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TAUBERIAN CONDITIONS UNDER WHICH STATISTICAL CONVERGENCE FOLLOWS FROM STATISTICAL SUMMABILITY BY WEIGHTED MEAN METHOD IN TWO-NORMED SPACES

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Abstract. In this paper, we introduce the statistical summability by the weighted mean method in a two-normed space and establish necessary and sufficient Tauberian conditions under which statistical convergence of a sequence (x_n) follows from its statistical summability by weighted mean method in a two-normed space. In particular, our Tauberian conditions are satisfied if (x_n) is statistically slowly oscillating in a two-normed space. The main theorem of this paper is the statistical extension of a Tauberian theorem in Çanak et al. [Acta Comment. Univ. Tartu. Math. 24, 49-57 (2020)].

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1. INTRODUCTION

The concept of statistical convergence as an extension to the concept of ordinary convergence was independently introduced by Fast [4] and Schoenberg [19].

A sequence (x_n) of real or complex numbers is said to be statistically convergent to some number l if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n+1} \left|\{k \leq n : |x_k - l| \geq \varepsilon\}\right| = 0,$$

where we denote by $|\mathcal{S}|$ the number of the elements in the set $\mathcal{S}$ of positive integers. In symbol, we write

$$st - \lim_{n\to\infty} x_n = l. \tag{1.1}$$

The concept of two-normed spaces was first defined by S. Gähler [8] and has been studied extensively by Cho, Diminnie, Freese, Gähler, Kim, White and many others ([3, 5, 6, 12, 13]).

Let X be a real vector space with $\dim X \geq 2$. A two-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies

(i) $\|x,y\| = 0$ if and only if x and y are linearly dependent;
(ii) $\|x,y\| = \|y,x\|$ for each $x,y \in X$;
(iii) $\|\alpha x,y\| = |\alpha|\|x,y\|$ for each $x,y \in X$ and $\alpha \in \mathbb{R}$;
(iv) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$ for each $x,y,z \in X$.

The pair $(X,\|\cdot,\cdot\|)$ is then called a two-normed space ([7]).

A standard example of a two-normed space is $X = \mathbb{R}^2$ being equipped with the two-norm $\|x,y\| :=$ area of the parallelogram spanned by the vectors x and y, which may be given explicitly by

$$\|x,y\| = |x_1y_2 - x_2y_1|, \text{ where } x = (x_1,x_2) \text{ and } y = (y_1,y_2).$$

Let (p_n) be a sequence of nonnegative numbers with $p_0 > 0$ such that

$$P_n = \sum_{k=0}^{n} p_k \to \infty, \quad n \to \infty. \tag{1.2}$$

The weighted means of a sequence (x_n) are defined by

$$\sigma_n = \frac{1}{P_n}\sum_{k=0}^{n} p_k x_k \quad (n = 0,1,2,...).$$

A sequence (x_n) is said to be summable to $l \in X$ by the weighted mean method determined by the sequence $p = (p_n)$ (or briefly summable $(\overline{N},p)$ to $l \in X$) if for every $y \in X$,

$$\lim_{n\to\infty} \|\sigma_n - l,y\| = 0. \tag{1.3}$$

We note that if $p_n = 1$ for all $n = 0,1,2,...$, $(\overline{N},p)$ summability method reduces to the Cesàro summability method.

Savaş and Sezer [18] introduced the concept of Cesàro summability method and obtained necessary and sufficient Tauberian conditions for Cesàro summability method in two-normed spaces. Following Savaş and Sezer [18], Çanak et al. [2] introduced the concept of weighted mean method of summability and established necessary and sufficient Tauberian conditions for the weighted mean summability of sequences in two-normed spaces.

Gürdal and Pehlivan [9] defined the concept of statistical convergence and obtained some properties of statistical convergence in two-normed spaces. A sequence (x_n) in two-normed space $(X,\|\cdot,\cdot\|)$ is said to be statistically convergent to some number l if for every $\varepsilon > 0$ and for every $z \in X$,

$$\lim_{n\to\infty} \frac{1}{n+1} |\{k \leq n : \|x_k - l,z\| \geq \varepsilon\}| = 0.$$

In symbol, we write

$$st - \lim_{n\to\infty} \|x_n,z\| = \|l,z\|. \tag{1.4}$$

A sequence (x_n) is said to be statistically summable to l in two-normed space $(X, \|\cdot,\cdot\|)$ by the weighted mean method determined by the sequence $p = (p_n)$ (or briefly statistically summable $(\overline{N}, p)$ to l in $(X, \|\cdot,\cdot\|)$ if for every $z \in X$,

$$st - \lim_{n\to\infty} \|\sigma_n, z\| = \|l, z\|. \tag{1.5}$$

Notice that statistical summability $(\overline{N}, p)$ reduces to the statistical summability $(C, 1)$ if $p_n = 1$ for all $n = 0, 1, 2, ...$; to the statistical logarithmic summability $(\ell, 1)$ if $p_n = 1/(n+1)$ for all $n = 0, 1, 2, ...$

Kolk [14] obtained necessary and sufficient Tauberian conditions under which statistical convergence of a bounded sequence implies its statistical summability by weighted means. But the converse implication is not true in general. Móricz and Orhan [16] obtained Tauberian conditions under which the converse implication is satisfied for sequences for real and complex numbers.

In 2020, Jena et al. [11] proved some Tauberian theorems for Cesàro summability of double sequences of fuzzy numbers. Subsequently, Parida et al. [17] established some new Tauberian theorems based on post-quantum calculus by means of statistical Cesàro summability mean of real-valued continuous functions of one variable under some appropriate conditions.

On the other hand, Srivastava et al. [21] obtained a new Korovkin-type approximation theorem on a Banach space and verified that their theorem both extends and improves most of the previous results. Later on, Srivastava et al. [25] introduced the notions of statistical convergence and statistical summability for martingale sequences of random variables by means of deferred Cesàro mean and proved new Korovkin-type approximation theorems with algebraic test functions for a martingale sequence over a Banach space via these notions.

For recent developments in Korovkin and Voronovskaya type approximation theorems given in different contents, we refer to [1, 20, 22–24, 26].

In Çanak et al. [2], Tauberian theorems for summability by weighted mean method were obtained in two-normed spaces. Recently, Loku et al. [15] established necessary and sufficient Tauberian conditions under which ordinary convergence of a sequence follows from its summability by Nörlund mean method in two-normed spaces. Although a special case of summability by Nörlund mean method is summability by weighted mean method, this present paper is roughly a statistical extension of the results in Çanak et al. [2]. Therefore, the results obtained in Loku et al. [15] and this present paper are completely independent of each other. But one can easily obtain a statistical extension of the results in Loku et al. [15] by using the techniques in this present paper.

In this paper, we investigate the problem under what conditions (1.4) follows from (1.5) in two-normed spaces.

2. Main Result

Theorem 1. *Let (p_n) be a sequence of nonnegative numbers such that $p_0 > 0$ and*

$$st - \liminf_{n\to\infty} \frac{P_{\lambda_n}}{P_n} > 1 \quad \text{for every} \quad \lambda > 1, \tag{2.1}$$

where $\lambda_n := [\lambda n]$ denotes the integer part of the product λn, and let (x_n) be a sequence in $(X, ||\cdot,\cdot||)$ which is statistically summable $(\overline{N}, p)$ to $l \in X$. Then (x_n) is statistically convergent to l if and only if for every $z \in X$, one of the following two conditions is satisfied:

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \left\| \frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k (x_k - x_n), z \right\| \ge \varepsilon \right\} \right| = 0, \tag{2.2}$$

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \left\| \frac{1}{P_n - P_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k (x_n - x_k), z \right\| \ge \varepsilon \right\} \right| = 0. \tag{2.3}$$

Following Hardy [10], a sequence (x_n) is said to be statistically slowly oscillating in two-norm if for every $\varepsilon > 0$ and for every $z \in X$,

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \max_{n<k\le\lambda_n} \|x_k - x_n, z\| \ge \varepsilon \right\} \right| = 0 \tag{2.4}$$

or equivalently,

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \max_{\lambda_n<k\le n} \|x_n - x_k, z\| \ge \varepsilon \right\} \right| = 0. \tag{2.5}$$

These conditions imply conditions (2.2) and (2.3), respectively.

Thus, the following corollary of Theorem 1 is obvious.

Corollary 1. *Let (p_n) be a sequence of nonnegative numbers such that $p_0 > 0$ and condition (2.1) is satisfied. If (1.5) and (2.4) hold, then (1.4) also holds.*

It is clear that (2.4) is satisfied if there exists a positive constant H such that for every $z \in X$, $n\|x_n - x_{n-1}, z\| \le H$ holds for every n large enough, say $n > n_1$.

For each $1 < n_1 \le n < k \le \lambda_n$ and each $z \in X$, we have

$$\|x_k - x_n, z\| \le \sum_{j=n+1}^{k} \|x_j - x_{j-1}, z\| \le H \sum_{j=n+1}^{k} \frac{1}{j} \le H \left(\frac{k-n}{n} \right) \le H(\lambda - 1). \tag{2.6}$$

It follows from (2.6) that

$$\max_{n<k\le\lambda_n} \|x_k - x_n, z\| \le \varepsilon \tag{2.7}$$

for each $\varepsilon > 0$ and $1 < \lambda \le 1 + \frac{\varepsilon}{H}$. For $n > n_1$ and for $z \in X$, the set

$$\left\{ n_1 < n \le N : \max_{n<k\le\lambda_n} \|x_k - x_n, z\| \ge \varepsilon \right\}$$

is empty. This shows that (2.4) is satisfied.

3. Auxiliary Results

Lemma 1 ([16]). *If (P_n) is a nondecreasing sequence of positive numbers, then conditions (2.1) and*

$$st-\liminf_{n\to\infty}\frac{P_n}{P_{\lambda_n}}>1 \quad \text{for every} \quad 0<\lambda<1 \tag{3.1}$$

are equivalent.

Lemma 2. *Let (p_n) be a sequence of nonnegative numbers such that $p_0>0$ and condition (2.1) is satisfied, and let (x_n) be a sequence in X which is statistically summable $(\overline{N},p)$ to l. Then for every $\lambda>0$ and for every $z\in X$,*

$$st-\lim_{n\to\infty}\|\sigma_{\lambda_n},z\|=\|l,z\|. \tag{3.2}$$

Lemma 2 is given for sequences of complex numbers in [16]. Since the proof of Lemma 2 is similar to that of proof of Lemma 1 in [16], we omit the proof of it.

Lemma 3 ([9]). *Let (x_n) and (y_n) be sequences in 2-normed space $(X,\|\cdot,\cdot\|)$ and $L,L'\in X$ and $a\in\mathbb{R}$. If $st\text{-}\lim_{n\to\infty}\|x_n,z\|=\|L,z\|$ and $st\text{-}\lim_{n\to\infty}\|y_n,z\|=\|L',z\|$, for every nonzero $z\in X$, then*

(i) *$st\text{-}\lim_{n\to\infty}\|x_n+y_n,z\|=\|L+L',z\|$, for each nonzero $z\in X$ and*
(ii) *$st\text{-}\lim_{n\to\infty}\|ax_n,z\|=\|aL,z\|$, for each nonzero $z\in X$.*

Lemma 4. *Let (p_n) be a sequence of nonnegative numbers such that $p_0>0$ and condition (2.1) is satisfied, and let (x_n) be a sequence in X which is statistically summable $(\overline{N},p)$ to l. Then,*

$$st-\lim_{n\to\infty}\left\|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n}p_kx_k,z\right\|=\|l,z\| \tag{3.3}$$

for every $\lambda>1$ and for every $z\in X$,

$$st-\lim_{n\to\infty}\left\|\frac{1}{P_n-P_{\lambda_n}}\sum_{k=\lambda_n+1}^{n}p_kx_k,z\right\|=\|l,z\| \tag{3.4}$$

for every $0<\lambda<1$ and for every $z\in X$.

Proof. Case $\lambda>1$. By definition, we have

$$\left\|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n}p_kx_k-l,z\right\|=\left\|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n}p_kx_k+\sigma_{\lambda_n}-\sigma_{\lambda_n}-l,z\right\|$$
$$=\left\|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=0}^{\lambda_n}p_kx_k-\frac{1}{P_{\lambda_n}-P_n}\sum_{k=0}^{n}p_kx_k+\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n}p_kx_k-\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n}p_kx_k-l,z\right\|$$
$$=\left\|\frac{P_n}{P_{\lambda_n}-P_n}\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n}p_kx_k-\frac{P_n}{P_{\lambda_n}-P_n}\frac{1}{P_n}\sum_{k=0}^{n}p_kx_k+\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n}p_kx_k-l,z\right\|$$

$$\leq \frac{P_n}{P_{\lambda_n}-P_n}\left\|\sigma_{\lambda_n}-\sigma_n,z\right\|+\left\|\sigma_{\lambda_n}-l,z\right\|.$$

Thus we find

$$\left\|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n}p_kx_k-l,z\right\|\leq \frac{P_n}{P_{\lambda_n}-P_n}\left\|\sigma_{\lambda_n}-\sigma_n,z\right\|+\left\|\sigma_{\lambda_n}-l,z\right\|. \tag{3.5}$$

By (2.1), we have

$$st-\limsup_{n\to\infty}\frac{P_n}{P_{\lambda_n}-P_n}=\left(st-\liminf_{n\to\infty}\frac{P_{\lambda_n}}{P_n}-1\right)^{-1}<\infty. \tag{3.6}$$

Now, (3.3) follows from (1.5), (3.2) and (3.5).

Case $0<\lambda<1$. By definition, we have

$$\left\|\frac{1}{P_n-P_{\lambda_n}}\sum_{k=\lambda_n+1}^{n}p_kx_k-l,z\right\|=\left\|\frac{1}{P_n-P_{\lambda_n}}\sum_{k=\lambda_n+1}^{n}p_kx_k+\sigma_n-\sigma_n-l,z\right\|$$
$$=\left\|\frac{1}{P_n-P_{\lambda_n}}\sum_{k=0}^{n}p_kx_k-\frac{1}{P_n-P_{\lambda_n}}\sum_{k=0}^{\lambda_n}p_kx_k+\frac{1}{P_n}\sum_{k=0}^{n}p_kx_k-\frac{1}{P_n}\sum_{k=0}^{n}p_kx_k-l,z\right\|$$
$$=\left\|\frac{P_{\lambda_n}}{P_n-P_{\lambda_n}}\frac{1}{P_n}\sum_{k=0}^{n}p_kx_k-\frac{P_{\lambda_n}}{P_n-P_{\lambda_n}}\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n}p_kx_k+\frac{1}{P_n}\sum_{k=0}^{n}p_kx_k-l,z\right\|$$
$$\leq \frac{P_{\lambda_n}}{P_n-P_{\lambda_n}}\left\|\sigma_n-\sigma_{\lambda_n},z\right\|+\left\|\sigma_n-l,z\right\|.$$

Thus we find

$$\left\|\frac{1}{P_n-P_{\lambda_n}}\sum_{k=\lambda_n+1}^{n}p_kx_k-l,z\right\|\leq \frac{P_{\lambda_n}}{P_n-P_{\lambda_n}}\left\|\sigma_n-\sigma_{\lambda_n},z\right\|+\left\|\sigma_n-l,z\right\|. \tag{3.7}$$

By (3.1), we have

$$st-\limsup_{n\to\infty}\frac{P_{\lambda_n}}{P_n-P_{\lambda_n}}=\left(st-\liminf_{n\to\infty}\frac{P_n}{P_{\lambda_n}}-1\right)^{-1}<\infty. \tag{3.8}$$

Now, (3.4) follows from (1.5), (3.2) and (3.7). □

4. Proof of the Main Result

Proof of Theorem 1. Necessity. Assume that (1.4) and (1.5) are satisfied. Let $\lambda>1$. By Lemmas 3 and 4, we have

$$st-\lim_{n\to\infty}\left\|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n}p_k(x_k-x_n),z\right\|$$

$$\leq st - \lim_{n\to\infty} \left\| \frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k x_k - l, z \right\| + st - \lim_{n\to\infty} \|x_n - l, z\| = 0$$

for every $z \in X$. This proves (2.2) even in a stronger form.

Let $0 < \lambda < 1$. We obtain in an analogous way that for every $z \in X$,

$$st - \lim_{n\to\infty} \left\| \frac{1}{P_n - P_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k(x_n - x_k), z \right\| = 0,$$

which is stronger than (2.3).

Sufficiency. Assume that (1.5), (2.1) and one of conditions (2.2) and (2.3) are satisfied. To this end, let $\varepsilon > 0$ be given. In case $\lambda > 1$, we rewrite the difference $x_n - l$ in the following form:

$$\begin{aligned} x_n - l &= \left[\frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k x_k - l \right] - \left[\frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k(x_k - x_n) \right] \\ &= A + B, \text{ say.} \end{aligned}$$

Then, we have

$$\{n \leq N : \|x_n - l, z\| \geq \varepsilon\} \subseteq \left\{ n \leq N : \|A, z\| \geq \frac{\varepsilon}{2} \right\} \cup \left\{ n \leq N : \|B, z\| \geq \frac{\varepsilon}{2} \right\}$$

for every $z \in X$. Hence,

$$|\{n \leq N : \|x_n - l, z\| \geq \varepsilon\}| \leq \left| \left\{ n \leq N : \|A, z\| \geq \frac{\varepsilon}{2} \right\} \right| + \left| \left\{ n \leq N : \|B, z\| \geq \frac{\varepsilon}{2} \right\} \right| \tag{4.1}$$

for every $z \in X$.

In case $0 < \lambda < 1$, we rewrite the difference $x_n - l$ in the following form:

$$\begin{aligned} x_n - l &= \left[\frac{1}{P_n - P_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k x_k - l \right] + \left[\frac{1}{P_n - P_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k(x_n - x_k) \right] \\ &= A' + B', \text{ say.} \end{aligned}$$

Then, we have

$$\{n \leq N : \|x_n - l, z\| \geq \varepsilon\} \subseteq \left\{ n \leq N : \|A', z\| \geq \frac{\varepsilon}{2} \right\} \cup \left\{ n \leq N : \|B', z\| \geq \frac{\varepsilon}{2} \right\}$$

for every $z \in X$. Hence,

$$|\{n \leq N : \|x_n - l, z\| \geq \varepsilon\}| \leq \left| \left\{ n \leq N : \|A', z\| \geq \frac{\varepsilon}{2} \right\} \right| + \left| \left\{ n \leq N : \|B', z\| \geq \frac{\varepsilon}{2} \right\} \right| \tag{4.2}$$

for every $z \in X$. By (2.2), for every $\eta > 0$ there exists some $\lambda > 1$ such that

$$\limsup_{N\to\infty} \frac{1}{N+1}\left|\left\{n \leq N : \left\|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n} p_k(x_k - x_n), z\right\| \geq \frac{\varepsilon}{2}\right\}\right| \leq \eta, \tag{4.3}$$

for every $z \in X$ or by (2.3), for every $\eta > 0$ there exists some $0 < \lambda < 1$ such that

$$\limsup_{N\to\infty} \frac{1}{N+1}\left|\left\{n \leq N : \left\|\frac{1}{P_n - P_{\lambda_n}}\sum_{k=\lambda_n+1}^{n} p_k(x_n - x_k), z\right\| \geq \frac{\varepsilon}{2}\right\}\right| \leq \eta \tag{4.4}$$

for every $z \in X$. Combining (4.1) and (4.4), in both cases we have by Lemma 4

$$\lim_{N\to\infty} \frac{1}{N+1}\left|\{n \leq N : \|x_n - l, z\| \geq \varepsilon\}\right| \leq \eta$$

for every $z \in X$. Since $\eta > 0$ is arbitrary, we necessarily have for every $\varepsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N+1}\left|\{n \leq N : \|x_n - l, z\| \geq \varepsilon\}\right| = 0$$

for every $z \in X$. This completes the proof of the Theorem.

5. Concluding Remark

Theorem 1 is the statistical extension of the Tauberian theorem given in [2]. Condition (2.2) is equivalent to the following: For given $\varepsilon > 0$ and $\eta > 0$, there exists some $\lambda > 1$ such that

$$\limsup_{N\to\infty} \frac{1}{N+1}\left|\left\{n \leq N : \left\|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n} p_k(x_k - x_n), z\right\| \geq \varepsilon\right\}\right| \leq \eta.$$

The equivalent form of condition (2.3) can be similarly given.

If conditions (1.4), (1.5) and (2.1) are satisfied, then we necessarily have

$$st - \lim_{n\to\infty}\left\|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n} p_k(x_k - x_n), z\right\| = 0 \tag{5.1}$$

for every $\lambda > 1$ and for every $z \in X$, and

$$st - \lim_{n\to\infty}\left\|\frac{1}{P_n - P_{\lambda_n}}\sum_{k=\lambda_n+1}^{n} p_k(x_n - x_k), z\right\| = 0 \tag{5.2}$$

for every $0 < \lambda < 1$ and for every $z \in X$.

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