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ON GENERALIZED PROJECTIVE CURVATURE TENSOR OF PARA-KENMOTSU MANIFOLDS

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Abstract. The object of the present paper is to generalize projective curvature tensor of para-Kenmotsu manifold with the help of a new generalized (0,2) symmetric tensor $\mathcal{Z}$ introduced by Mantica and Suh [8]. Various geometric properties of generalized projective curvature tensor of para-Kenmotsu manifold have been studied. It is shown that a generalized projectively ϕ-symmetric para-Kenmotsu manifold is an Einstein manifold.

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1. INTRODUCTION

A relevant tensor from the differential geometric point of view is the projective curvature tensor. Under a geodesic conserving transformation on a semi-Riemannian manifold, the projective curvature tensor P is invariant. In 1982 Szabo [15,16] studied Riemannian space satisfying $R(X,Y)\cdot R = 0$. In De and Samui [4] studied $P\cdot R = 0$, $R\cdot P = 0$ and $P\cdot S = 0$ in an LP-Sasakian manifold. The projective curvature tensor is therefore a measure of a Riemannian manifold's failure to be of constant curvature. Afterwards several researchers have carried out the study of projective curvature tensor in a variety of directions such as [10, 13, 14].

Several years ago, the notion of paracontact metric structures were introduced in [7]. Since the publication of [19], paracontact metric manifolds have been studied by many authors in recent years. The importance of para-Kenmotsu geometry, have been pointed out especially in the last years by several papers highlighting the exchanges with the theory of para-K*ä*hler manifolds and its role in semi-Riemannian geometry and mathematical physics [3, 5, 6, 11].

In this paper, we consider the generalized projective curvature tensor of para-Kenmotsu manifolds and study some properties of generalized projective curvature tensor. The organisation of the paper is as follows: After preliminaries on para-Kenmotsu manifold in section 2, we describe briefly the generalized projective curvature tensor on para-Kenmotsu manifold in section 3 and also we study some properties of generalized projective curvature tensor in para-Kenmotsu manifold. In section 4, we study a generalized projectively semi-symmetric para-Kenmotsu manifold is an Einstein manifold. Further in the section 5, we show that a generalized projectively Ricci semi-symmetric para-Kenmotsu manifold is either Einstein manifold or $\psi = 1-n$ on it. The last section is devoted to the study of the generalized projectively ϕ-symmetric para-Kenmotsu manifold is an Einstein manifold.

2. Preliminaries

The notion of an almost para-contact manifold was introduced by I. Sato [12]. An n-dimensional differentiable manifold M^n is said to have almost para-contact structure (ϕ, ξ, η), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field known as characteristic vector field and η is a 1-form satisfying the following relations

$$\phi^2(X) = X - \eta(X)\xi, \tag{2.1}$$

$$\eta(\phi X) = 0, \tag{2.2}$$

$$\phi(\xi) = 0, \tag{2.3}$$

and

$$\eta(\xi) = 1. \tag{2.4}$$

A differentiable manifold with almost para-contact structure (ϕ, ξ, η) is called an almost para-contact manifold. Further, if the manifold M^n has a semi Riemannian metric g satisfying

$$\eta(X) = g(X, \xi) \tag{2.5}$$

and

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y). \tag{2.6}$$

then the structure (ϕ, ξ, η, g) satisfying conditions (2.1) to (2.6) is called an almost para-contact Riemannian structure and the manifold M^n with such a structure is called an almost para-contact Riemannian manifold [1, 12].

On a para-Kenmotsu manifold [2, 11], the following relations hold:

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.7}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.8}$$

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{2.9}$$

$$\eta(R(X,Y,Z)) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.10}$$

$$R(X,Y,\xi) = \eta(X)Y - \eta(Y)X, \tag{2.11}$$

$$R(X,\xi,Y) = -R(\xi,X,Y) = g(X,Y)\xi - \eta(Y)X, \tag{2.12}$$

$$S(\phi X, \phi Y) = -(n-1)g(\phi X, \phi Y), \tag{2.13}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.14}$$

$$Q\xi = -(n-1)\xi, \tag{2.15}$$

$$r = -n(n-1), \tag{2.16}$$

for any vector fields X,Y,Z , where Q is the Ricci operator that is $g(QX,Y) = S(X,Y)$, S is the Ricci tensor and r is the scalar curvature.

In [2], Blaga have given an example on para-Kenmotsu manifold:

Example 1. We consider the three dimensional manifold $M^3 = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$, where (x,y,z) are the standard co-ordinates in $\mathbb{R}^3$. The vector fields

$$e_1 := \frac{\partial}{\partial x}, e_2 := \frac{\partial}{\partial y}, e_3 := -\frac{\partial}{\partial z}$$

are linearly independent at each point of the manifold.

Define

$$\begin{aligned} \phi &:= \frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial x} \otimes dy, \xi := -\frac{\partial}{\partial z}, \eta := -dz, \\ g &:= dx \otimes dx - dy \otimes dy + dz \otimes dz. \end{aligned}$$

Then it follows that

$$\begin{gathered} \phi e_1 = e_2, \phi e_2 = e_1, \phi e_3 = 0, \\ \eta(e_1) = 0, \eta(e_2) = 0, \eta(e_3) = 1. \end{gathered}$$

Let ∇ be the Levi-Civita connetion with respect to metric g. Then, we have

$$[e_1, e_2] = 0, [e_2, e_3] = 0, [e_3, e_1] = 0$$

The Riemannian connection ∇ of the metric g is deduced from Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &\quad - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]). \end{aligned}$$

Then Koszul's formula yields

$$\begin{aligned} &\nabla_{e_1} e_1 = -e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = e_1, \\ &\nabla_{e_2} e_1 = 0, \nabla_{e_2} e_2 = e_3, \nabla_{e_2} e_3 = e_2, \\ &\nabla_{e_3} e_1 = e_1, \nabla_{e_3} e_2 = e_2, \nabla_{e_3} e_3 = 0. \end{aligned}$$

These results shows that the manifold satisfies

$$\nabla_X \xi = X - \eta(X)\xi,$$

for $\xi = e_3$. Hence the manifold under consideration is para-Kenmotsu manifold of dimension three.

3. GENERALIZED PROJECTIVE CURVATURE TENSOR OF PARA-KENMOTSU MANIFOLD

In this section, we give a brief account of generalized projective curvature tensor of para-Kenmotsu manifold and studied various geometric properties of it.

The projective curvature tensor is defined by R.S. Mishra [9]

$$P(X,Y,U) = R(X,Y,U) - \frac{1}{(n-1)}[S(Y,U)X - S(X,U)Y], \tag{3.1}$$

where S is a Ricci tensor, such a tensor field P is known as projective curvature tensor.

Also, the type $(0,4)$ tensor field $'P$ is given by

$$\begin{aligned} 'P(X,Y,U,V) = {}'R(X,Y,U,V) - \frac{1}{(n-1)}[S(Y,U)g(X,V) \\ - S(X,U)g(Y,V)] \end{aligned} \tag{3.2}$$

where

$$'P(X,Y,U,V) = g(P(X,Y,U),V)$$

and

$$'R(X,Y,U,V) = g(R(X,Y,U),V)$$

for the arbitrary vector fields X,Y,U,V.

Differentiating covariantly equation (3.1) with respect to W, we get

$$\begin{aligned} (\nabla_W P)(X,Y)U) = (\nabla_W R)(X,Y)U) - \frac{1}{(n-1)}[(\nabla_W S)(Y,U)X \\ - (\nabla_W S)(X,U)Y]. \end{aligned} \tag{3.3}$$

A new generalized $(0,2)$ symmetric tensor $\mathcal{Z}$ is defined by Mantica and Suh [8]

$$\mathcal{Z}(X,Y) = S(X,Y) + \psi g(X,Y), \tag{3.4}$$

where ψ is an arbitrary scalar function.

From equation (3.4), we have

$$\mathcal{Z}(\phi X, \phi Y) = S(\phi X, \phi Y) + \psi g(\phi X, \phi Y), \tag{3.5}$$

which on using equations (2.6) and (2.13), gives

$$\mathcal{Z}(\phi X, \phi Y) = [\psi - (n-1)][-g(X,Y) + \eta(X)\eta(Y)]. \tag{3.6}$$

From equation (3.2), we have

$$\begin{aligned} 'P(X,Y,U,V) = {}'R(X,Y,U,V) - \frac{1}{(n-1)}[S(Y,U)g(X,V) \\ - S(X,U)g(Y,V)]. \end{aligned} \tag{3.7}$$

From equation (3.4) above equation reduces to

$$'P(X,Y,U,V) = {}'R(X,Y,U,V) - \frac{1}{(n-1)}[\mathcal{Z}(Y,U)g(X,V) \tag{3.8}$$

$$- \mathcal{Z}(X,U)g(Y,V)] + \frac{\psi}{(n-1)}[g(Y,U)g(X,V) - g(Y,V)g(X,U)],$$

Let

$$'P^*(X,Y,U,V) = 'R(X,Y,U,V) - \frac{1}{(n-1)}[\mathcal{Z}(Y,U)g(X,V) - \mathcal{Z}(X,U)g(Y,V)], \tag{3.9}$$

In the above equation, we get

$$'P^*(X,Y,U,V) = 'P(X,Y,U,V) - \frac{\psi}{(n-1)}[g(X,V)g(Y,U) - g(Y,V)g(X,U)]. \tag{3.10}$$

Thus $'P^*$ defined in equation (3.9) is called generalized projective curvature tensor of para-Kenmotsu manifold.

If ψ=0, then from eqauation (3.10), we have

$$'P^*(X,Y,U,V) = 'P(X,Y,U,V). \tag{3.11}$$

Lemma 1. *If the scalar function ψ vanishes on para-Kenmotsu manifold, then the projective curvature tensor and generalized projective curvature tensor are identicle.*

Lemma 2. *Generalized projective curvature tensor of para-Kenmotsu manifold satisfies Bianchi's first identity.*

Remark 1. Generalized projective curvature tensor $'P^*$ of para-Kenmotsu manifold is

(a) skew symmetric in first two slots.
(b) skew symmetric in last two slots.
(c) symmetric in pair of slots.

Proposition 1. *Generalized projective curvature tensor of para-Kenmotsu manifold satisfies the following identities:*

$$(a) \qquad P^*(\xi,Y,U) = -P^*(Y,\xi,U) = \left[\frac{1-n-\psi}{n-1}\right] g(Y,U)\xi - \frac{1}{(n-1)}S(Y,U)\xi + \frac{\psi}{(n-1)}\eta(U)Y, \tag{3.12}$$

$$(b) \qquad P^*(X,Y,\xi) = -\frac{\psi}{(n-1)}[\eta(Y)X - \eta(X)Y] \tag{3.13}$$

$$(c)\ \eta(P^*(U,V,Y)) = \left[-1 - \frac{\psi}{(n-1)}\right][g(V,Y)\eta(U) - g(U,Y)\eta(V)] - \frac{1}{(n-1)}[S(V,Y)\eta(U) - S(U,Y)\eta(V)]. \tag{3.14}$$

4. Generalized Projectively semi-symmetric para-Kenmotsu manifold

Definition 1. Para-Kenmotsu manifold is said to be semi-symmetric if it satisfies the condition

$$R(X,Y)\cdot R=0, \tag{4.1}$$

where $R(X,Y)$ is considered as the derivative of the tensor algebra at each point of the manifold.

Definition 2. Para-Kenmotsu manifold is said to be generalized projectively semi-symmetric if it satisfies the condition

$$R(X,Y)\cdot P^{*}=0, \tag{4.2}$$

where P^{*} is generalized projective curvature tensor and $R(X,Y)$ is considered as the derivative of the tensor algebra at each point of the manifold.

Theorem 1. *A generalized projectively semi-symmetric para-Kenmotsu manifold is an Einstein manifold.*

Proof. Consider

$$(R(\xi,X)\cdot P^{*})(U,V,Y)=0,$$

for any $X,Y,U,V\in T_LM$, where P^{*} is generalized projective curvature tensor.

Then we have

$$\begin{aligned}0=&\,R(\xi,X,P^{*}(U,V,Y))-P^{*}(R(\xi,X,U),V,Y)\\&-P^{*}(U,R(\xi,X,V),Y)-P^{*}(U,V,R(\xi,X,Y)).\end{aligned} \tag{4.3}$$

In view of the equation (2.12) above equation takes the form

$$\begin{aligned}0=&\,\eta(P^{*}(U,V,Y))X-{}'P^{*}(U,V,Y,X)\xi-\eta(U)P^{*}(X,V,Y)\\&+g(X,U)P^{*}(\xi,V,Y)-\eta(V)P^{*}(U,X,Y)+g(X,V)P^{*}(U,\xi,Y)\\&-\eta(Y)P^{*}(U,V,X)+g(X,Y)P^{*}(U,V,\xi).\end{aligned}$$

Taking inner product of above equation with ξ and using equations (2.4), (3.10), (3.12), (3.13), (3.14), we get

$$\begin{aligned}0=&-{}'P(U,V,Y,X)+\frac{\psi}{(n-1)}[g(X,U)g(Y,V)-g(X,V)g(Y,U)]\\&+\left[1+\frac{\psi}{(n-1)}\right][g(X,V)\eta(U)\eta(Y)-g(X,U)\eta(V)\eta(Y)]\\&+\left[\frac{1-n-\psi}{(n-1)}\right]g(X,U)g(Y,V)-\left[\frac{1-n-\psi}{(n-1)}\right]g(X,V)g(Y,U)\\&+\frac{\psi}{(n-1)}g(X,U)\eta(Y)\eta(V)-\frac{\psi}{(n-1)}g(X,V)\eta(Y)\eta(U)\end{aligned}$$

$$+\frac{1}{(n-1)}g(X,V)S(Y,U)-\frac{1}{(n-1)}g(X,U)S(Y,V)$$
$$+\frac{1}{(n-1)}[S(X,V)\eta(U)\eta(Y)-S(X,U)\eta(V)\eta(Y)].$$

By virtue of equation (3.2) above equation reduces to

$$\begin{aligned}'R(U,V,Y,X) &= \frac{\psi}{(n-1)}[g(X,U)g(Y,V)-g(X,V)g(Y,U)]\\ &+\left[1+\frac{\psi}{(n-1)}\right][g(X,V)\eta(U)\eta(Y)-g(X,U)\eta(V)\eta(Y)]\\ &+\frac{1}{(n-1)}[S(X,V)\eta(Y)\eta(U)-S(X,U)\eta(Y)\eta(V)]\\ &+\left[\frac{1-n-\psi}{(n-1)}\right]g(X,U)g(Y,V)-\left[\frac{1-n-\psi}{(n-1)}\right]g(X,V)g(Y,U)\\ &+\frac{\psi}{(n-1)}g(X,U)\eta(Y)\eta(V)-\frac{\psi}{(n-1)}g(X,V)\eta(Y)\eta(U).\end{aligned}$$

Let $\{e_i : i=1,2......n\}$ be an orthonormal basis vector putting $X=U=e_i$ in above equation and taking summation over i, we get

$$S(Y,V)=-(n-1)g(Y,V).$$

This shows that generalized projectively semi-symmetric para-Kenmotsu manifold is an Einstein manifold. □

5. Generalized Projectively Ricci semi-symmetric para-Kenmotsu manifold

Definition 3. Para-Kenmotsu manifold M is said to be Ricci semi-symmetric if the condition

$$R(X,Y)\cdot S=0, \tag{5.1}$$

holds for all $X,Y \in T_LM$.

Definition 4. Para-Kenmotsu manifold is said to be generalized projectively Ricci semi-symmetric if the condition

$$P^*(X,Y)\cdot S=0, \tag{5.2}$$

holds for all X,Y, where P^* is generalized projective curvature tensor of para-Kenmotsu manifold.

Theorem 2. *A generalized projectively Ricci semi-symmetric para-Kenmotsu manifold is either Einstein manifold or $\psi = 1-n$ on it.*

Proof. Consider
$$(P^*(\xi,X)\cdot S)(U,V)=0,$$
which gives
$$S(P^*(\xi,X,U),V)+S(U,P^*(\xi,X,V))=0.$$
Using equations (2.14) and (3.12) in above equation, we get
$$\left[\frac{n-1+\psi}{(n-1)}\right][S(X,V)\eta(U)+S(X,U)\eta(V)]$$
$$-(1-n-\psi)[g(X,U)\eta(V)+g(X,V)\eta(U)]=0.$$
Putting $U=\xi$ in the above equation and using (2.4), (2.5) and (2.14), we get
$$(n-1+\psi)[S(X,V)+g(X,V)(n-1)]=0,$$
which gives either $\psi=1-n$ or
$$S(X,V)=-(n-1)g(X,V).$$
This shows that generalized projectively Ricci semi-symmetric para-Kenmotsu manifold is an Einstein manifold. □

6. Generalized projectively ϕ-symmetric para-Kenmotsu manifold

Definition 5. A para-Kenmotsu manifold M^n is said to be locally ϕ-symmetric if
$$\phi^2((\nabla_W R)(X,Y,U))=0, \tag{6.1}$$
for all vector fields X,Y,U,W orthogonal to ξ.

This notion was introduced by Takahashi for Sasakian manifold [17].

Definition 6. A para-Kenmotsu manifold is said to be ϕ-symmetric if
$$\phi^2((\nabla_W R)(X,Y,U))=0, \tag{6.2}$$
for arbitrary vector fields X,Y,U,W.

This notion was also introduced by Takahashi for Sasakian manifold [18]. Also analogous to these definitons, we define

Definition 7. A para-Kenmotsu manifold M^n is said to be generalized projective locally ϕ-symmetric para-Kenmotsu manifold if
$$\phi^2((\nabla_W P^*)(X,Y,U))=0, \tag{6.3}$$
for all vector fields X,Y,U,W orthogonal to ξ.

And also

Definition 8. A para-Kenmotsu manifold M^n is said to be generalized projectively ϕ-symmetric para-Kenmotsu manifold if

$$\phi^2((\nabla_W P^*)(X,Y,U)) = 0, \tag{6.4}$$

for arbitary vector fields X,Y,U,W.

Theorem 3. *A generalized projectively ϕ-symmetric para Kenmotsu manifold is an Einstein manifold.*

Proof. Taking covariant derivative of equation (3.10) with respect to vector field W, we obtain

$$(\nabla_W P^*)(X,Y,U) = (\nabla_W P)(X,Y,U) - \frac{dr(\psi)}{(n-1)}[g(Y,U)X - g(X,U)Y)], \tag{6.5}$$

Using equation (3.3) in the above equation, we yields

$$(\nabla_W P^*)(X,Y,U) = (\nabla_W R)(X,Y,U) - \frac{dr(\psi)}{(n-1)}[g(Y,U)X - g(X,U)Y)] - \frac{1}{(n-1)}[(\nabla_W S)(Y,U)X - (\nabla_W S)(X,U)Y], \tag{6.6}$$

Assume that the manifold is generalized projectively ϕ-symmetric, then from equation (6.4), we have

$$\phi^2((\nabla_W P^*)(X,Y,U)) = 0,$$

which on using equation (2.1), gives

$$(\nabla_W P^*)(X,Y,U) = \eta((\nabla_W P^*)(X,Y,U))\xi. \tag{6.7}$$

Using equation (6.6) in above equation, we get

$$\begin{aligned}(\nabla_W R)(X,Y,U) &- \frac{dr(\psi)}{(n-1)}[g(Y,U)X - g(X,U)Y)] \\ &- \frac{1}{(n-1)}[(\nabla_W S)(Y,U)X - (\nabla_W S)(X,U)Y] \\ = \eta((\nabla_W R)(X,Y,U))\xi &- \frac{dr(\psi)}{(n-1)}[g(Y,U)\eta(X) - g(X,U)\eta(Y))]\xi \\ &- \frac{1}{(n-1)}[(\nabla_W S)(Y,U)\eta(X) - (\nabla_W S)(X,U)\eta(Y)]\xi,\end{aligned} \tag{6.8}$$

Taking inner product of the above equation with V, we get

$$g((\nabla_W R)(X,Y,U),V) - \frac{dr(\psi)}{(n-1)}[g(Y,U)g(X,V) - g(X,U)g(Y,V)] - \frac{1}{(n-1)}[(\nabla_W S)(Y,U)g(X,V) - (\nabla_W S)(X,U)g(Y,V)]$$

$$= \eta((\nabla_W R)(X,Y,U))\eta(V) - \frac{dr(\psi)}{(n-1)}[g(Y,U)\eta(X) \tag{6.9}$$
$$- g(X,U)\eta(Y)]\eta(V) - \frac{1}{(n-1)}[(\nabla_W S)(Y,U)\eta(X) - (\nabla_W S)(X,U)\eta(Y)]\eta(V),$$

Putting $X = V = e_i$ and taking summation over i, we obtain

$$-dr(\psi)g(Y,U) = \eta((\nabla_W R)(e_i,Y,U))\eta(e_i) - \frac{1}{(n-1)}[(\nabla_W S)(Y,U) \tag{6.10}$$
$$- (\nabla_W S)(e_i,U)\eta(Y)\eta(e_i)] - \frac{dr(\psi)}{(n-1)}[g(Y,U) - \eta(Y)\eta(U)],$$

Taking $U = \xi$ in the above equation, we have

$$\eta((\nabla_W R)(e_i,Y,\xi))\eta(e_i) + dr(\psi)\eta(Y) - \frac{1}{(n-1)}[(\nabla_W S)(Y,\xi) \tag{6.11}$$
$$- (\nabla_W S)(e_i,\xi)\eta(e_i)\eta(Y)] = 0.$$

Now

$$\eta((\nabla_W R)(e_i,Y,\xi)\eta(e_i) = g((\nabla_W R)(e_i,Y,\xi),\xi)g(e_i,\xi). \tag{6.12}$$

Also

$$g((\nabla_W R)(e_i,Y,\xi),\xi) = g(\nabla_W R(e_i,Y,\xi),\xi) - g(R(\nabla_W e_i,Y,\xi),\xi)$$
$$- g(R(e_i,\nabla_W Y,\xi),\xi) - g(R(e_i,Y,\nabla_W \xi),\xi). \tag{6.13}$$

Since $\{e_i\}$ is an orthonormal basis, so $\nabla_X e_i = 0$ and using equation (2.11), we get

$$g(R(e_i,\nabla_W Y,\xi),\xi) = 0.$$

As

$$g(R(e_i,Y,\xi),\xi) + g(R(\xi,\xi,Y),e_i) = 0,$$

we have

$$g(\nabla_W R(e_i,Y,\xi),\xi) + g(R(e_i,Y,\xi),\nabla_W \xi) = 0.$$

Using this fact, we get

$$g((\nabla_W R)(e_i,Y,\xi),\xi) = 0. \tag{6.14}$$

Using equation (6.14) in (**??**), we have

$$(\nabla_W S)(Y,\xi) = (n-1)dr(\psi)\eta(Y). \tag{6.15}$$

Taking $Y = \xi$ in above equation and using equations (2.4) and (2.14), we get

$$dr(\psi) = 0, \tag{6.16}$$

which shows that r is constant. Now we have

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi),$$

Then by using (2.8), (2.9), (2.14) in the above equation, it follows that

$$(\nabla_W S)(Y,\xi) = -S(Y,W) - (n-1)g(Y,W). \tag{6.17}$$

So from equation (6.15), (6.16) and (6.17), This shows that

$$S(Y,W) = -(n-1)g(Y,W), \tag{6.18}$$

which shows that M^n is an Einstein manifold. □

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