Miskolc Mathematical Notes
Vol. 23 (2022), No. 1, pp. 117–129
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2022.3751

THE EXISTENCE OF ONE NON-TRIVIAL WEAK SOLUTION OF GENERALIZED YAMABE EQUATIONS

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Received 02 March, 2021

Abstract. In this work, we study the nonlinear problem on compact d-dimensional ($d \geq 3$) Riemannian manifolds with respect to absence of boundary. The existence of one non-trivial weak solution is established, and its application to solve Emden-Fowler equations which contain infinity nonlinear terms. We also introduce an example to illustrate the results obtained, which can be applied to many of the problems resulting in astrophysics, conformal Riemannian geometry, gas combustion, isothermal stationary gas sphere and in the theory of thermionic emission.

2010 *Mathematics Subject Classification:* 35J60; 35B30; 35B40

Keywords: Yamabe problem, Emden-Fowler equation, sublinear eigenvalue problem, one non-trivial solutions, Riemannian manifolds

1. INTRODUCTION

In the last decades, many researchers have focused on searching mathematical analysis of differential equations. The field of analysis of the Riemannian manifolds is fertile and under great development, and it is an important tool for solving geometrical problems. On the other hand, geometry can help solve some of the problems associated with the analysis (see [2, 12, 16]).

We consider a compact d-dimensional Riemannian manifold with absence of boundary $(\mathcal{M}, g)$, where $d \geq 3$, Δ_g is the Laplace-Beltrami operator and suppose that $\alpha, K \in C^\infty(\mathcal{M})$ are positive functions. Assume the locally Hölder continuous function $f : \mathbb{R} \to \mathbb{R}$ with *sublinear* growth and $\lambda > 0$. In this work, we investigate the existence of the (classical) solutions of the eigenvalue problem

$$-\Delta_g w + \alpha(\sigma) w = \lambda K(\sigma) f(w), \;\; \sigma \in \mathcal{M}, \;\; w \in H_1^2(\mathcal{M}). \tag{1.1}$$

This problem is a generalization of the Yamabe equation (see [3]):

$$4\frac{d-1}{d-2}\Delta_g \varphi + R\varphi = \mu \varphi^{q-1} \text{in } \mathcal{M},$$

The fifth author extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through research groups program under grant (R.G.P-2/1/42).

where $2 < q < 2d/(d-2)$ with a curvature scalar R in $\mathcal{M}$. In [20], Yamabe showed in detail that for embedded manifolds there always exists such an equivalent metric corresponding to any given scale on the manifold. By applying heterogeneous methods, we get a well determined open interval of values λ for any problem (1.1) accepts at least one nontrivial solution. Later, Aubin in [2] proved the theorem for all manifolds of dimension $d \geq 6$ that were not conformably flat.

In 1984, by applying the variational methods Schoen settled the problem completely in [18], and get a well determined open interval of values λ for any problem (1.1) accepting at least one nontrivial solution.

A case of problem (1.1) is

$$-\Delta_h w + s(1-s-d)w = \lambda K(\sigma) f(w), \quad \sigma \in \mathrm{S}^d, \quad w \in H_1^2(\mathrm{S}^d), \tag{1.2}$$

where S^d is the unit sphere in $\mathbb{R}^{d+1}$, the standard metric h is induced by the embedding $\mathrm{S}^d \hookrightarrow \mathbb{R}^{d+1}$, the constant s verifies $1-d < s < 0$, and Δ_h is the Laplace-Beltrami operator on (S^d, h).

Using a suitable switching of coordinates, the results of the existence in (1.2) give us the existence of solutions to the Emden-Fowler equation

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s} u), \quad x \in \mathbb{R}^{d+1} \backslash \{0\}, \tag{1.3}$$

and we refer to Remark 3 and Corollary 2.

In addition, when $s = -d/2$ or $s = -d/2+1$, then the problem (1.2) has a smooth positive solution, and the answer $f(t) = |t|^{\frac{4}{d-2}} t$ can be considered convergent to the problem of Yamabe on S^d (for more details see [2, 9–11, 14, 15, 19, 20]).

In [19], the authors have solved the problem by applying minimization or minimum methods, provided that $f(t) = |\mathrm{t}|^{p-1}\mathrm{t}$, with $p > 1$. In [13] the authors were concerned with the existence of multiple solutions to problem (1.1) to obtain the solutions of Emden-Fowler equation with regard to infinity as nonlinear terms. Likewise, in [13], two non-trivial solutions to problem (1.1) were obtained, which took λ large enough.

There are also several works related to this type of problem, for example see ([1, 4–8]). Starting from these works, in this paper we will take a new path, and this is by attaching subproblems at infinity (previously studied and developed in [17]), where we get an open interval of positive parameters for which problem (1.1) admits at least one nontrivial solution.

The paper has been divided into the following parts: in the second section we lay out some preliminary and necessary facts and definitions. In the third section we examine the existence of at least one non-trivial solution to problem (1.1). In the final section we present some key results, with the solution to the Emden-Fowler equation, after which we propose an applied example for our theorems.

2. PRELIMINARIES

Starting from works [2] and [10], we present some basic concepts in Riemannian geometry.

Consider the smooth compact d-dimensional $(d \geq 3)$ Riemannian manifold $(\mathcal{M}, g)$ without boundary and let g_{ij} be the components of the metric g. Denote the space of smooth functions given on $\mathcal{M}$ by $C^\infty(\mathcal{M})$. Let the positive function $\alpha \in C^\infty(\mathcal{M})$ and $\|\alpha\|_\infty := \max_{\sigma\in\mathcal{M}} \alpha(\sigma)$. For every $w \in C^\infty(\mathcal{M})$, set

$$\|w\|^2_{H^2_\alpha} := \int_{\mathcal{M}} |\nabla w(\sigma)|^2 d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma)|w(\sigma)|^2 d\sigma_g,$$

where ∇w is the covariant derivative of w, and $d\sigma_g$ is the Riemannian measure. In local coordinates $(x^1,\dots,x^d)$, the components of ∇w are given by

$$(\nabla^2 w)_{ij} = \frac{\partial^2 w}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial w}{\partial x^k},$$

where

$$\Gamma^k_{ij} := \frac{1}{2}(\frac{\partial g_{1j}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k})g^{lk},$$

are Christoffel symbols and g^{lk} are the elements of the inverse matrix of g.

In what follows, we adopt the Einstein summation convention.

Furthermore, the element of measure $d\sigma_g$ supposes the form $d\sigma_g = \sqrt{\det g}\,dx$, where dx stands for the Lebesgue volume element of $\mathbb{R}^d$. Then, let

$$\mathrm{Vo1}_g(M) := \int_{\mathcal{M}} d\sigma_g.$$

If $(\mathcal{M}, g) = (\mathrm{S}^d, h)$, where S^d is the unit sphere in $\mathbb{R}^{d+1}$ and h is the standard metric induced by the embedding $\mathrm{S}^d \hookrightarrow \mathbb{R}^{d+1}$, we set

$$\omega_d := \mathrm{Vo1}_h(\mathrm{S}^d) := \int_{\mathrm{S}^d} d\sigma_h.$$

$H^2_\alpha(\mathcal{M})$ is the Sobolev space given as the completion of $C^\infty(\mathcal{M})$ with respect to the norm $\|\cdot\|_{H^2_\alpha}$. Hence, the Hilbert space $H^2_\alpha(\mathcal{M})$ is endowed with the inner product

$$\langle w_1,\ w_2\rangle_{H^2_\alpha} = \int_{\mathcal{M}} \langle \nabla w_1,\ \nabla w_2\rangle_g d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma)\langle w_1,\ w_2\rangle_g d\sigma_g,\ w_1,\ w_2 \in H^2_\alpha(\mathcal{M}),$$

where $\langle .,.\rangle_g$ is the inner product on covariant tensor fields associated to g.

As $\alpha > 0$, the norm $\|\cdot\|_{H^2_\alpha}$ is equivalent with the norm

$$\|w\|_{H^2_1} := (\int_{\mathcal{M}} |\nabla w(\sigma)|^2 d\sigma_g + \int_{\mathcal{M}} |w(\sigma)|^2 d\sigma_g)^{1/2}.$$

Furthermore, if $w \in H^2_\alpha(\mathcal{M})$, the following inequalities hold:

$$\min\{1, \min_{\sigma\in M} \alpha(\sigma)^{1/2}\}\|w\|_{H^2_1} \leq \|w\|_{H^2_\alpha} \leq \max\{1,\ \|\alpha\|^{1/2}_\infty\}\|w\|_{H^2_1}. \tag{2.1}$$

From the Rellich-Kondrachov theorem for compact manifolds without boundary one has

$$H^2_\alpha(\mathcal{M}) \hookrightarrow L^q(\mathcal{M}),$$

for any $q \in [1, 2d/(d-2)]$, the embedding is compact, whenever $q \in [1, 2d/(d-2)]$).

Then, $\exists S_q > 0$ so that

$$\|w\|_q \leq S_q \|w\|_{H^2_\alpha}, \text{for all } w \in H^2_\alpha(\mathcal{M}). \tag{2.2}$$

Now, we suppose that the nonlinear f satisfies the following structural condition: $f : \mathbb{R} \to \mathbb{R}$ is a locally Hölder continuous function sublinear at infinity, as follows

$$\lim_{|t|\to+\infty} \frac{f(t)}{t} = 0. \tag{2.3}$$

Let $K \in C^\infty(\mathcal{M})$ a positive function. Recall that $w \in H^2_1(\mathcal{M})$ is a *weak solution* of problem (1.1) if

$$\int_{\mathcal{M}} \langle \nabla w,\, \nabla v\rangle_g d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma)\langle w,\, v\rangle_g d\sigma_g - \lambda \int_{\mathcal{M}} K(\sigma) f(w(\sigma)) v(\sigma) d\sigma_g = 0$$

for every $v \in H^2_1(\mathcal{M})$. Moreover, because of the regularity supposition on f, the weak solutions are classical.

Remark 1. In (1.1), Δ_g represents the Laplace-Beltrami operator applied to a function $w \in H^2_1(\mathcal{M})$, and is given (locally) by the expression

$$\Delta_g w = g^{ij}\left(\frac{\partial^2 w}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial w}{\partial x^k}\right).$$

For a fixed $\lambda > 0$, the function $w_\lambda(\sigma) = c \in \mathbb{R}\backslash\{0\}$ is a solution of (P_λ) if and only if the function $\sigma \mapsto \lambda K(\sigma)/\alpha(\sigma)$ is constant. In this case, non-trivial constant solutions of (1.1) appear as fixed points of the function $t \mapsto k_\lambda f(t)$, where k_λ denotes the constant value $\lambda K(\sigma)/\alpha(\sigma)$.

In order to obtain one non-trivial solution of (1.1) not only in the case of constant solution but also other solutions, we use the variational methods. The main tool is a critical point theorem and we recall it here in a convenient form. The result is obtained by Ricceri in [17].

Theorem 1 ([17]). *Let X be a reflexive real Banach space, and let Φ, Ψ be two Gâteaux differentiable functionals such that Φ is (strongly) continuous, sequentially weakly lower semicontinuous and coercive. Moreover, suppose that Ψ is sequentially weakly upper semicontinuous. For any $\rho > \inf_X \Phi$, put*

$$\varphi(\rho) = \inf_{u\in\Phi^{-1}(]-\infty,\rho[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,\rho[)} \Psi(v)\right) - \Psi(u)}{\rho - \Phi(u)}.$$

Hence, for every $\rho > \inf_X \Phi$*, and any* $\lambda \in \left]0, \frac{1}{\varphi(\rho)}\right[$*, the restriction of* $J_\lambda := \Phi - \lambda\Psi$ *to* $\Phi^{-1}(]-\infty,\rho[)$ *accepts a global minimum, which is a critical point (local minimum) of* J_λ *in* X*.*

This result is a refinement of the variational principle of Ricceri, for more information see [17].

3. Main results

We give our main result in this section. Set

$$\kappa_\alpha := \left(\frac{2}{\|\alpha\|_{L^1(\mathcal{M})}}\right)^{1/2}$$

and

$$K_1 := \frac{S_1}{\sqrt{2}}\|\alpha\|_{L^1(\mathcal{M})},\ K_2 := \frac{S_q^q}{2^{\frac{2-q}{2}} - q}\|\alpha\|_{L^1(\mathcal{M})}.$$

Moreover, let

$$F(\xi) := \int_0^\xi f(t)dt$$

for any $\xi \in \mathbb{R}$.

The main abstract theorem in this work is the following.

Theorem 2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function with* $f(t) \neq 0$ *in* $\mathbb{R}$*, such that* (h_∞) *holds and suppose that*

(h1) $\exists a_1, a_2 > 0$ *such that* $|f(t)| \leq a_1 + a_2|t|^{q-1}$*, for any* $t \in \mathbb{R}$*, where* $q \in]1, 2d/(d-2)[$.

Then $\exists\lambda^* > 0$ *given by*

$$\lambda^* := \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2\|K\|_\infty} \sup_{\gamma>0}\left(\frac{\bar{\gamma}}{a_1K_1 + a_2K_2\bar{\gamma}^{q-1}}\right)$$

such that, for any parameter λ *belonging to* $\Lambda_\gamma :=]0, \lambda^*[$*, problem (1.1) possesses at least one non-trivial weak solution* u_λ *in* $H_1^2(\mathcal{M})$*. Moreover,*

$$\lim_{\lambda\to 0^+}\|u_\lambda\| = 0$$

and the function $\lambda \longmapsto J_\lambda(u_\lambda)$ *is negative and strictly decreasing in* $]0, \lambda^*[$.

Proof. Our goal is to use Theorem 1. Let $X := H_1^2(\mathcal{M})$ and let the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined by

$$\Phi(w) := \frac{\|w\|^2_{H^2_\alpha}}{2},$$

$$\Psi(w) := \int_{\mathcal{M}} K(\sigma)F(w(\sigma))d\sigma_g$$

for every $w \in X$. $\Phi : X \to \mathbb{R}$ is a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*. Ψ is well defined, continuously Gâteaux differentiable and has compact derivative. Further, one has

$$\Phi'(w)(v) = \int_{\mathcal{M}} \langle \nabla w, \nabla v \rangle_g d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma) \langle w(\sigma), v(\sigma) \rangle_g d\sigma_g,$$

and

$$\Psi'(w)(v) = \int_{\mathcal{M}} K(\sigma) f(w(\sigma)) v(\sigma) d\sigma_g,$$

for any $w, v \in X$. Fix $\lambda > 0$. A critical point of the functional $J_\lambda := \Phi - \lambda \Psi$ is a function $w \in X$ such that

$$\Phi'(w)(v) - \lambda \Psi'(w)(v) = 0,$$

for any $v \in X$. Then, the critical points of the functional J_λ are weak solutions (classical solutions) of problem (1.1).

Now, since (h1) and $\Phi(0) = \Psi(0) = 0$ hold, we have

$$F(\xi) \le a_1 |\xi| + a_2 \frac{|\xi|^q}{q}, \tag{3.1}$$

for every $\xi \in \mathbb{R}$. Since $0 < \lambda < \lambda^*$, there exists $\bar{\gamma}$ such that

$$\lambda < \lambda^*(\bar{\gamma}) := \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2\|K\|_\infty} \left(\frac{\bar{\gamma}}{a_1 K_1 + a_2 K_2 \bar{\gamma}^{q-1}} \right). \tag{3.2}$$

Let $\rho \in]0, +\infty[$ and let the function

$$\chi(\rho) := \frac{\sup_{w \in \Phi^{-1}(]-\infty, \rho])} \Psi(w)}{\rho}.$$

Recalling (3.1), it follows that

$$\Psi(w) = \int_{\mathcal{M}} K(\sigma) F(w(\sigma)) d\sigma_g \le \|K\|_\infty \left(a_1 \|w\|_{L^1(\mathcal{M})} + \frac{a_2}{q} \|w\|^q_{L^q(\mathcal{M})} \right).$$

Hence, for any $w \in X$ such that $w \in \Phi^{-1}(]-\infty, \rho])$ owing to (3.2), we find

$$\Psi(w) \le \|K\|_\infty \left((2\rho)^{1/2} S_1 a_1 + \frac{2^{q/2} S_q^q a_2}{q} \rho^{q/2} \right).$$

Then, by using the definition of Φ, we have

$$\sup_{w \in \Phi^{-1}(]-\infty, \rho])} \Psi(w) \le \|K\|_\infty \left((2\rho)^{1/2} S_1 a_1 + \frac{2^{q/2} S_q^q a_2}{q} \rho^{q/2} \right). \tag{3.3}$$

From (3.3), the following inequality holds

$$\chi(\rho) \leq \|K\|_\infty \left(\sqrt{\frac{2}{\rho}} S_1 a_1 + \frac{2^{q/2} S_q^q a_2}{q} \rho^{q/2-1} \right)$$

for every $\rho > 0$.
Hence, in particular

$$\chi(\bar{\gamma}^2) \leq \|K\|_\infty \left(\frac{\sqrt{2}}{\bar{\gamma}} S_1 a_1 + \frac{2^{q/2} S_q^q a_2}{q} \bar{\gamma}^{q-2} \right). \tag{3.4}$$

Now, observe that

$$\varphi\left(\bar{\gamma}^2\right) := \inf_{u \in \Phi^{-1}\left(\left]-\infty, \bar{\gamma}^2\right[\right)} \frac{\left(\sup_{v \in \Phi^{-1}\left(\left]-\infty, \bar{\gamma}^2\right[\right)} \Psi(v)\right)}{\rho - \Phi(u)} \leq \chi\left(\bar{\gamma}^2\right),$$

because $u_0 \in \Phi^{-1}\left(\left]-\infty, \bar{\gamma}^2\right[\right)$ and $\Phi(u_0) = \Psi(u_0) = 0$, where $u_0 \in X$ is the identically zero function. Finally, recalling to (3.3), the last inequality with (3.4) gives

$$\begin{aligned} \varphi\left(\bar{\gamma}^2\right) \leq \chi\left(\bar{\gamma}^2\right) &\leq \|K\|_\infty \left(\frac{\sqrt{2}}{\bar{\gamma}} S_1 a_1 + \frac{2^{q/2} S_q^q a_2}{q} \bar{\gamma}^{q-2} \right) \\ &= \frac{2\|K\|_\infty}{\|\alpha\|_{L^1(\mathcal{M})}} \left(a_1 \frac{K_1}{\bar{\gamma}} + a_2 K_2 \bar{\gamma}^{q-2} \right) < \frac{1}{\lambda}. \end{aligned}$$

In other words,

$$\lambda \in \left]0, \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2\|K\|_\infty} \left(\frac{\bar{\gamma}}{a_1 K_1 + a_2 K_2 \bar{\gamma}^{q-1}} \right) \right[\subseteq \left]0, \frac{1}{\varphi\left(\bar{\gamma}^2\right)}\right[.$$

By Theorem 1, $\exists u_\lambda \in \Phi^{-1}\left(\left]-\infty, \bar{\gamma}^2\right[\right)$ such that

$$J'_\lambda(u_\lambda) = \Phi'(u_\lambda) - \lambda \Psi'(u_\lambda) = 0,$$

and, u_λ is a global minimum of the restriction of J_λ to $\Phi^{-1}\left(\left]-\infty, \bar{\gamma}^2\right[\right)$. Further, since $f(t) \neq 0$ in $\mathbb{R}$, the function u_λ cannot be trivial, i.e. $u_\lambda \neq 0$.

Then, for any $\lambda \in]0, \lambda^*[$ problem (1.1) accepts a non-trivial solution $u_\lambda \in X$. Now, we prove that $\|u_\lambda\| \to 0$ as $\lambda \to 0^+$ and the function $\lambda \to J_\lambda(u_\lambda)$ is negative and strictly decreasing in $]0, \lambda^*[$.

For our aim, let us consider $\overline{\lambda} \in]0, \lambda^*[$. Moreover, let $\bar{\gamma} > 0$ such that $\overline{\lambda} < \lambda^*(\bar{\gamma})$. Arguing as before, for every $\lambda \in]0, \lambda^*(\bar{\gamma})[$ the functional $J_\lambda(u_\lambda)$ admits a non-trivial critical point $u_\lambda \in \Phi^{-1}\left(\left]-\infty, \bar{\gamma}^2\right[\right)$. At this point, since Φ is coercive and $u_\lambda \in \Phi^{-1}\left(\left]-\infty, \bar{\gamma}^2\right[\right)$ for every $\lambda \in]0, \lambda^*(\bar{\gamma})[$, $\exists \tau > 0$ such that

$$\|u_\lambda\| \leq \tau,$$

for any $\lambda \in]0, \lambda^*(\bar{\gamma})[$. So, since Ψ' is a compact operator, $\exists M > 0$ such that

$$|\Psi(u_\lambda)| \le \left\|\Psi'(u_\lambda)\right\|_{X^*} \|u_\lambda\| < M\,\tau^2, \tag{3.5}$$

for every $\lambda \in]0, \lambda^*(\bar{\gamma})[$.

Now $J'_\lambda(u_\lambda) = 0$, for every $\lambda \in]0, \lambda^*(\bar{\gamma})[$ and in particular

$$J'_\lambda(u_\lambda)(u_\lambda) = 0,$$

that is,

$$\|u_\lambda\|^2_{H^2_\alpha} = \lambda \int_{\mathcal{M}} K(\sigma) f(u_\lambda(\sigma)) u_\lambda(\sigma)\, dg_\sigma, \tag{3.6}$$

for every $\lambda \in]0, \lambda^*(\bar{\gamma})[$. Hence, from (3.5) and (3.6) it follows that

$$\lim_{\lambda \to 0^+} \Phi(u_\lambda) = 0,$$

implying

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{H^2_1} = 0.$$

Further, the map $\lambda \longmapsto J_\lambda(u_\lambda)$ is negative in $]0, \lambda^*(\bar{\gamma})[$, since the restriction of the functional J_λ to $\Phi^{-1}\left(\left]-\infty, \bar{\gamma}^2\right[\right)$ accepts a global minimum, which is a critical point (local minimum) of J_λ in X.

Finally, observe that

$$J_\lambda(u) = \lambda\left(\frac{\Phi(u)}{\lambda} - \Psi(u)\right),$$

for every $u \in X$ and fix $0 < \lambda_1 < \lambda_2 < \lambda^*(\bar{\gamma})$. Moreover, put

$$m_{\lambda_1} := \left(\frac{\Phi(u_{\lambda_1})}{\lambda_1} - \Psi(u_{\lambda_1})\right) = \inf_{u \in \Phi^{-1}\left(\left]-\infty, \bar{\gamma}^2\right[\right)} \left(\frac{\Phi(u)}{\lambda_1} - \Psi(u)\right),$$

and

$$m_{\lambda_2} := \left(\frac{\Phi(u_{\lambda_2})}{\lambda_2} - \Psi(u_{\lambda_2})\right) = \inf_{u \in \Phi^{-1}\left(\left]-\infty, \bar{\gamma}^2\right[\right)} \left(\frac{\Phi(u)}{\lambda_2} - \Psi(u)\right).$$

Since $m_{\lambda_i} < 0$, (for $i = 1, 2$), and $m_{\lambda_2} < m_{\lambda_1}$ according $\lambda_1 < \lambda_2$. Then the mapping $\lambda \longmapsto J_\lambda(u_\lambda)$ is strictly decreasing in $]0, \lambda^*(\bar{\gamma})[$ owing to

$$J_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \le \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = J_{\lambda_1}(u_{\lambda_1}).$$

This concludes the proof. □

Remark 2. Theorem 2 ensures that f has the global growth given by (h1). Hence, for any parameter λ belonging to the real interval $\Lambda_\gamma :=]0, \lambda^*[$, where

$$\lambda^* = \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2\|K\|_\infty} \sup_{\gamma > 0}\left(\frac{\bar{\gamma}}{a_1 K_1 + a_2 K_2 \bar{\gamma}^{q-1}}\right),$$

problem (1.1) accepts at least one non-trivial solution $u_\lambda \in X$. Furthermore, $\|u_\lambda\|_{H_1^2} \to 0$ as $\lambda \to 0^+$. Hence, a straightforward computation shows that

$$\lambda^* := \begin{cases} +\infty, & \text{if } 1 < q < 2, \\ \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2\|K\|_\infty K_1 a_1}, & \text{if } q = 2, \\ \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2\|K\|_\infty}\left(\frac{\bar{\gamma}_{\max}}{a_1K_1+a_2K_2\bar{\gamma}_{\max}^{q-1}}\right), & \text{if } q \in \ ]2, 2d\,/(d-2)[, \end{cases}$$

where

$$\bar{\gamma}_{\max} := \left(\frac{a_1K_1}{(q-2)\,a_2K_2}\right)^{1/(q-1)}.$$

Remark 3. The sharp Sobolev inequalities are important in the study of PDEs, especially in the study of those arising in geometry and physics. There has been much work on such inequalities and their applications. A concrete upper bound for the constants S_q in Theorem 2 is essential for a concrete evaluation of the interval Λ_γ.

In the case $(\mathcal{M}, g) = (\mathbb{S}^d, h)$, if $q \in [1, 2d/(d-2)[$, we have

$$S_q \leq \frac{\kappa_q}{\min\{1, \min_{\sigma \in \mathbb{S}^d} \alpha(\sigma)^{1/2}\}}, \tag{3.6}$$

where

$$\kappa_q := \begin{cases} \omega_d^{\frac{2-q}{2q}}, & \text{if} q \in [1,\ 2[, \\ \max\left\{\left(\frac{q-2}{d\ \omega_d^{\frac{q-2}{q}}}\right)^{\frac{1}{2}},\ \frac{1}{\omega_d^{\frac{q-2}{2q}}}\right\}, & \text{if } q \in [2,\ \frac{2d}{d-2}[. \end{cases}$$

Indeed, in Beckner [7], it is shown that for every $2 \leq q < 2d/(d-2)$ and every $w \in H_1^2(\mathbb{S}^d)$, we have

$$\left(\int_{\mathbb{S}^d} |w(\sigma)|^q d\sigma_h\right)^{2/q} \leq \frac{q-2}{d\omega_d^{1-2/q}} \int_{\mathbb{S}^d} |\nabla w(\sigma)|^2 d\sigma_h + \frac{1}{\omega_d^{1-2/q}} \int_{\mathbb{S}^d} |w(\sigma)|^2 d\sigma_h.$$

See also Theorem 4.28 in Hebey [10]. Then,

$$\|w\|_{L^q(\mathbb{S}^d)} \leq \max\left\{\left(\frac{q-2}{d\ \omega_d^{\frac{q-2}{q}}}\right)^{\frac{1}{2}},\ \frac{1}{\omega_d^{\frac{q-2}{2q}}}\right\}\left(\int_{\mathbb{S}^d} |\nabla w(\sigma)|^2 d\sigma_h + \int_{\mathbb{S}^d} |w(\sigma)|^2 d\sigma_h\right)^{1/2},$$

for every $w \in H_1^2(\mathbb{S}^d)$. And, if $q \in [1, 2\ [$, as a consequence of Hölder inequality, it follows that

$$\|w\|_{L^q(\mathbb{S}^d)} \leq \omega_d^{\frac{2-q}{2q}}\ \|w\|_{L^2(\mathbb{S}^d)}, \text{ for all } w \in L^2(\mathbb{S}^d).$$

The thesis is achieved by taking into account that

$$\|w\|_{L^2(\mathbb{S}^d)} \leq \|w\|_{H_1^2} \leq \frac{\|w\|_{H_\alpha^2}}{\min\{1, \min_{\sigma \in \mathbb{S}^d} \alpha(\sigma)^{1/2}\}}, \text{ for every } w \in H_1^2(\mathbb{S}^d).$$

4. Applications to nonlinear eigenvalue problems and Emden-Fowler equations

We set $\alpha, K \in C^\infty(\mathbb{S}^d)$ be positive and let

$$K_1^\star := \frac{\kappa_1 \|\alpha\|_{L^1(\mathbb{S}^d)}}{\sqrt{2}\min\{1, \min_{\sigma\in\mathbb{S}^d}\alpha(\sigma)^{1/2}\}}. \tag{4.1}$$

Further, for $q \in [1, \frac{2d}{d-2}[$ we will denote

$$K_2^\star := \frac{\kappa_q^q \|\alpha\|_{L^1(\mathbb{S}^d)}}{2^{\frac{2-q}{2}} q \min\{1, \min_{\sigma\in\mathbb{S}^d}\alpha(\sigma)^{q/2}\}}. \tag{4.2}$$

By Theorem 2, and according to Remark 2, we find the existence of at least one non-trivial solution for nonlinear eigenvalues problems on the unit sphere $\mathbb{S}^d$.

Corollary 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that (h_∞) and $(h1)$ hold. Moreover, there exists a positive number λ_1^* given by*

$$\lambda_1^* := \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2\|K\|_\infty} \sup_{\gamma>0}\left(\frac{\bar{\gamma}}{a_1 K_1^* + a_2 K_2^* \bar{\gamma}^{q-1}}\right)$$

such that, for every parameter λ belonging to $\Lambda_\gamma :=]0, \lambda_1^*[$, the problem

$$-\Delta_h w + \alpha(\sigma) w = \lambda K(\sigma) f(\omega),\ \sigma \in \mathbb{S}^d, w \in H_1^2(\mathbb{S}^d) \tag{4.3}$$

possesses at least one non-trivial weak solution u_λ in $H_1^2(\mathbb{S}^d)$. Further,

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0.$$

Now, consider the following parametrized Emden-Fowler problem that arises in astrophysics, conformal Riemannian geometry, gas combustion, the theory of thermionic emission and isothermal stationary gas sphere :

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s} u), x \in \mathbb{R}^{d+1}\backslash\{0\}. \tag{4.4}$$

Equation (4.4) has been studied when f has the form $f(t) = |t|^{p-1} t, p > 1$, see Cotsiolis and Iliopoulos [9], Vázquez and Véron [5].

Remark 4. The solutions of (4.4) are being expressed in the particular form

$$u(x) = r^s w(\sigma), \tag{4.5}$$

where

$$(r, \sigma) := (|x|, x/|x|) \in (0, \infty) \times \mathbb{S}^d$$

are the spherical coordinates in $\mathbb{R}^{d+1}\backslash\{0\}$ and the smooth function w given on $\mathbb{S}^d$.

Also, in [5] the authors used this type of transformation, where studied the asymptotic of a special form of (4.4). Throughout (4.5), with that in mind

$$\Delta u = r^{-d}\frac{\partial}{\partial r}(r^{d}\frac{\partial u}{\partial r}) + r^{-2}\Delta_h u,$$

(4.4) reduces to

$$-\Delta_h w + s(1-s-d)w = \lambda K(\sigma) f(w)\ ,\ \sigma \in \mathrm{S}^d,\ w \in H_1^2(\mathrm{S}^d),$$

(see also [13] Kristály and Rădulescu).

According to Remark 3, we give the following result.

Corollary 2. *Suppose that d and s are two constants such that $1-d<s<0$. Moreover, let the positive function $K \in C^\infty(\mathrm{S})$ and $f\colon \mathbb{R} \to \mathbb{R}$ as in Corollary 1. Hence, for every parameter λ belonging to*

$$\Lambda_\gamma^{s,d} := \left]0, \frac{s(1-s-d)\omega_d}{2\|K\|_\infty(a_1\frac{K_1^\star}{\gamma} + a_2 K_2^\star \gamma^{q-1})}\right[$$

the following problem

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s}u), x \in \mathbb{R}^{d+1}\backslash\{0\} \tag{4.6}$$

accepts at least one non-trivial weak solution.

Proof. Choose $(\mathcal{M}, g) = (\mathrm{S}^d, h)$, and $\alpha(\sigma) := s(1-s-d)$ for any $\sigma \in \mathrm{S}^d$ in Corollary 1. Clearly $\alpha \in C^\infty(\mathrm{S}^d)$ and, by $1-d<s<0$, we have $\alpha > 0$ on S^d. Hence, for any $\lambda \in \Lambda_\gamma^{s,d}$, the problem

$$-\Delta_h w + s(1-s-d)w = \lambda K(\sigma) f(w)\ ,\ \sigma \in \mathrm{S}^d,\ w \in H_1^2(\mathrm{S}^d)$$

has at least one non-trivial weak solution $w_\lambda \in H_1^2(\mathrm{S}^d)$. According to (4.5), we have that $u_\lambda(x) = |x|^s w_\lambda(x/|x|)$ is a solution of (4.6). □

Example 1. Let $\left(\mathrm{S}^d, h\right)$ with $d \geq 3$, and $K \in \mathrm{S}^d$. Moreover, consider the following equation

$$-\Delta_h w + w = \lambda K(\sigma)\left(|w|^{r-2} w + |w|^{s-2} w\right) \tag{4.7}$$

for every $\sigma \in \mathrm{S}^d$ and $w \in H_1^2(\mathrm{S}^d)$, where $1<r<2$ and $2<s<2^*$. Hence, for any

$$\lambda \in \left]0, \frac{\omega_d^{1/2}\bar{\gamma}_{\max}}{4\|K\|_\infty\left(K_1 + K_2\bar{\gamma}_{\max}^{s-1}\right)}\right[,$$

where

$$\bar{\gamma}_{\max} := \left(\frac{K_1}{K_2(s-2)}\right)^{1/(s-1)}$$

problem (4.7) accepts at least one non-trivial weak solution $w_\lambda \in H_1^2(\mathrm{S}^d)$ such that $\|u_\lambda\|_{H_1^2} \to 0$ as $\lambda \to 0^+$. To show this, we can use Theorem 2 with

$$f(t) = \left(|t|^{r-2} t + |t|^{s-2} t\right)$$

for every $t \in \mathbb{R}$. Then $|f(t)| \leq 2\left(1 + |t|^{s-1}\right)$ can be verified $\forall t \in \mathbb{R}$.

So, all the suppositions of Theorem 2 are verified and the conclusion follows.

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