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ON FEJÉR-HERMITE-HADAMARD INEQUALITIES FOR FUNCTIONS INVOLVING FRACTIONAL INTEGRALS

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Abstract. In this paper, some new estimations of Hermite-Hadamard-Fejér type inequalities for higher order differentiable preinvex functions are established via fractional integral operators. The main motivation point of the study is that our findings, which include generalized preinvex functions for Hermite-Hadamard-Fejer type inequalities by means of fractional integral operators, are the rare results in the literature. The coincidence of the special cases of the our main theorems with the earlier works in the literature is also demonstrated as a verification of our results.

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1. INTRODUCTION

Many researchers have concentrated on the concept of convexity, which has an important place in mathematics as well as in many other disciplines with its applications and features. We will start by recalling two basic inequalities built on the concept of convexity, which is the task of establishing the strong link between inequality theory and convex analysis. The Hermite-Hadamard inequality, which produces bounds for the Cauchy mean value of a convex function, and the Hermite-Hadamard-Fejér inequality, which is a weighted version of this inequality, have very useful and effective applications in mathematical analysis, engineering, statistics, physics and numerical integration.

Assume that g is a convex mapping such that $g: V \subseteq \mathbb{R} \to \mathbb{R}$ and $\ell_1, \ell_2 \in V$ with $\ell_1 < \ell_2$, then the inequality

$$g\left(\frac{\ell_1+\ell_2}{2}\right) \leq \frac{1}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2} g(x)dx \leq \frac{g(\ell_1)+g(\ell_2)}{2}. \tag{1.1}$$

is known as the celebrated **Hermite-Hadamard inequality** in the literature.

In [12], a motivated generalization of inequality (1.1) is established by Fejér as follow:

$$g\left(\frac{\ell_1+\ell_2}{2}\right)\int_{\ell_1}^{\ell_2} f(x)dx \leq \int_{\ell_1}^{\ell_2} f(x)g(x)dx \leq \frac{g(\ell_1)+g(\ell_2)}{2}\int_{\ell_1}^{\ell_2} f(x)dx, \tag{1.2}$$

holds, where $f:[\ell_1;\ell_2]\to\mathbb{R}$ is a nonnegative, integrable function and it is symmetric about $x=\frac{\ell_1+\ell_2}{2}$.

Although fractional analysis is historically as old as classical analysis, it has not only rapidly increased its popularity in recent years, but also the properties and applications of new concepts that have been gained to the literature by fractional analysis have made a breakthrough especially in applied sciences. Fractional analysis makes it attractive for many researchers, with its advantages and perfect compatibility with classical analysis for some concepts whose deficiencies are discussed according to classical analysis. Defining fractional derivatives and integral operators, revealing their properties, using them in modeling and solving equations dealing with various natural phenomena, memory effect caused by locality and singularity of kernel structures, and effective results obtained in applications are the factors that carry the fractional analysis forward. To provide further on fractional analysis, see the papers [1–4, 6–11, 13, 14, 18, 21, 23–25, 27]. Now let's get to know one of the cornerstones of fractional analysis, the Riemann-Liouville fractional integral integrals, and the fractional version of the Hadamard inequality that includes these famous operators.

Theorem 1. *(See [22]) Let $g:[\ell_1,\ell_2]\subseteq\mathbb{R}\to\mathbb{R}$ with $0\leq\ell_1<\ell_2$ and $g\in L[\ell_1,\ell_2]$. If g is positive and convex function on $[\ell_1,\ell_2]$, then the following inequalities hold:*

$$g\left(\frac{\ell_1+\ell_2}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(\ell_2-\ell_1)^{\alpha}}\left[J^{\alpha}_{\ell_2}g(\ell_1)+J^{\alpha}_{\ell_1}g(\ell_2)\right]\leq\frac{g(\ell_1)+g(\ell_2)}{2}.$$

with $ß>0$.

Here, the left and right-sided Riemann-Liouville fractional integrals of the order ß $\in\mathbb{R}^+$ denoted by the symbols $J^{ß}_{\ell_1}$ and $J^{ß}_{\ell_2}$ that are defined in [15]

$$J^{ß}_{\ell_1}g(x)=\frac{1}{\Gamma(ß)}\int_{\ell_1}^{x}(x-t)^{ß-1}g(t)dt,\ 0\leq\ell_1<x\leq\ell_2$$

and

$$J^{ß}_{\ell_2}g(x)=\frac{1}{\Gamma(ß)}\int_{x}^{\ell_2}(t-x)^{ß-1}g(t)dt,\ 0\leq\ell_1\leq x<\ell_2.$$

If we choose $ß=1$, the fractional integral becomes the classical integral.

Researchers who have worked on the concept of convexity have introduced many new types of convex functions, made definitions of convexity in multidimensional spaces, and contributed to the spread of the concept to a wide spectrum from operator convexity to convexity defined in $n-$dimensional space. As a result of these efforts, many new classes of functions, their properties and countless inequalities built on them have been revealed. In this context, we will continue with the concept of preinvexity, which is a generalization of convexity. But first, let's remind the definition of invex sets.

Definition 1. (See [5]) A set $V \subseteq \mathbb{R}$ is invex with respect to the map $\Im : V \times V \to \mathbb{R}$ if for every $\ell_1, \ell_2 \in V$ and $t \in [0,1]$, $\ell_2 + t\Im(\ell_1, \ell_2) \in V$. The invex set V is also called an $\Im$-connected set. Every convex set is an invex set but its converse is not true.

Definition 2. (See [29]) Let $V \subseteq \mathbb{R}$ be an invex set and a function $g : V \to \mathbb{R}$ is said to be preinvex w.r.t. $\Im$ if

$$g(\ell_2 + t\Im(\ell_1, \ell_2)) \leq tg(\ell_1) + (1-t)g(\ell_2).$$

$\forall \ell_1, \ell_2 \in V$ and $t \in [0,1]$.

If $\Im(\ell_1, \ell_2) = \ell_1 - \ell_2$, then in classical sense , the preinvex functions coincide convex functions. Recently, the following identity has been proved by Sikander et. al in [20] for $\check{n}-$times differentiable preinvex functions.

Lemma 1. *Let $V \subseteq [0,\infty)$ be an open invex subset with respect to $\Im : V \times V \to \mathbb{R}$. Suppose $g : V \to \mathbb{R}$ is a function such that $g^{(\check{n})}$ exists on V and $g^{(\check{n})}$ is integrable on $[\ell_1, \ell_1 + \Im(\ell_2, \ell_1)]$ for $\check{n} \in \mathbb{N}, \check{n} \geq 1$, then for every $\ell_1, \ell_2 \in V$ with $\Im(\ell_2, \ell_1) > 0$, the following equality holds:*

$$\begin{aligned}
&\frac{g(\ell_1) + g(\ell_1 + \Im(\ell_2, \ell_1))}{2} \\
&\quad - \frac{\Gamma(\beta+1)}{2(\Im(\ell_2, \ell_1))^{\beta}} \left[J^{\beta}_{\ell_1^+} g(\ell_1 + \Im(\ell_2, \ell_1)) + J^{\beta}_{(\ell_1 + \Im(\ell_2, \ell_1))^-} g(\ell_1) \right] \\
&= \sum_{\kappa=1}^{\check{n}-1} \frac{\Gamma(\beta+1)(\Im(\ell_2, \ell_1))^{\kappa}}{2\,\Gamma(\beta+\kappa+1)} \left[(-1)^{\kappa-1} g^{(\kappa)}(\ell_1 + \Im(\ell_2, \ell_1)) - g^{(\kappa)}(\ell_1) \right] \\
&\quad - \frac{(\Im(\ell_2, \ell_1))^{\check{n}}\ \Gamma(\beta+1)}{2\,\Gamma(\beta+\check{n})} \\
&\qquad \times \int_0^1 \left[(1-t)^{\beta+\check{n}-1} + (-1)^{\check{n}} t^{\beta+\check{n}-1} \right] g^{(\check{n})}(\ell_1 + t\Im(\ell_2, \ell_1)) dt,
\end{aligned}$$

for $\beta > 0$ and $\check{n} \geq 1$

Similar results can be found in [16, 17, 26, 28, 30].

The main purpose of this study is to create integral identities that include Riemann-Liouville integral operators that have the potential to produce Hermite-Hadamard-Fejér type inequalities for $\check{n}-$times differentiable preinvex functions and to obtain integral inequalities that present new approaches based on these identities.

2. MAIN RESULTS

In the sequel of the paper, we will denote

$$\|f\|_{\infty} = \sup_{x\in[\ell_1,\ell_1+\Im(\ell_2,\ell_1)]} |f(x)|,$$

where $f : [\ell_1;\ell_1+\Im(\ell_2,\ell_1)] \to \mathbb{R}$ is a continuous function, $g^{(\check{n})}$ is the $\check{n}$-th derivative of g w.r.t. variable t and $Ļ[\ell_1,\ell_2]$ is the classes of all real-valued Riemann integrable mappings defined on $[\ell_1,\ell_2]$.

Lemma 2. *Suppose that $V \subseteq \mathbb{R}$ is an open invex set and $\Im : V\times V \to \mathbb{R}$ is a mapping. Assume that $g : V \to \mathbb{R}$ is a differentiable mapping such that $g^{(\check{n})} \in L[\ell_1,\ell_1+\Im(\ell_2,\ell_1)]$ where $\Im(\ell_2,\ell_1) > 0$. If $w : [\ell_1,\ell_1+\Im(\ell_2,\ell_1)] \to [0,\infty)$ is an integrable mapping, then we have the following equality:*

$$\begin{aligned}
&\sum_{\kappa=0}^{\check{n}-1} \frac{g^{(\kappa)}(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{2^{\kappa}\Im(\ell_2,\ell_1)^{\beta+\check{n}-\kappa}} \\
&\quad\times\left[(-1)^{\check{n}-\kappa-1} J^{\beta}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-} f(\ell_1) + (-1)^{\check{n}+1} J^{\beta}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+} f(\ell_1+\Im(\ell_2,\ell_1))\right] \\
&\quad+(-1)^{\check{n}} \frac{1}{(\Im(\ell_2,\ell_1))^{\beta+\check{n}}} \\
&\quad\times\left[J^{\beta}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-}(gf)(\ell_1) + J^{\beta}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+}(gf)(\ell_1+\Im(\ell_2,\ell_1))\right] \\
&= \frac{1}{\Gamma(\beta)}\int_0^1 w(t) g^{(\check{n})}(\ell_1+t\Im(\ell_2,\ell_1))dt, \qquad (2.1)
\end{aligned}$$

for $\forall \ell_1,\ell_2 \in V$, where

$$w(t) = \begin{cases} \underbrace{\int_0^t\int_0^t\cdots\int_0^t}_{\check{n}-integrals} u^{\beta-1} f(\ell_1+u\Im(\ell_2,\ell_1))(du)^{\check{n}}, & t\in\left[0,\frac{1}{2}\right). \\ \underbrace{\int_1^t\int_1^t\cdots\int_1^t}_{\check{n}-integrals} (1-u)^{\beta-1} f(\ell_1+u\Im(\ell_2,\ell_1))(du)^{\check{n}}, & t\in\left[\frac{1}{2},1\right]. \end{cases}$$

Proof. Consider

$$\begin{aligned}
&\int_0^1 w(t) g^{(\check{n})}(\ell_1+t\Im(\ell_2,\ell_1))dt \\
&\qquad= \int_0^{\frac{1}{2}} \left(\underbrace{\int_0^t\int_0^t\cdots\int_0^t}_{\check{n}-integrals} u^{\beta-1} f(\ell_1+u\Im(\ell_2,\ell_1))(du)^{\check{n}} \right) \\
&\qquad\quad\times g^{(\check{n})}(\ell_1+t\Im(\ell_2,\ell_1))dt
\end{aligned}$$

$$+\int_{\frac{1}{2}}^{1}\left(\underbrace{\int_1^t\int_1^t\ldots\int_1^t}_{\check{n}-integrals}(1-u)^{ß-1}f(\ell_1+u\Im(\ell_2,\ell_1))(du)^{\check{n}}\right)$$
$$\times g^{(\check{n})}(\ell_1+t\Im(\ell_2,\ell_1))dt=I_1+I_2$$

From the first integral, we have

$$I_1=\int_0^{\frac{1}{2}}\left(\underbrace{\int_0^t\int_0^t\ldots\int_0^t}_{\check{n}-integrals}u^{ß-1}f(\ell_1+u\Im(\ell_2,\ell_1))(du)^{\check{n}}\right)$$
$$\times g^{(\check{n})}(\ell_1+t\Im(\ell_2,\ell_1))dt$$
$$=\frac{1}{\Im(\ell_2,\ell_1)}\left(\underbrace{\int_0^t\int_0^t\ldots\int_0^t}_{\check{n}-1\ \ integrals}u^{ß-1}f(\ell_1+u\Im(\ell_2,\ell_1))(du)^{\check{n}-1}\right)$$
$$\times g^{(\check{n}-1)}(\ell_1+t\Im(\ell_2,\ell_1)\Big|_0^{\frac{1}{2}}$$
$$-\frac{1}{\Im(\ell_2,\ell_1)}\int_0^{\frac{1}{2}}\left(\underbrace{\int_0^t\int_0^t\ldots\int_0^t}_{\check{n}-1\ \ integrals}u^{ß-1}f(\ell_1+u\Im(\ell_2,\ell_1))(du)^{\check{n}}\right)$$
$$\times g^{(\check{n}-1)}(\ell_1+t\Im(\ell_2,\ell_1))dt$$
$$=\frac{g^{(\check{n}-1)}(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{\Im(\ell_2,\ell_1)}$$
$$\times\left(\underbrace{\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\ldots\int_0^{\frac{1}{2}}}_{\check{n}\ \ integrals}t^{ß-1}f(\ell_1+t\Im(\ell_2,\ell_1))(dt)^{\check{n}}\right)$$
$$-\frac{1}{\Im(\ell_2,\ell_1)}\int_0^{\frac{1}{2}}\left(\underbrace{\int_0^t\int_0^t\ldots\int_0^t}_{\check{n}-1\ \ integrals}u^{ß-1}f(\ell_1+u\Im(\ell_2,\ell_1))(du)^{\check{n}-1}\right)$$
$$\times g^{(\check{n}-1)}(\ell_1+t\Im(\ell_2,\ell_1))dt.$$

On generalizing the result, we have

$$\begin{aligned}
I_1 = {} & \frac{g^{(\check{n}-1)}(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{\Im(\ell_2,\ell_1)}\left(\underbrace{\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\cdots\int_0^{\frac{1}{2}}}_{\check{n}-integrals} t^{\text{ß}-1} f(\ell_1+t\Im(\ell_2,\ell_1))\,(dt)^{\check{n}}\right) \\
& -\frac{g^{(\check{n}-2)}(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^2}\left(\underbrace{\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\cdots\int_0^{\frac{1}{2}}}_{\check{n}-1\ \ integrals} t^{\text{ß}-1} f(\ell_1+t\Im(\ell_2,\ell_1))\,(dt)^{\check{n}-1}\right) \\
& \vdots \\
& +(-1)^{\check{n}-2}\frac{g'(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{\check{n}-1}}\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^{\text{ß}-1} f(\ell_1+t\Im(\ell_2,\ell_1))\,(dt)^2\right) \\
& +(-1)^{\check{n}-1}\frac{g(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{\check{n}}}\left(\int_0^{\frac{1}{2}} t^{\text{ß}-1} f(\ell_1+t\Im(\ell_2,\ell_1))dt\right) \\
& +(-1)^{\check{n}}\frac{1}{(\Im(\ell_2,\ell_1))^{\check{n}}}\left(\int_0^{\frac{1}{2}} t^{\text{ß}-1} g(\ell_1+t\Im(\ell_2,\ell_1)) f(\ell_1+t\Im(\ell_2,\ell_1))\right) dt.
\end{aligned}$$

After simplification

$$\begin{aligned}
I_1 = {} & \frac{g^{(\check{n}-1)}(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{2^{\check{n}-1}\Im(\ell_2,\ell_1)}\left(\int_0^{\frac{1}{2}} t^{\text{ß}-1} f(\ell_1+t\Im(\ell_2,\ell_1))dt\right) \\
& -\frac{g^{(\check{n}-2)}(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{2^{\check{n}-2}(\Im(\ell_2,\ell_1))^2}\left(\int_0^{\frac{1}{2}} t^{\text{ß}-1} f(\ell_1+t\Im(\ell_2,\ell_1))dt\right) \\
& \vdots \\
& +(-1)^{\check{n}-2}\frac{g'(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{2(\Im(\ell_2,\ell_1))^{\check{n}-1}}\left(\int_0^{\frac{1}{2}} t^{\text{ß}-1} f(\ell_1+t\Im(\ell_2,\ell_1))dt\right) \\
& +(-1)^{\check{n}-1}\frac{g(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{\check{n}}}\left(\int_0^{\frac{1}{2}} t^{\text{ß}-1} f(\ell_1+t\Im(\ell_2,\ell_1))dt\right) \\
& +(-1)^{\check{n}}\frac{1}{(\Im(\ell_2,\ell_1))^{\check{n}}}\left(\int_0^{\frac{1}{2}} t^{\text{ß}-1} g(\ell_1+t\Im(\ell_2,\ell_1)) f(\ell_1+t\Im(\ell_2,\ell_1))dt\right).
\end{aligned}$$

On substituting $x=\ell_1+t\Im(\ell_2,\ell_1)$, we get

$$\begin{aligned}
I_1 = {} & \frac{g^{(\check{n}-1)}(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{2^{\check{n}-1}\Im(\ell_2,\ell_1)^{\text{ß}+1}} \int_{\ell_1}^{\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1)} (x-\ell_1)^{\text{ß}-1} f(x)dx \\
& - \frac{g^{(\check{n}-2)}(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{2^{\check{n}-2}(\Im(\ell_2,\ell_1))^{\text{ß}+2}} \int_{\ell_1}^{\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1)} (x-\ell_1)^{\text{ß}-1} f(x)dx \\
& \vdots \\
& + (-1)^{\check{n}-2} \frac{g'(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{2(\Im(\ell_2,\ell_1))^{\text{ß}+\check{n}-1}} \int_{\ell_1}^{\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1)} (x-\ell_1)^{\text{ß}-1} f(x)dx \\
& + (-1)^{\check{n}-1} \frac{g(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{\text{ß}+\check{n}}} \int_{\ell_1}^{\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1)} (x-\ell_1)^{\text{ß}-1} f(x)dx \\
& + (-1)^{\check{n}} \frac{1}{(\Im(\ell_2,\ell_1))^{\text{ß}+\check{n}}} \int_{\ell_1}^{\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1)} (x-\ell_1)^{\text{ß}-1} g(x)f(x)dx.
\end{aligned}$$

Using the definition of Riemann-Liouville fractional integrals, we have

$$\begin{aligned}
I_1 = {} & \frac{g^{(\check{n}-1)}(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))\Gamma(\text{ß})}{2^{\check{n}-1}\Im(\ell_2,\ell_1)^{\text{ß}+1}} J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-} f(\ell_1) \\
& - \frac{g^{(\check{n}-2)}(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))\Gamma(\text{ß})}{2^{\check{n}-2}(\Im(\ell_2,\ell_1))^{\text{ß}+2}} J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-} f(\ell_1) \\
& \vdots \\
& + (-1)^{\check{n}-2} \frac{g'(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))\Gamma(\text{ß})}{2(\Im(\ell_2,\ell_1))^{\text{ß}+\check{n}-1}} J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-} f(\ell_1) \\
& + (-1)^{\check{n}-1} \frac{g(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))\Gamma(\text{ß})}{(\Im(\ell_2,\ell_1))^{\text{ß}+\check{n}}} J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-} f(\ell_1) \\
& + (-1)^{\check{n}} \frac{\Gamma(\text{ß})}{(\Im(\ell_2,\ell_1))^{\text{ß}+\check{n}}} J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-} (gf)(\ell_1).
\end{aligned}$$

After summing the above expressions, we get

$$\begin{aligned}
I_1 = {} & \sum_{\kappa=0}^{\check{n}-1} (-1)^{\check{n}-\kappa-1} \frac{g^{(\kappa)}(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{2^{\kappa}\Im(\ell_2,\ell_1)^{\text{ß}+\check{n}-\kappa}} \Gamma(\text{ß}) J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-} f(\ell_1) \\
& + (-1)^{\check{n}} \frac{1}{(\Im(\ell_2,\ell_1))^{\text{ß}+\check{n}}} \Gamma(\text{ß}) J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-} (gf)(\ell_1). \qquad (2.2)
\end{aligned}$$

By a similar argument for the second integral, we have

$$I_2 = (-1)^{\check{n}+1} \sum_{\kappa=0}^{\check{n}-1} \frac{g^{(\kappa)}(\ell_1 + \frac{1}{2}\Im(\ell_2,\ell_1))}{2^{\kappa}\Im(\ell_2,\ell_1)^{\text{ß}+\check{n}-\kappa}} \Gamma(\text{ß}) J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+} f(\ell_1 + \Im(\ell_2,\ell_1))$$
$$+ (-1)^{\check{n}} \frac{1}{(\Im(\ell_2,\ell_1))^{\text{ß}+\check{n}}} \Gamma(\text{ß}) J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+} (gf)(\ell_1 + \Im(\ell_2,\ell_1)). \tag{2.3}$$

Upon adding (2.1)and (2.2), we get the required result. Thus, the lemma is proved. □

Remark 1. If we set $\check{n} = 1$, we get Lemma 1 of [19].

$$\begin{aligned} &\frac{g(\ell_1 + \frac{1}{2}\Im(\ell_2,\ell_1))}{\Im(\ell_2,\ell_1)^{\text{ß}+1}} \\ &\quad \times \left[J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-} f(\ell_1) + J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+} f(\ell_1 + \Im(\ell_2,\ell_1)) \right] \\ &\quad - \frac{1}{(\Im(\ell_2,\ell_1))^{\text{ß}+1}} \\ &\quad \times \left[J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-} (gf)(\ell_1) + J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+} (gf)(\ell_1 + \Im(\ell_2,\ell_1)) \right] \\ &= \frac{1}{\Gamma(\text{ß})} \int_0^1 w(t) g'(\ell_1 + t\Im(\ell_2,\ell_1)) dt, \end{aligned}$$

where

$$w(t) = \begin{cases} \int_0^t u^{\text{ß}-1} f(\ell_1 + u\Im(\ell_2,\ell_1)) du, & t \in \left[0, \frac{1}{2}\right). \\ \int_1^t (1-u)^{\text{ß}-1} f(\ell_1 + u\Im(\ell_2,\ell_1)) du, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Lemma 3. *If $g : [\ell_1, \ell_1 + \Im(\ell_2,\ell_1)] \to \mathbb{R}$ is an integrable function which is also symmetric about $\ell_1 + \frac{1}{2}\Im(\ell_2,\ell_1)$ with $\ell_1 < \ell_1 + \Im(\ell_2,\ell_1)$,then the following identity holds*

$$\begin{aligned} J^{\beta}_{\ell_1+} g(\ell_1 + \Im(\ell_2,\ell_1)) &= J^{\beta}_{(\ell_1+\Im(\ell_2,\ell_1))-} g(\ell_1) \\ &= \frac{1}{2} \left[J^{\beta}_{\ell_1+} g(\ell_1 + \Im(\ell_2,\ell_1)) + J^{\beta}_{(\ell_1+\Im(\ell_2,\ell_1))-} g(\ell_1) \right], \end{aligned} \tag{2.4}$$

where $\beta > 0$

Proof. Since g is symmetric about $\ell_1 + \frac{1}{2}\Im(\ell_2,\ell_1)$, we have $g(2\ell_1 + \Im(\ell_2,\ell_1) - x) = g(x)$, for all $x \in [\ell_1, \ell_1 + \Im(\ell_2,\ell_1)]$. Taking into account $2\ell_1 + \Im(\ell_2,\ell_1) - t = x$, we can write

$$\begin{aligned} J^{\text{ß}}_{\ell_1+} g(\ell_1 + \Im(\ell_2,\ell_1)) &= \frac{1}{\Gamma(\text{ß})} \int_{\ell_1}^{\ell_1+\Im(\ell_2,\ell_1)} [(\ell_1 + \Im(\ell_2,\ell_1) - t]^{\text{ß}-1} g(t) dt \\ &= \frac{1}{\Gamma(\text{ß})} \int_{\ell_1}^{\ell_1+\Im(\ell_2,\ell_1)} (x - \ell_1)^{\text{ß}-1} g(2\ell_1 + \Im(\ell_2,\ell_1) - x) dx \end{aligned}$$

$$= \frac{1}{\Gamma(ß)} \int_{\ell_1}^{\ell_1+\Im(\ell_2,\ell_1)} (x-\ell_1)^{ß-1} g(x)dx = J^{ß}_{(\ell_1+\Im(\ell_2,\ell_1))-} g(\ell_1).$$

The proof is completed. □

Lemma 4. *Let $V \subseteq \mathbb{R}$ be an open invex set and $\Im : V \times V \to \mathbb{R}$ be a mapping. Suppose that $g : V \to \mathbb{R}$ is a differentiable mapping such that $g^{(\check{n})} \in L[\ell_1, \ell_1 + \Im(\ell_2,\ell_1)]$ where $\Im(\ell_2,\ell_1) > 0$. If $w : [\ell_1, \ell_1+\Im(\ell_2,\ell_1)] \to [0,\infty)$ is an integrable mapping, then we have the following equality*:

$$\sum_{\kappa=0}^{\check{n}-1} \left[\frac{(-1)^{\check{n}+1} g^{(\kappa)}(\ell_1) + (-1)^{\check{n}-\kappa-1} g^{(\kappa)}(\ell_1+\Im(\ell_2,\ell_1))}{2(\Im(\ell_2,\ell_1))^{\beta+\check{n}-\kappa}} \right]$$
$$\times \left[J^{\beta}_{(\ell_1+\Im(\ell_2,\ell_1))-} f(\ell_1) + J^{\beta}_{\ell_1+} f(\ell_1+\Im(\ell_2,\ell_1)) \right]$$
$$+ (-1)^{\check{n}} \left[\frac{J^{\beta}_{(\ell_1+\Im(\ell_2,\ell_1))-}(gf)(\ell_1) + J^{\beta}_{\ell_1+}(gf)(\ell_1+\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{\beta+\check{n}}} \right]$$
$$= \frac{1}{\Gamma(\beta)} \int_0^1 w(t) g^{(\check{n})}(\ell_1 + t\Im(\ell_2,\ell_1))dt, \tag{2.5}$$

for $\forall\, \ell_1,\ell_2 \in V$, where

$$w(t) = \underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n}\ integrals} u^{\beta-1} f(\ell_1 + u\Im(\ell_2,\ell_1))(du)^{\check{n}}$$
$$+ \underbrace{\int_1^t \int_1^t .. \int_1^t}_{\check{n}\ integrals} (1-u)^{\beta-1} f(\ell_1+u\Im(\ell_2,\ell_1))(du)^{\check{n}}, \qquad t \in [0,1].$$

Proof. We will start with

$$\int_0^1 w(t) g^{(\check{n})}(\ell_1 + t\Im(\ell_2,\ell_1))dt$$
$$= \int_0^1 \left(\underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n}-\ integrals} u^{ß-1} f(\ell_1+u\Im(\ell_2,\ell_1))(du)^{\check{n}} \right)$$
$$\times g^{(\check{n})}(\ell_1 + t\Im(\ell_2,\ell_1))dt$$
$$+ \int_0^1 \left(\underbrace{\int_1^t \int_1^t .. \int_1^t}_{\check{n}\ -integrals} (1-u)^{ß-1} f(\ell_1+u\Im(\ell_2,\ell_1))(du)^{\check{n}} \right)$$

$$\times g^{(\check{n})}(\ell_1 + t\Im(\ell_2,\ell_1))dt = I_1 + I_2.$$

By evaluating the first integral, we have

$$\begin{aligned}
I_1 &= \int_0^1 \left(\underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{\check{n}\ integrals} u^{ß-1} f(\ell_1 + u\Im(\ell_2,\ell_1))\,(du)^{\check{n}} \right) \\
&\quad \times g^{(\check{n})}(\ell_1 + t\Im(\ell_2,\ell_1))dt \\
&= \frac{1}{\Im(\ell_2,\ell_1)} \left(\underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{\check{n}-1\ integrals} u^{ß-1} f(\ell_1 + u\Im(\ell_2,\ell_1))\,(du)^{\check{n}} \right) \\
&\quad \times g^{(\check{n}-1)}(\ell_1 + t\Im(\ell_2,\ell_1)\Big|_0^1 \\
&\quad - \frac{1}{\Im(\ell_2,\ell_1)} \int_0^1 \left(\underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{\check{n}-1\ integrals} u^{ß-1} f(\ell_1 + u\Im(\ell_2,\ell_1))\,(du)^{\check{n}} \right) \\
&\quad \times g^{(\check{n}-1)}(\ell_1 + t\Im(\ell_2,\ell_1))dt.
\end{aligned}$$

Namely,

$$\begin{aligned}
I_1 &= \frac{g^{(\check{n}-1)}(\ell_1 + \Im(\ell_2,\ell_1))}{\Im(\ell_2,\ell_1)} \left(\underbrace{\int_0^1 \int_0^1 \cdots \int_0^1}_{\check{n}\ integrals} t^{ß-1} f(\ell_1 + t\Im(\ell_2,\ell_1))\,(dt)^{\check{n}} \right) \\
&\quad - \frac{1}{\Im(\ell_2,\ell_1)} \int_0^1 \left(\underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{\check{n}-1\ integrals} u^{ß-1} f(\ell_1 + u\Im(\ell_2,\ell_1))\,(du)^{\check{n}-1} \right) \\
&\quad \times g^{(\check{n}-1)}(\ell_1 + t\Im(\ell_2,\ell_1))dt.
\end{aligned}$$

On generalizing the result, we have

$$I_1 = \frac{g^{(\check{n}-1)}(\ell_1 + (\ell_2,\ell_1))}{\Im(\ell_2,\ell_1)} \left(\underbrace{\int_0^1 \int_0^1 \cdots \int_0^1}_{\check{n}\ integrals} t^{ß-1} f(\ell_1 + t\Im(\ell_2,\ell_1))\,(dt)^{\check{n}} \right)$$

$$
\begin{aligned}
&-\frac{g^{(\check{n}-2)}(\ell_1+\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^2}\left(\underbrace{\int_0^1\int_0^1\ldots\int_0^1}_{\check{n}-1\ \ integrals} t^{\text{ß}-1}f(\ell_1+t\Im(\ell_2,\ell_1))\,(dt)^{\check{n}-1}\right)\\
&+\frac{g^{(\check{n}-3)}(\ell_1+\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^3}\left(\underbrace{\int_0^1\int_0^1\ldots\int_0^1}_{\check{n}-2\ \ integrals} t^{\text{ß}-1}f(\ell_1+t\Im(\ell_2,\ell_1))\,(dt)^{\check{n}-2}\right)\\
&\vdots\\
&+(-1)^{\check{n}-2}\frac{g'(\ell_1+\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{\check{n}-1}}\left(\int_0^1\int_0^1 t^{\text{ß}-1}f(\ell_1+t\Im(\ell_2,\ell_1))\,(dt)^2\right)\\
&+(-1)^{\check{n}-1}\frac{g(\ell_1+\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{\check{n}}}\left(\int_0^1 t^{\text{ß}-1}f(\ell_1+t\Im(\ell_2,\ell_1))dt\right)\\
&+(-1)^{\check{n}}\frac{1}{(\Im(\ell_2,\ell_1))^{\check{n}}}\left(\int_0^1 t^{\text{ß}-1}g(\ell_1+t\Im(\ell_2,\ell_1))f(\ell_1+t\Im(\ell_2,\ell_1))\right)dt.
\end{aligned}
$$

After simplifications, we obtain

$$
\begin{aligned}
I_1=&\frac{g^{(\check{n}-1)}(\ell_1+\Im(\ell_2,\ell_1))}{\Im(\ell_2,\ell_1)}\left(\int_0^1 t^{\text{ß}-1}f(\ell_1+t\Im(\ell_2,\ell_1))dt\right)\\
&-\frac{g^{(\check{n}-2)}(\ell_1+\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^2}\left(\int_0^1 t^{\text{ß}-1}f(\ell_1+t\Im(\ell_2,\ell_1))dt\right)\\
&+\frac{g^{(\check{n}-3)}(\ell_1+\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^3}\left(\int_0^1 t^{\text{ß}-1}f(\ell_1+t\Im(\ell_2,\ell_1))dt\right)\\
&\vdots\\
&+(-1)^{\check{n}-2}\frac{g'(\ell_1+\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{\check{n}-1}}\left(\int_0^1 t^{\text{ß}-1}f(\ell_1+t\Im(\ell_2,\ell_1))dt\right)\\
&+(-1)^{\check{n}-1}\frac{g(\ell_1+\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{\check{n}}}\left(\int_0^1 t^{\text{ß}-1}f(\ell_1+t\Im(\ell_2,\ell_1))dt\right)\\
&+(-1)^{\check{n}}\frac{1}{(\Im(\ell_2,\ell_1))^{\check{n}}}\left(\int_0^1 t^{\text{ß}-1}g(\ell_1+t\Im(\ell_2,\ell_1))f(\ell_1+t\Im(\ell_2,\ell_1))dt\right).
\end{aligned}
$$

After substituting $x=\ell_1+t\Im(\ell_2,\ell_1)$, we have

$$
I_1=\frac{g^{(\check{n}-1)}(\ell_1+\Im(\ell_2,\ell_1))}{\Im(\ell_2,\ell_1)^{\text{ß}+1}}\int_{\ell_1}^{\ell_1+\Im(\ell_2,\ell_1)}(x-\ell_1)^{\text{ß}-1}f(x)dx
$$

$$
\begin{aligned}
&-\frac{g^{(\check{n}-2)}(\ell_1+\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{ß+2}}\int_{\ell_1}^{\ell_1+\Im(\ell_2,\ell_1)}(x-\ell_1)^{ß-1}f(x)dx\\
&+\frac{g^{(\check{n}-3)}(\ell_1+\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{ß+3}}\int_{\ell_1}^{\ell_1+\Im(\ell_2,\ell_1)}(x-\ell_1)^{ß-1}f(x)dx\\
&\vdots\\
&+(-1)^{\check{n}-2}\frac{g'(\ell_1+\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{ß+\check{n}-1}}\int_{\ell_1}^{\ell_1+\Im(\ell_2,\ell_1)}(x-\ell_1)^{ß-1}f(x)dx\\
&+(-1)^{\check{n}-1}\frac{g(\ell_1+\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{ß+\check{n}}}\int_{\ell_1}^{\ell_1+\Im(\ell_2,\ell_1)}(x-\ell_1)^{ß-1}f(x)dx\\
&+(-1)^{\check{n}}\frac{1}{(\Im(\ell_2,\ell_1))^{ß+\check{n}}}\int_{\ell_1}^{\ell_1+\Im(\ell_2,\ell_1)}(x-\ell_1)^{ß-1}g(x)f(x)dx.
\end{aligned}
$$

Using the definition of Riemann-Liouville fractional integrals, we can easily see that

$$
\begin{aligned}
I_1=&\frac{g^{(\check{n}-1)}(\ell_1+\Im(\ell_2,\ell_1))\Gamma(ß)}{\Im(\ell_2,\ell_1)^{ß+1}}J^{ß}_{(\ell_1+\Im(\ell_2,\ell_1))-}f(\ell_1)\\
&-\frac{g^{(\check{n}-2)}(\ell_1+\Im(\ell_2,\ell_1))\Gamma(ß)}{(\Im(\ell_2,\ell_1))^{ß+2}}J^{ß}_{(\ell_1+\Im(\ell_2,\ell_1))-}f(\ell_1)\\
&+\frac{g^{(\check{n}-3)}(\ell_1+\Im(\ell_2,\ell_1))\Gamma(ß)}{(\Im(\ell_2,\ell_1))^{ß+3}}J^{ß}_{(\ell_1+\Im(\ell_2,\ell_1))-}f(\ell_1)\\
&\vdots\\
&+(-1)^{\check{n}-2}\frac{g'(\ell_1+\Im(\ell_2,\ell_1))\Gamma(ß)}{(\Im(\ell_2,\ell_1))^{ß+\check{n}-1}}J^{ß}_{(\ell_1+\Im(\ell_2,\ell_1))-}f(\ell_1)\\
&+(-1)^{\check{n}-1}\frac{g(\ell_1+\Im(\ell_2,\ell_1))\Gamma(ß)}{(\Im(\ell_2,\ell_1))^{ß+\check{n}}}J^{ß}_{(\ell_1+\Im(\ell_2,\ell_1))-}f(\ell_1)\\
&+(-1)^{\check{n}}\frac{\Gamma(ß)}{(\Im(\ell_2,\ell_1))^{ß+\check{n}}}J^{ß}_{(\ell_1+\Im(\ell_2,\ell_1))-}(gf)(\ell_1).
\end{aligned}
$$

After summing the above expressions, we get

$$
\begin{aligned}
I_1=&\sum_{\kappa=0}^{\check{n}-1}(-1)^{\check{n}-\kappa-1}\frac{g^{(\kappa)}(\ell_1+\Im(\ell_2,\ell_1))}{\Im(\ell_2,\ell_1)^{ß+\check{n}-\kappa}}\Gamma(ß)J^{ß}_{(\ell_1+\Im(\ell_2,\ell_1))-}f(\ell_1)\\
&+(-1)^{\check{n}}\frac{1}{(\Im(\ell_2,\ell_1))^{ß+\check{n}}}\Gamma(ß)J^{ß}_{(\ell_1+\Im(\ell_2,\ell_1))-}(gf)(\ell_1). \qquad (2.6)
\end{aligned}
$$

By proceeding a similar method for the second integral, we deduce

$$I_2 = (-1)^{\check{n}+1} \sum_{\kappa=0}^{\check{n}-1} \frac{g^{(\kappa)}(\ell_1)}{\Im(\ell_2,\ell_1)^{\text{ß}+\check{n}-\kappa}} \Gamma(\text{ß}) J_{\ell_1+}^{\text{ß}} f(\ell_1 + \Im(\ell_2,\ell_1)) + (-1)^{\check{n}} \frac{1}{(\Im(\ell_2,\ell_1))^{\text{ß}+\check{n}}} \Gamma(\text{ß}) J_{\ell_1+}^{\text{ß}} (gf)(\ell_1 + \Im(\ell_2,\ell_1)). \tag{2.7}$$

Upon adding (2.6) and (2.7) and using lemma 3, we get the required result. The lemma is proved. □

Remark 2. If we set $\check{n} = 1$, the identity reduces to the Lemma 3 of [19].

$$\left[\frac{g(\ell_1) + g(\ell_1 + \Im(\ell_2,\ell_1))}{2(\Im(\ell_2,\ell_1))^{\text{ß}+1}}\right] \times \left[J_{(\ell_1+\Im(\ell_2,\ell_1))-}^{\text{ß}} f(\ell_1) + J_{\ell_1+}^{\text{ß}} f(\ell_1 + \Im(\ell_2,\ell_1))\right] - \left[\frac{J_{(\ell_1+\Im(\ell_2,\ell_1))-}^{\text{ß}} (gf)(\ell_1) + J_{\ell_1+}^{\text{ß}} (gf)(\ell_1 + \Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{\text{ß}+1}}\right] = \frac{1}{\Gamma(\text{ß})} \int_0^1 w(t) g'(\ell_1 + t\Im(\ell_2,\ell_1)) dt,$$

where

$$w(t) = \int_0^t u^{\text{ß}-1} f(\ell_1 + u\Im(\ell_2,\ell_1)) du + \int_1^t (1-u)^{\text{ß}-1} f(\ell_1 + u\Im(\ell_2,\ell_1)) du, \qquad t \in [0,1].$$

Theorem 2. *Let $V \subseteq \mathbb{R}$ be an open invex set and $\Im : V \times V \to \mathbb{R}^\beta$ be a mapping. Suppose that $g : V \to \mathbb{R}$ is a differentiable mapping such that $g^{(\check{n})} \in L[\ell_1, \ell_1 + \Im(\ell_2,\ell_1)]$ where $\Im(\ell_2,\ell_1) > 0$. If $f : [\ell_1, \ell_1 + \Im(\ell_2,\ell_1)] \to [0,\infty)$ is an integrable mapping and it is also symmetric w.r.t. $\ell_1 + \frac{1}{2}\Im(\ell_2,\ell_1)$ and $\left|g^{(\check{n})}\right|$ is preinvex function on V, then the following inequality holds:*

$$\left| \sum_{\kappa=0}^{\check{n}-1} \frac{g^{(\kappa)}(\ell_1 + \frac{1}{2}\Im(\ell_2,\ell_1))}{2^\kappa \Im(\ell_2,\ell_1)^{\beta+\check{n}-\kappa}} \times \left[(-1)^{\check{n}-\kappa-1} J_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-}^{\beta} f(\ell_1) + (-1)^{\check{n}+1} J_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+}^{\beta} f(\ell_1 + \Im(\ell_2,\ell_1))\right] + (-1)^{\check{n}} \frac{1}{(\Im(\ell_2,\ell_1))^{\beta+\check{n}}} \times \left[J_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-}^{\beta} (gf)(\ell_1) + J_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+}^{\beta} (gf)(\ell_1 + \Im(\ell_2,\ell_1))\right] \right|$$

$$\leq \frac{\|f\|_\infty}{\Gamma(\beta+\check{n}+1)}\frac{1}{2^{\beta+\check{n}}}\left[\left|g^{(\check{n})}(\ell_1)\right|+\left|g^{(\check{n})}(\ell_2)\right|\right], \quad \textit{when } \check{n} \textit{ is odd integer}$$
$$\leq \frac{\|f\|_\infty}{\Gamma(\beta+\check{n}+2)}\frac{1}{2^{\beta+\check{n}}}\left[\left|g^{(\check{n})}(\ell_1)\right|-\left|g^{(\check{n})}(\ell_2)\right|\right], \quad \textit{when } \check{n} \textit{ is even integer}$$

for $\forall\, \ell_1, \ell_2 \in V$.

Proof. Applying modulus on both sides of (2.1), we can write

$$\begin{aligned}
&\Bigg|\sum_{\kappa=0}^{\check{n}-1}\frac{g^{(\kappa)}(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{2^{\kappa}\Im(\ell_2,\ell_1)^{\text{ß}+\check{n}-\kappa}} \\
&\quad\times\left[(-1)^{\check{n}-\kappa-1}J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-}f(\ell_1)+(-1)^{\check{n}+1}J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+}f(\ell_1+\Im(\ell_2,\ell_1))\right] \\
&\quad+(-1)^{\check{n}}\frac{1}{(\Im(\ell_2,\ell_1))^{\text{ß}+\check{n}}}\Big[J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-}(gf)(\ell_1) \\
&\qquad+J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+}(gf)(\ell_1+\Im(\ell_2,\ell_1))\Big]\Bigg| \\
&=\frac{1}{\Gamma(\text{ß})}\left|\int_0^{1/2}w(t)\,g^{(\check{n})}(\ell_1+t\Im(\ell_2,\ell_1))dt+\frac{1}{\Gamma(\text{ß})}\int_{1/2}^{1}w(t)\,g^{(\check{n})}(\ell_1+t\Im(\ell_2,\ell_1))dt\right|.
\end{aligned}$$

From preinvexity of $\left|g^{(\check{n})}\right|$ on V and using the fact $\|f\|_\infty = sup_{x\in[\ell_1,\ell_2]}|f(x)|$, we have

$$\begin{aligned}
&\Bigg|\sum_{\kappa=0}^{\check{n}-1}\frac{g^{(\kappa)}(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{2^{\kappa}\Im(\ell_2,\ell_1)^{\text{ß}+\check{n}-\kappa}} \\
&\quad\times\left[(-1)^{\check{n}-\kappa-1}J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-}f(\ell_1)+(-1)^{\check{n}+1}J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+}f(\ell_1+\Im(\ell_2,\ell_1))\right] \\
&\quad+(-1)^{\check{n}}\frac{1}{(\Im(\ell_2,\ell_1))^{\text{ß}+\check{n}}}\Big[J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-}(gf)(\ell_1) \\
&\qquad+J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+}(gf)(\ell_1+\Im(\ell_2,\ell_1))\Big]\Bigg| \\
&\leq\frac{\|f\|_\infty}{\Gamma(\text{ß})}\int_0^{1/2}\left(\underbrace{\int_0^t\int_0^t\cdots\int_0^t}_{\check{n}\ integrals}u^{\text{ß}-1}(du)^{\check{n}}\right)\left[(1-t)\left|g^{(\check{n})}(\ell_1)\right|+t\left|g^{(\check{n})}(\ell_2)\right|\right]dt \\
&\quad+\frac{\|f\|_\infty}{\Gamma(\text{ß})}\int_{1/2}^{1}\left(\underbrace{\int_1^t\int_1^t\cdots\int_1^t}_{\check{n}\ integrals}(1-u)^{\text{ß}-1}(du)^{\check{n}}\right)\left[(1-t)\left|g^{(\check{n})}(\ell_1)\right|+t\left|g^{(\check{n})}(\ell_2)\right|\right]dt \\
&=I_1+I_2 \qquad (2.8)
\end{aligned}$$

From the first term of (2.8), we get

$$I_1 = \frac{\|f\|_\infty}{\Gamma(ß)} \int_0^{1/2} \left(\underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n}-integrals} u^{ß-1} (du)^{\check{n}} \right) \left[(1-t) \left| g^{(\check{n})}(\ell_1) \right| + t \left| g^{(\check{n})}(\ell_2) \right| \right] dt$$
$$= \frac{\|f\|_\infty}{\Gamma(ß+\check{n}-1)} \int_0^{1/2} u^{ß+\check{n}-2} \int_u^{1/2} \left[(1-t) \left| g^{(\check{n})}(\ell_1) \right| + t \left| g^{(\check{n})}(\ell_2) \right| \right] dt du$$
$$= \frac{\|f\|_\infty}{\Gamma(ß+\check{n}-1)} \int_0^{1/2} u^{ß+\check{n}-2} \left[\left| g^{(\check{n})}(\ell_1) \right| \left(\frac{(1-u)^2}{2} - \frac{1}{8} \right) + \left| g^{(\check{n})}(\ell_2) \right| \left(\frac{1}{8} - \frac{u^2}{2} \right) \right] du.$$

By making use the change of variable $x = \ell_1 + u\Im(\ell_2, \ell_1)$ for $u \in [0,1]$, it is obvious to see that

$$I_1 = \frac{\|f\|_\infty}{\Gamma(ß+\check{n}-1)} \frac{\left| g^{(\check{n})}(\ell_1) \right|}{\Im(\ell_2, \ell_1)}$$
$$\times \int_{\ell_1}^{\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1)} \left(\frac{1}{2} \left(1 - \frac{x-\ell_1}{\Im(\ell_2,\ell_1)} \right)^2 - \frac{1}{8} \right) \left(\frac{x-\ell_1}{\Im(\ell_2,\ell_1)} \right)^{ß+\check{n}-2} dx$$
$$+ \frac{\|f\|_\infty}{\Gamma(ß+\check{n}-1)} \frac{\left| g^{(\check{n})}(\ell_2) \right|}{\Im(\ell_2, \ell_1)}$$
$$\times \int_{\ell_1}^{\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1)} \left(\frac{1}{8} - \frac{1}{2} \left(\frac{x-\ell_1}{\Im(\ell_2,\ell_1)} \right)^2 \right) \left(\frac{x-\ell_1}{\Im(\ell_2,\ell_1)} \right)^{ß+\check{n}-2} dx. \tag{2.9}$$

From the second term of (2.8), we have

$$I_2 = \frac{\|f\|_\infty}{\Gamma(ß)} \int_{1/2}^1 \left(\underbrace{\int_1^t \int_1^t \dots \int_1^t}_{\check{n}-integrals} (1-u)^{ß-1} (du)^{\check{n}} \right) \left[(1-t) \left| g^{(\check{n})}(\ell_1) \right| + t \left| g^{(\check{n})}(\ell_2) \right| \right] dt$$
$$= \frac{(-1)^{\check{n}-1} \|f\|_\infty}{\Gamma(ß+\check{n}-1)} \int_{1/2}^1 \left(\int_1^t (1-u)^{ß+\check{n}-2} du \right) \left[(1-t) \left| g^{(\check{n})}(\ell_1) \right| + t \left| g^{(\check{n})}(\ell_2) \right| \right] dt$$
$$= \frac{(-1)^{\check{n}-1} \|f\|_\infty}{\Gamma(ß+\check{n}-1)} \|f\|_\infty \int_{1/2}^1 (1-u)^{ß+\check{n}-2} \left(\int_{1/2}^u \left[(1-t) \left| g^{(\check{n})}(\ell_1) \right| + t \left| g^{(\check{n})}(\ell_2) \right| \right] dt \right) du$$
$$= \frac{(-1)^{\check{n}-1} \|f\|_\infty}{\Gamma(ß+\check{n}-1)} \|f\|_\infty \int_{1/2}^1 (1-u)^{ß+\check{n}-2} \left[\left| g^{(\check{n})}(\ell_1) \right| \left(\frac{1}{8} - \frac{(1-u)^2}{2} \right) \right.$$
$$\left. + \left| g^{(\check{n})}(\ell_2) \right| \left(\frac{u^2}{2} - \frac{1}{8} \right) \right] du.$$

By the change of variable $x = \ell_1 + (1-u)\Im(\ell_2,\ell_1)$, we get

$$\begin{aligned}
I_2 &= \frac{(-1)^{\check{n}-1}\|f\|_\infty}{\Gamma(ß+\check{n}-1)}\frac{\left|g^{(\check{n})}(\ell_1)\right|}{\Im(\ell_2,\ell_1)} \\
&\quad\times\int_{\ell_1}^{\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1)}\left(\frac{1}{8}-\frac{1}{2}\left(\frac{x-\ell_1}{\Im(\ell_2,\ell_1)}\right)^2\right)\left(\frac{x-\ell_1}{\Im(\ell_2,\ell_1)}\right)^{ß+\check{n}-2}dx \\
&\quad+\frac{(-1)^{\check{n}-1}\|f\|_\infty}{\Gamma(ß+\check{n}-1)}\|f\|_\infty\frac{\left|g^{(\check{n})}(\ell_2)\right|}{\Im(\ell_2,\ell_1)} \\
&\quad\times\int_{\ell_1}^{\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1)}\left(\frac{1}{2}\left(1-\frac{x-\ell_1}{\Im(\ell_2,\ell_1)}\right)^2-\frac{1}{8}\right)\left(\frac{x-\ell_1}{\Im(\ell_2,\ell_1)}\right)^{ß+\check{n}-2}dx. \qquad (2.10)
\end{aligned}$$

Case (i) When n is odd integer.
Adding (2.9) and (2.10) based on (2.8), we have

$$\begin{aligned}
&\left|\sum_{\kappa=0}^{\check{n}-1}\frac{g^{(\kappa)}(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{2^\kappa\Im(\ell_2,\ell_1)^{ß+\check{n}-\kappa}}\right. \\
&\quad\times\left[(-1)^{\check{n}-\kappa-1}J^{ß}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-}f(\ell_1)+(-1)^{\check{n}+1}J^{ß}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+}f(\ell_1+\Im(\ell_2,\ell_1))\right] \\
&\quad+(-1)^{\check{n}}\frac{1}{(\Im(\ell_2,\ell_1))^{ß+\check{n}}}\left[J^{ß}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-}(gf)(\ell_1)\right. \\
&\qquad\left.\left.+J^{ß}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+}(gf)(\ell_1+\Im(\ell_2,\ell_1))\right]\right| \\
&\le\frac{\|f\|_\infty}{\Gamma(ß+\check{n}+1)}\frac{1}{2^{ß+\check{n}}}\left[\left|g^{(\check{n})}(\ell_1)\right|+\left|g^{(\check{n})}(\ell_2)\right|\right].
\end{aligned}$$

Case (ii) When n is even integer.
Adding (2.9) and (2.10) based on (2.8), we have

$$\begin{aligned}
&\left|\sum_{\kappa=0}^{\check{n}-1}\frac{g^{(\kappa)}(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{2^\kappa\Im(\ell_2,\ell_1)^{ß+\check{n}-\kappa}}\right. \\
&\quad\times\left[(-1)^{\check{n}-\kappa-1}J^{ß}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-}f(\ell_1)+(-1)^{\check{n}+1}J^{ß}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+}f(\ell_1+\Im(\ell_2,\ell_1))\right] \\
&\quad+(-1)^{\check{n}}\frac{1}{(\Im(\ell_2,\ell_1))^{ß+\check{n}}}\left[J^{ß}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-}(gf)(\ell_1)\right. \\
&\qquad\left.\left.+J^{ß}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+}(gf)(\ell_1+\Im(\ell_2,\ell_1))\right]\right| \\
&\le\frac{\|f\|_\infty}{\Gamma(ß+\check{n}+2)}\frac{1}{2^{ß+\check{n}}}\left[\left|g^{(\check{n})}(\ell_1)\right|-\left|g^{(\check{n})}(\ell_2)\right|\right].
\end{aligned}$$

The proof is completed. □

Remark 3. If we choose $\check{n}=1$, the result coincides with the Theorem 3 of [19].

$$\left|\frac{g(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))}{\Im(\ell_2,\ell_1)^{\text{ß}+1}} \times\left[J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-}f(\ell_1)+J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+}f(\ell_1+\Im(\ell_2,\ell_1))\right] +\frac{1}{(\Im(\ell_2,\ell_1))^{\text{ß}+2}}\left[J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))-}(gf)(\ell_1)+J^{\text{ß}}_{(\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1))+}(gf)(\ell_1+\Im(\ell_2,\ell_1))\right]\right|$$
$$\leq\frac{\|f\|_\infty}{\Gamma(\text{ß}+2)}\frac{1}{2^{\text{ß}+1}}\left[|g'(\ell_1)|+|g'(\ell_2)|\right].$$

Theorem 3. *Let $V\subseteq\mathbb{R}$ be an open invex set and $\Im:V\times V\to\mathbb{R}^{\beta}$ be a mapping. Suppose that $g:V\to\mathbb{R}$ is a differentiable mapping such that $g^{(\check{n})}\in L[\ell_1,\ell_1+\Im(\ell_2,\ell_1)]$ where $\Im(\ell_2,\ell_1)>0$. If $f:[\ell_1,\ell_1+\Im(\ell_2,\ell_1)]\to[0,\infty)$ is an integrable mapping and it is also symmetric w.r.t. $\ell_1+\frac{1}{2}\Im(\ell_2,\ell_1)$ and $\left|g^{(\check{n})}\right|$ is preinvex function on V, then the following inequality holds:*

$$|J|=\left|\sum_{\kappa=0}^{\check{n}-1}\left[\frac{(-1)^{\check{n}+1}g^{(\kappa)}(\ell_1)+(-1)^{\check{n}-\kappa-1}g^{(\kappa)}(\ell_1+\Im(\ell_2,\ell_1))}{2(\Im(\ell_2,\ell_1))^{\beta+\check{n}-\kappa}}\right] \times\left[J^{\beta}_{(\ell_1+\Im(\ell_2,\ell_1))-}f(\ell_1)+J^{\beta}_{\ell_1+}f(\ell_1+\Im(\ell_2,\ell_1))\right] +(-1)^{\check{n}}\left[\frac{J^{\beta}_{(\ell_1+\Im(\ell_2,\ell_1))-}(gf)(\ell_1)+J^{\beta}_{\ell_1+}(gf)(\ell_1+\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{\beta+\check{n}}}\right]\right|$$
$$\leq\frac{\|f\|_\infty}{(\beta+\check{n})\Gamma(\beta+\check{n}-1)}\left[\left|g^{(\check{n})}(\ell_1)\right|+\left|g^{(\check{n})}(\ell_2)\right|\right]\quad ;\ \text{when } \check{n} \text{ is odd integer}$$
$$\leq\frac{\|f\|_\infty}{(\beta+\check{n})(\beta+\check{n}+1)\Gamma(\beta+\check{n}-1)}\left[\left|g^{(\check{n})}(\ell_1)\right|-\left|g^{(\check{n})}(\ell_2)\right|\right];\quad \text{when } \check{n} \text{ is even integer}$$

for $\forall\,\ell_1,\ell_2\in V$ *.*

Proof. Applying modulus on both sides of (2.5), one can write

$$\left|\sum_{\kappa=0}^{\check{n}-1}\left[\frac{(-1)^{\check{n}+1}g^{(\kappa)}(\ell_1)+(-1)^{\check{n}-\kappa-1}g^{(\kappa)}(\ell_1+\Im(\ell_2,\ell_1))}{2(\Im(\ell_2,\ell_1))^{\text{ß}+\check{n}-\kappa}}\right] \times\left[J^{\text{ß}}_{(\ell_1+\Im(\ell_2,\ell_1))-}f(\ell_1)+J^{\text{ß}}_{\ell_1+}f(\ell_1+\Im(\ell_2,\ell_1))\right] +(-1)^{\check{n}}\left[\frac{J^{\text{ß}}_{(\ell_1+\Im(\ell_2,\ell_1))-}(gf)(\ell_1)+J^{\text{ß}}_{\ell_1+}(gf)(\ell_1+\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{\text{ß}+\check{n}}}\right]\right|$$

$$= \left| \frac{1}{\Gamma(\text{ß})} \int_0^1 \left(\underbrace{\int_0^t \int_0^t \ldots \int_0^t}_{\check{n} \ integrals} u^{\text{ß}-1} f(\ell_1 + u\Im(\ell_2,\ell_1))(du)^{\check{n}} \right.\right.$$

$$\left.\left. + \underbrace{\int_1^t \int_1^t .. \int_1^t}_{\check{n} \ integrals} (1-u)^{\text{ß}-1} f(\ell_1 + u\Im(\ell_2,\ell_1))(du)^{\check{n}} \right) g^{(\check{n})}(\ell_1 + t\Im(\ell_2,\ell_1)) dt \right|.$$

From preinvexity of $\left|g^{(\check{n})}\right|$ on V, we have

$$|J| = \left| \sum_{\kappa=0}^{\check{n}-1} \left[\frac{(-1)^{\check{n}+1} g^{(\kappa)}(\ell_1) + (-1)^{\check{n}-\kappa-1} g^{(\kappa)}(\ell_1 + \Im(\ell_2,\ell_1))}{2(\Im(\ell_2,\ell_1))^{\text{ß}+\check{n}-\kappa}} \right] \right.$$
$$\times \left[J^{\text{ß}}_{(\ell_1+\Im(\ell_2,\ell_1))-} f(\ell_1) + J^{\text{ß}}_{\ell_1+} f(\ell_1 + \Im(\ell_2,\ell_1)) \right]$$
$$\left. + (-1)^{\check{n}} \left[\frac{J^{\text{ß}}_{(\ell_1+\Im(\ell_2,\ell_1))-}(gf)(\ell_1) + J^{\text{ß}}_{\ell_1+}(gf)(\ell_1 + \Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{\text{ß}+\check{n}}} \right] \right|$$

$$= \frac{1}{\Gamma(\text{ß})} \int_0^1 \left(\underbrace{\int_0^t \int_0^t \ldots \int_0^t}_{\check{n} \ integrals} u^{\text{ß}-1} |f(\ell_1 + u\Im(\ell_2,\ell_1))| (du)^{\check{n}} \right.$$
$$\left. + \underbrace{\int_1^t \int_1^t .. \int_1^t}_{\check{n} \ integrals} (1-u)^{\text{ß}-1} |f(\ell_1 + u\Im(\ell_2,\ell_1))| (du)^{\check{n}} \right)$$
$$\left((1-t) \left| g^{(\check{n})}(\ell_1) \right| + t \left| g^{(\check{n})}(\ell_2) \right| \right) dt$$

After simplification and letting $\|f\|_\infty = \sup_{x\in[\ell_1,\ell_2]} |f(x)|$, we have

$$|J| \le \frac{\|f\|_\infty}{\Gamma(\text{ß}+\check{n}-1)} \int_0^1 u^{\text{ß}+\check{n}-2} \left(\int_0^u \left((1-t)\left|g^{(\check{n})}(\ell_1)\right| + t\left|g^{(\check{n})}(\ell_2)\right| \right) dt \right) du$$
$$+ \frac{(-1)^{\check{n}-1}\|f\|_\infty}{\Gamma(\text{ß}+\check{n}-1)} \int_0^1 (1-u)^{\text{ß}+\check{n}-2} \left(\int_u^1 \left((1-t)\left|g^{(\check{n})}(\ell_1)\right| + t\left|g^{(\check{n})}(\ell_2)\right| \right) dt \right) du$$

Case (i) n is odd integer.
Adding (2.9) and (2.10) based on (2.8), we have

$$|J| \le \frac{\|f\|_\infty}{(\text{ß}+\check{n})\Gamma(\text{ß}+\check{n}-1)} \left[\left|g^{(\check{n})}(\ell_1)\right| + \left|g^{(\check{n})}(\ell_2)\right| \right].$$

Case (ii) n is even integer.

Adding (2.9) and (2.10) based on (2.8), we have

$$|J| \leq \frac{\|f\|_\infty}{(ß+\check{n})(ß+\check{n}+1)\Gamma(ß+\check{n}-1)}\left[\left|g^{(\check{n})}(\ell_1)\right|-\left|g^{(\check{n})}(\ell_2)\right|\right].$$

The proof is completed. □

Corollary 1. *If we set $\check{n}=1$, we have the following result*

$$\left|\left[\frac{g(\ell_1)+g(\ell_1+\Im(\ell_2,\ell_1))}{2(\Im(\ell_2,\ell_1))^{\beta+1}}\right]\left[J^{\beta}_{(\ell_1+\Im(\ell_2,\ell_1))-}f(\ell_1)+J^{\beta}_{\ell_1+}f(\ell_1+\Im(\ell_2,\ell_1))\right]\right.$$
$$\left.-\left[\frac{J^{\beta}_{(\ell_1+\Im(\ell_2,\ell_1))-}(gf)(\ell_1)+J^{\beta}_{\ell_1+}(gf)(\ell_1+\Im(\ell_2,\ell_1))}{(\Im(\ell_2,\ell_1))^{\beta+1}}\right]\right|$$
$$\leq \frac{\|f\|_\infty}{(\beta+1)\Gamma(\beta)}\left[\left|g'(\ell_1)\right|+\left|g'(\ell_2)\right|\right].$$

3. Conclusion

We have obtained new lower and upper bounds of Hermite-Hadamard-Fejér type inequalities for $\check{n}-$times differentiable preinvex functions. We have also elaborated the results with special cases. Authors expect that the results of our paper will be inspiring for interested readers to prove new integral inequalities via some certain fractional integral operators.

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