0(x) = 2 - ix$ and $GL_1(x) = x + 2i$.

Furthermore, it is clear that $GL_n(x) = L_n(x) + iL_{n-1}(x)$. Setting $x = 1$ in the eq. (1.3), the Gaussian Lucas number GL_n can be obtained.

The Binet's formulas for the Gaussian Lucas polynomials are given by

$$GL_n(x) = \alpha^{n-1}(x)(\alpha(x) + i) + \beta^{n-1}(x)(\beta(x) + i), \tag{1.4}$$

where $\alpha(x)$ and $\beta(x)$ are same as defined in eq. (1.2).

Quaternions, four-dimensional hyper-complex numbers, introduced by Sir William Rowan Hamilton in 1843. These numbers have found widespread application in quantum physics, computer graphics, robotics and signal processing.

A quaternion q is of the form

$$q = q_0 + q_1 i + q_2 j + q_3 k = (q_0, q_1, q_2, q_3),$$

where q_0, q_1, q_2, q_3 are real numbers, and i, j, k are quaternionic units which satisfy the equalities

$$i^2 = j^2 = k^2 = ijk = -1, ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik. \tag{1.5}$$

The set of all quaternions denoted by $\mathbb{H}$ is a non-commutative associative algebra over the real numbers. For a survey on quaternions, we refer the reader to [6, 15].

In [8], Horadam defined the Fibonacci and Lucas quaternions as

$$FQ_n = F_n + F_{n+1} i + F_{n+2} j + F_{n+3} k$$

and

$$LQ_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k,$$

respectively, where F_n is the nth Fibonacci number, L_n is the nth Lucas number, and i, j, k are quaternionic units which satisfy the rules (1.5).

There have been many studies in literature on Fibonacci and Lucas quaternions, see for example [2, 3, 9, 13, 14], among others.

The main objective of this paper is to define and study Gaussian Fibonacci and Lucas quaternion polynomials. We shall give recurrence relations, Binet's formulas, generating functions and summation formulas involving these quaternion polynomials.

2. THE GAUSSIAN FIBONACCI QUATERNION POLYNOMIALS

In this section, we first give the definitions of Gaussian Fibonacci quaternion polynomials and Gaussian Lucas quaternion polynomials. We then obtain some results for these quaternion polynomials.

For $n \geq 0$, Gaussian Fibonacci quaternion polynomials $\{GFQ_n(x)\}_{n=0}^{\infty}$ and Gaussian Lucas quaternion polynomials $\{GLQ_n(x)\}_{n=0}^{\infty}$ are defined by

$$GFQ_n(x) = (GF_n(x), GF_{n+1}(x), GF_{n+2}(x), GF_{n+3}(x)) \tag{2.1}$$

and

$$GLQ_n(x) = (GL_n(x), GL_{n+1}(x), GL_{n+2}(x), GL_{n+3}(x)), \tag{2.2}$$

respectively, where $GF_n(x)$ is the nth Gaussian Fibonacci polynomial, $GL_n(x)$ is the nth Gaussian Lucas polynomial.

It is easy to see that the nth Gaussian Fibonacci quaternion polynomial is defined recursively by

$$GFQ_n(x) = xGFQ_{n-1}(x) + GFQ_{n-2}(x), \quad n \geq 2, \tag{2.3}$$

with initial conditions

$$GFQ_0(x) = (i, 1, x+i, x^2+1+i) = 2i + (x-1)j + (x^2+2)k \tag{2.4}$$

and

$$\begin{aligned} GFQ_1(x) &= (1, x+i, x^2+1+i, x^3+2x+i(x+1)) \\ &= xi + (x^2-x)j + (x^3+2x+1)k. \end{aligned} \tag{2.5}$$

Similarly, the nth Gaussian Lucas quaternion polynomial is defined recursively by

$$GLQ_n(x) = xGLQ_{n-1}(x) + GLQ_{n-2}(x), \quad n \geq 2, \tag{2.6}$$

with initial conditions

$$GLQ_0(x) = (x^3+4x)k \tag{2.7}$$

and

$$GLQ_1(x) = (x^2+4)i + (x^4+5x^2+4)k. \tag{2.8}$$

It must be noted that if we set $x = 1$ in eqs. (2.1) or (2.3), we obtain Gaussian Fibonacci quaternions (see, [3,5])

$$GFQ_n = GFQ_{n-1} + GFQ_{n-2}, \quad n \geq 2, \tag{2.9}$$

with initial conditions

$$GFQ_0 = (i, 1, 1+i, 2+i) = 2i + 3k,$$
$$GFQ_1 = (1, 1+i, 2+i, 3+2i) = i + 4k.$$

and if we set $x = 1$ in eqs. (2.2) or (2.6), we obtain Gaussian Lucas quaternions (see, [4])

$$GLQ_n = GLQ_{n-1} + GLQ_{n-2}, \quad n \geq 2, \tag{2.10}$$

with initial conditions

$$GLQ_0 = (2-i, 1+2i, 3+i, 4+3i) = 5k,$$
$$GLQ_1 = (1+2i, 3+i, 4+3i, 7+4i) = 5i + 10k.$$

Let $\alpha(x)$ and $\beta(x)$ be the roots of the characteristic equation $t^2 - xt - 1 = 0$ on the recurrence relation (2.3) of Gaussian Fibonacci quaternion polynomials. Here, $\alpha(x) = \frac{x+\sqrt{x^2+4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2+4}}{2}$. These roots satisfy the following rules:

$$\alpha(x) + \beta(x) = x, \quad \alpha(x) - \beta(x) = \sqrt{x^2+4}, \quad \alpha(x)\beta(x) = -1,$$
$$\frac{\alpha(x)}{\beta(x)} = -\alpha^2(x), \quad \frac{\beta(x)}{\alpha(x)} = -\beta^2(x). \tag{2.11}$$

We now give the Binet's formulas for the Gaussian Fibonacci and Lucas quaternion polynomials in the following theorem.

Theorem 1. *The Binet's formulas for the Gaussian Fibonacci and Lucas quaternion polynomials are given by*

$$GFQ_n(x) = \frac{GFQ_1(x)(\alpha^n(x) - \beta^n(x)) + GFQ_0(x)(\alpha^{n-1}(x) - \beta^{n-1}(x))}{\alpha(x) - \beta(x)} \tag{2.12}$$

and

$$GLQ_n(x) = \frac{GLQ_1(x)(\alpha^n(x) - \beta^n(x)) + GLQ_0(x)(\alpha^{n-1}(x) - \beta^{n-1}(x))}{\alpha(x) - \beta(x)} \tag{2.13}$$

respectively.

Proof. From the general solution for the recurrence relation and using initial conditions the desired results can be obtained easily. □

If we set $x = 1$ in eq. (2.12), we obtain the Binet's formula for the Gaussian Fibonacci quaternions (see, [3,5]) as follow:

$$GFQ_n = \frac{GFQ_1(\alpha^n - \beta^n) + GFQ_0(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta}.$$

Moreover, if we set $x = 1$ in eq. (2.13), we obtain the Binet's formula for the Gaussian Lucas quaternions (see, [4]) as follow:

$$GLQ_n = \frac{GLQ_1(\alpha^n - \beta^n) + GLQ_0(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta}.$$

Now, the ordinary generating functions, exponential generating functions and Poisson generating functions for the Gaussian Fibonacci quaternion polynomials and Gaussian Lucas quaternion polynomials are given in the following results, respectively.

Theorem 2. *The ordinary generating functions for the Gaussian Fibonacci quaternion polynomials and Gaussian Lucas quaternion polynomials are given by*

$$\begin{aligned} f(t) &= \frac{GFQ_0(x) + (GFQ_1(x) - xGFQ_0(x))t}{1 - xt - t^2} \\ &= \frac{2i + (x-1)j + (x^2+2)k + (-xi + k)t}{1 - xt - t^2}, \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} g(t) &= \frac{GLQ_0(x) + (GLQ_1(x) - xGLQ_0(x))t}{1 - xt - t^2} \\ &= \frac{(x^3+4x)k + (x^2+4)(i+k)t}{1 - xt - t^2}, \end{aligned} \tag{2.15}$$

respectively.

Proof. Let $f(t)$ be the generating function for the Gaussian Fibonacci quaternion polynomials. Then we write

$$f(t) = \sum_{n=0}^{\infty} GFQ_n(x)t^n = GFQ_0(x) + GFQ_1(x)t + \ldots + GFQ_n(x)t^n + \ldots \quad . \tag{2.16}$$

Multiplying the eq. (2.16) with $-xt$ and $-t^2$, respectively, we get

$$-xtf(t) = -xGFQ_0(x)t - xGFQ_1(x)t^2 - \ldots - xGFQ_{n-1}(x)t^n - \ldots \tag{2.17}$$

and

$$-t^2 f(t) = -GFQ_0(x)t^2 - GFQ_1(x)t^3 - \ldots - GFQ_{n-2}(x)t^n - \ldots \quad . \tag{2.18}$$

Then adding the eqs. (2.16), (2.17) and (2.18), we obtain

$$\begin{aligned} (1 - xt - t^2)f(t) &= GFQ_0(x) + (GFQ_1(x) - xGFQ_0(x))t \\ &\quad + \sum_{n=2}^{\infty} (GFQ_n(x) - xGFQ_{n-1}(x) - GFQ_{n-2}(x))t^n. \end{aligned}$$

From the eq. (2.3), we get

$$(1 - xt - t^2)f(t) = GFQ_0(x) + (GFQ_1(x) - xGFQ_0(x))t.$$

Using the eqs. (2.4) and (2.5), we have

$$(1-xt-t^2)f(t) = 2i+(x-1)j+(x^2+2)k+(-xi+k)t$$

which completes the proof of the first statement.

The second statement of the theorem can be proved in a similar manner. $\square$

If we set $x=1$ in eq. (2.14), we obtain the ordinary generating function for the Gaussian Fibonacci quaternions (see, [3,5]) as follow:

$$f(t) = \frac{2i+3k+(k-i)t}{1-t-t^2}.$$

Furthermore, if we set $x=1$ in eq. (2.15), we obtain the ordinary generating function for the Gaussian Lucas quaternions (see, [4]) as follow:

$$g(t) = \frac{5k+5(i+k)t}{1-t-t^2}.$$

Theorem 3. *The exponential generating functions for the Gaussian Fibonacci quaternion polynomials and Gaussian Lucas quaternion polynomials are given by*

$$F(t) = \frac{(GFQ_1(x)-\beta(x)GFQ_0(x))e^{\alpha(x)t}-(GFQ_1(x)-\alpha(x)GFQ_0(x))e^{\beta(x)t}}{\alpha(x)-\beta(x)},$$

and

$$G(t) = \frac{(GLQ_1(x)-\beta(x)GLQ_0(x))e^{\alpha(x)t}-(GLQ_1(x)-\alpha(x)GLQ_0(x))e^{\beta(x)t}}{\alpha(x)-\beta(x)},$$

respectively.

Proof. Using the Binet's formula (2.12) of the Gaussian Fibonacci quaternion polynomials, we have

$$\begin{aligned}
F(t) &= \sum_{n=0}^{\infty} GFQ_n(x)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{GFQ_1(x)(\alpha^n(x)-\beta^n(x))}{\alpha(x)-\beta(x)}\frac{t^n}{n!} + \sum_{n=0}^{\infty} \frac{GFQ_0(x)(\alpha^{n-1}(x)-\beta^{n-1}(x))}{\alpha(x)-\beta(x)}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(\alpha(x)GFQ_1(x)+GFQ_0(x))(\alpha(x)t)^n}{\alpha(x)(\alpha(x)-\beta(x))n!} \\
&\quad - \sum_{n=0}^{\infty} \frac{(\beta(x)GFQ_1(x)+GFQ_0(x))(\beta(x)t)^n}{\beta(x)(\alpha(x)-\beta(x))n!} \\
&= \frac{(\alpha(x)GFQ_1(x)+GFQ_0(x))}{\alpha(x)(\alpha(x)-\beta(x))}\sum_{n=0}^{\infty}\frac{(\alpha(x)t)^n}{n!} \\
&\quad - \frac{(\beta(x)GFQ_1(x)+GFQ_0(x))}{\beta(x)(\alpha(x)-\beta(x))}\sum_{n=0}^{\infty}\frac{(\beta(x)t)^n}{n!}
\end{aligned}$$

$$= \frac{(\alpha(x)GFQ_1(x) + GFQ_0(x))}{1+\alpha^2(x)} e^{\alpha(x)t} + \frac{(\beta(x)GFQ_1(x) + GFQ_0(x))}{1+\beta^2(x)} e^{\beta(x)t}$$

$$= \frac{(GFQ_1(x) - \beta(x)GFQ_0(x))e^{\alpha(x)t} - (GFQ_1(x) - \alpha(x)GFQ_0(x))e^{\beta(x)t}}{\alpha(x) - \beta(x)}.$$

Thus, the proof of the first statement is completed. The second statement can be proved using the Binet's formula (2.13) of the Gaussian Lucas quaternion polynomials in a similar manner. □

Theorem 4. *The Poisson generating functions for the Gaussian Fibonacci quaternion polynomials and Gaussian Lucas quaternion polynomials are given by*

$$\mathcal{F}(t) = \frac{(GFQ_1(x) - \beta(x)GFQ_0(x))e^{\alpha(x)t} - (GFQ_1(x) - \alpha(x)GFQ_0(x))e^{\beta(x)t}}{e^t(\alpha(x) - \beta(x))},$$

and

$$\mathcal{G}(t) = \frac{(GLQ_1(x) - \beta(x)GLQ_0(x))e^{\alpha(x)t} - (GLQ_1(x) - \alpha(x)GLQ_0(x))e^{\beta(x)t}}{e^t(\alpha(x) - \beta(x))},$$

respectively.

Proof. Since $\mathcal{F}(t) = e^{-t}F(t)$ and $\mathcal{G}(t) = e^{-t}G(t)$, the proof of the theorem is clear. □

Theorem 5. *For $n \geq 2$, we have*

$$\sum_{k=1}^{n} GFQ_k(x) = \frac{1}{x}(GFQ_{n+1}(x) + GFQ_n(x) - GFQ_1(x) - GFQ_0(x)), \tag{2.19}$$

and

$$\sum_{k=1}^{n} GLQ_k(x) = \frac{1}{x}(GLQ_{n+1}(x) + GLQ_n(x) - GLQ_1(x) - GLQ_0(x)). \tag{2.20}$$

Proof. From eq. (1.1), we can write $GFQ_{n-1}(x) = \frac{1}{x}(GFQ_n(x) - GFQ_{n-2}(x))$. By the telescoping sum, we get

$$\sum_{k=1}^{n} GFQ_k(x) = \frac{1}{x}\sum_{k=2}^{n+1} GFQ_k(x) - \frac{1}{x}\sum_{k=0}^{n-1} GFQ_k(x)$$
$$= \frac{1}{x}(GFQ_{n+1}(x) + GFQ_n(x) - GFQ_1(x) - GFQ_0(x))$$

which completes the proof of the eq. (2.19).

Since $GLQ_{n-1}(x) = \frac{1}{x}(GLQ_n(x) - GLQ_{n-2}(x))$ from eq. (1.3)), eq. (2.20)) can be obtained in a similar manner. □

If we set $x = 1$ in eq. (2.20), we have [4]

$$\sum_{k=0}^{n} GLQ_k = GLQ_{n+2} - 5(i + 2k).$$

Corollary 1. *For $n \geq 2$, we have*

$$\sum_{k=1}^{n} GFQ_{2k}(x) = \frac{1}{x}(GFQ_{2n+1}(x) - GFQ_1(x)),$$

$$\sum_{k=1}^{n} GFQ_{2k-1}(x) = \frac{1}{x}(GFQ_{2n}(x) - GFQ_0(x)),$$

$$\sum_{k=1}^{n} GLQ_{2k}(x) = \frac{1}{x}(GLQ_{2n+1}(x) - GLQ_1(x)),$$

$$\sum_{k=1}^{n} GLQ_{2k-1}(x) = \frac{1}{x}(GLQ_{2n}(x) - GLQ_0(x)).$$

We now define the matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$ as follows:

$$\mathbf{Q} = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} GFQ_2(x) & GFQ_1(x) \\ GFQ_1(x) & GFQ_0(x) \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} GLQ_2(x) & GLQ_1(x) \\ GLQ_1(x) & GLQ_0(x) \end{pmatrix}.$$

Theorem 6. *For $n \geq 1$, we have*

$$\mathbf{Q^n R} = \begin{pmatrix} GFQ_{n+2}(x) & GFQ_{n+1}(x) \\ GFQ_{n+1}(x) & GFQ_n(x) \end{pmatrix}$$

and

$$\mathbf{Q^n S} = \begin{pmatrix} GLQ_{n+2}(x) & GLQ_{n+1}(x) \\ GLQ_{n+1}(x) & GLQ_n(x) \end{pmatrix}.$$

Proof. The results can be obtained easily using the mathematical induction on n. $\square$

Theorem 7 (Cassini's Identity)**.** *For $n \geq 1$, we have*

$$GFQ_{n+1}(x)GFQ_{n-1}(x) - GFQ_n^2(x) = (-1)^{n-1}(GFQ_2(x)GFQ_0(x) - GFQ_1^2(x)),$$

and

$$GLQ_{n+1}(x)GLQ_{n-1}(x) - GLQ_n^2(x) = (-1)^{n-1}(GLQ_2(x)GLQ_0(x) - GLQ_1^2(x)).$$

Proof. It is clear that $\det \mathbf{Q^{n-1}} = (-1)^{n-1}$, $\det \mathbf{R} = GFQ_2(x)GFQ_0(x) - GFQ_1^2(x)$ and $\det \mathbf{S} = GLQ_2(x)GLQ_0(x) - GLQ_1^2(x)$. Taking $\det \mathbf{Q^{n-1}R}$ and $\det \mathbf{Q^{n-1}S}$, we can obtain the desired results, respectively. $\square$

3. Conclusion

In this paper, we study the Gaussian Fibonacci and Lucas quaternion polynomials. We give some results including recurrence relations, Binet's formulas, generating functions and summation formulas for these quaternion polynomials.

It must be noted that for $x = 1$, the results for the Gaussian Fibonacci quaternion polynomials and Gaussian Lucas quaternion polynomials given in this study correspond to the Gaussian Fibonacci quaternions [3, 5] and Gaussian Lucas quaternions [4], respectively.

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Author's address

Tülay Yağmur

Aksaray University, Mathematics Department, 68100 Aksaray, Turkey

E-mail address: tulayyagmurr@gmail.com; tulayyagmur@aksaray.edu.tr