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n-FINE RINGS

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Abstract. A ring R is said to be n-fine if every nonzero element in R can be written as a sum of a nilpotent and n units in R. The class of these rings contains fine rings and n-good rings in which each element is a sum of n units. Fundamental properties of such rings are obtained. One of the main results of this paper is that the $m\times m$ matrix ring $M_m(R)$ over any arbitrary ring R is 2-fine. Furthermore, the $m\times m$ matrix ring $M_m(R)$ over a n-fine ring R is n-fine.

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1. Introduction

All rings in this paper are assumed to be associative with identity. For a ring R, our terminology and notations are mainly in agreement with [4]. For instance, $U(R)$ is the multiplicative group of units of R, $Nil(R)$ is the set of all nilpotent elements of R and $Id(R)$ is the set of all idempotents of R. If R is commutative, then $Nil(R) = N(R)$ is the nil-radical of R. We denote by $M_m(R)$ the ring of $m\times m$ matrices over R with the identity I_m.

In the last four decades, an additive theory has emerged in the study of these three interesting sets. A ring R is called n-good if every element of R is a sum of n units. Many mathematicians for instance Vámos and Ashrafi studied 2-good rings extensively (see [1, 6, 7]). In 1977, Nicholson defined a ring element $a\in R$ to be clean if it can be written in the form $e+u$ where $e\in Id(R)$ and $u\in U(R)$ [5]. If every $a\in R$ is clean, R is said to be a clean ring. The interest in the clean property of rings stems from its close connection to exchange rings, since clean is a concise property that implies exchange. Prompted by this, Xiao and Tong [9] called a ring R n-clean if every element of R is the sum of an idempotent and n units (see also [8]). The class of these rings contains clean rings and n-good rings. Recently, G. Călugăreanu and T. Y. Lam defined a property in [2] related to 2-good in the following way: a nonzero element $a\in R$ is fine if $a=t+u$ where $t\in Nil(R)$ and $u\in U(R)$. The ring R is called fine if every nonzero element of R is fine. It is shown that fine rings form a proper class of simple ring [2, Theorem 2.3].

Guided by these definitions, we introduce in this work the following definition. Given a positive integer n, we call a ring R n-fine if every nonzero element of R can be written as the sum of a nilpotent and n units in R. It is clear that fine rings are 1-fine. In section 2, some fundamental properties of n-fine rings are studied. We shall prove every n-fine ring is $(n+1)$-good. Furthermore, we prove that the class of n-fine rings is closed under factor rings and direct products. The main result of this section states that a ring R is n-fine if and only if every factor ring of R is n-fine if and only if every indecomposable factor ring of R is n-fine. Then in Section 3 we will look at matrix rings and, more generally, endomorphism rings of free modules of infinite rank. In fact, over any ring R, we give an explicit 2-fine decomposition for generic matrices of orders 2 and 3. In the main theorem, we prove that over any ring R, the matrix ring $M_m(R)$ is 2-fine for each $m \geq 2$. As a consequence, we will revisit Henriksen's result that a proper matrix ring over any ring has unit sum number at most 3. An example shows that there exists a 2-fine ring that is not fine. This shows that n-fine rings are a proper generalization of fine rings. We also show that if R is n-fine, then so is the matrix ring $M_m(R)$ for any integer $m \geq 1$. Moreover, we prove that for any ring R, the endomorphism ring of a free R-module of rank at least 2 is 2-fine.

2. BASIC PROPERTIES OF n-FINE RINGS

Definition 1. Let n be a positive integer. A nonzero element x of R is called n-fine if $x = t + u_1 + \ldots + u_n$ where t is a nilpotent element of R and $u_1, \ldots, u_n$ are units in R. A ring R is called n-fine if every nonzero element of R is n-fine.

Proposition 1. *Let R be a ring. Then the following statements hold:*

(1) *If R is n-fine, then it is also m-fine for all $m \geq n$.*
(2) *If $Nil(R)$ is additively closed (in particular, if R is a commutative ring), then the sum of n-fine and m-fine elements of R is $(n+m)$-fine.*
(3) *Every n-good ring is n-fine; if R is n-fine, then R is $(n+1)$-good.*

Proof.

(1) Let r be a nonzero element of R and let $m > n$. Then, we can write $r = (r-(m-n).1) + (m-n).1$ and expressing $(r-(m-n).1)$ as a sum of a nilpotent element and n units of R gives a representation of r as a sum of a nilpotent and m units.
(2) It suffices to notice that if $Nil(R)$ is additively closed, then the sum of two nilpotent elements is a nilpotent element.
(3) It is clear that every n-good ring is n-fine. For the second statement, let r be a nonzero element of R. By hypothesis, $r-1 = t + u_1 + \ldots + u_n$ where $t \in Nil(R)$ and $u_1, \ldots, u_n \in U(R)$. Hence, $r = (1+t) + u_1 + \ldots + u_n$. Since, $(1+t) \in U(R)$. Then, r is $(n+1)$-good.

□

In the next two lemmas we consider the effect of some ring operations on our invariants

Lemma 1. *Let R be a ring, I an ideal of R and let $J(R)$ denote the Jacobson radical of R. If $a \in R$ is n-fine, then so is $\overline{a} \in R/I$. The converse also holds if $I \subseteq J(R)$ and $J(R)$ is nil.*

Proof. The first part of the statement is clear since the image of a unit (resp., a nilpotent) is again a unit (resp., a nilpotent).
Assume now that $I \subseteq J(R)$ and $J(R)$ is nil. Let $a \in R$ be such that $\overline{a}$ is n-fine in R/I. Then there are unit elements $\overline{u_i} \in R/I$, $1 \leq i \leq n$ and a nilpotent element $\bar{t} \in R/I$ such that $\overline{a} = \bar{t} + \overline{u_1} + \ldots + \overline{u_n}$. Then $u_i \in U(R)$ for all $1 \leq i \leq n$, and $h = a - (t + u_1 + \ldots + u_n) \in J(R)$. So, $u_1 + h \in U(R)$. Also, $t^m \in I$ for some integer $m \geq 2$. Hence, there is an integer $m' \geq 2$ such that $t^{mm'} = 0$ since $I \subseteq J(R)$ is nil; that is t is a nilpotent element of R. It follows that $a = t + (u_1 + h) + u_2 + \ldots + u_n$. This shows that a is n-fine. □

Lemma 2. *Let n be a positive integer. The following hold:*

(1) *A homomorphic image of a n-fine ring is n-fine.*
(2) *A direct product $\prod R_\alpha$ of rings $\{R_\alpha\}$ is n-fine if and only if so is each $\{R_\alpha\}$.*

Proof.

(1) The proof of (1) is clear.
(2) Suppose that each $\{R_\alpha\}$ is n-fine. Let $x = (x_\alpha) \in \prod R_\alpha$. For each α, write $x_\alpha = t_\alpha + u_\alpha^1 + \ldots + u_\alpha^n$, where $u_\alpha^i \in U(R_\alpha)$ for all $1 \leq i \leq n$ and $t_\alpha \in Nil(R_\alpha)$. Then, $x = t + u^1 + \ldots + u^n$, where $u_i = (u_\alpha^i) \in U(\prod R_\alpha)$ for all $1 \leq i \leq n$ and $t = (t_\alpha) \in Nil(\prod R_\alpha)$. Hence, $\prod R_\alpha$ is n-fine.
The converse immediately follows from Lemma 1.

□

We next determine when a polynomial ring or power series ring is a n-fine ring.

Proposition 2. *Let R be a nonzero commutative ring and n be a positive integer.*

(1) *$R[X]$ is never a n-fine ring.*
(2) *$R[[x]]$ is n-fine if and only if so is R.*

Proof.

(1) Note that $Nil(R[X]) = \{r_0 + r_1X + \ldots + r_nX^n |\ r_i \in \sqrt{0}\ (i = 0, \ldots, n)\}$ and $U(R[X]) = \{r_0 + r_1X + \ldots + r_nX^n |\ r_0 \in U(R), r_i \in \sqrt{0}\ (i = 1, \ldots, n)\}$. If X is n-fine, we may let

$$X = t + (u_1 + r_1X + \ldots) + (u_2 + r_2X + \ldots) + \ldots + (u_n + r_nX + \ldots),$$

where $t \in Nil(R)$, $u_1, \ldots, u_n \in U(R)$ and $r_1, \ldots, r_n \in \sqrt{0} \subseteq J(R)$ Jacobson radical of R, for each $1 \leq i \leq n$. Then $\sum_{i=1}^n r_i = 1 \in J(R)$, which is a contradiction. Thus $R[X]$ is not a n-fine ring.

(2) By Lemma 2(1), $R[[x]]$ is a n-fine ring, and this gives that $R = R[[x]]/(x)$ is a n-fine ring. Conversely, suppose that R is n-fine. Let $f = \sum_{i=0}^{\infty} r_i x^i \in R[[x]]$. Write $r_0 = t + u_1 + \ldots + u_n$, where $t \in Nil(R)$ and $u_1, \ldots, u_n$ are units in R. Then, $f = t + (u_1 + r_1 x + r_2 x^2 + \ldots) + u_2 + \ldots + u_n$, where $t \in Nil(R) \subseteq Nil(R[[x]])$ and $(u_1 + r_1 x + r_2 x^2 + \ldots) \in U(R[[x]])$, $u_i \in U(R) \subseteq U(R[[x]])$ for all $2 \leq i \leq n$. Thus, $R[[x]]$ is n-fine.

□

The next result shows that the n-fine property needs to be checked only for indecomposable rings.

Theorem 1. *Let R be a nonzero ring. Then the following are equivalent:*

(1) *R is n-fine;*
(2) *Every factor ring of R is n-fine;*
(3) *Every indecomposable factor ring of R is n-fine.*

Proof.

(1) ⇒ (2) Follows from Lemma 1.

(2) ⇒ (3) It is clear.

(3) ⇒ (1) Suppose that (3) holds and R is not n-fine. Then there is an element $a \in R$ which is not n-fine. Now let S be the set of all proper ideals I of R such $\overline{a}$ is not n-fine in R/I. Clearly $0 \in S$, and the set S is not empty. If $\{I_\alpha, \alpha \in A\}$ is a chain in S, let $I = \bigcup_{\alpha \in A} I_\alpha$. We prove that $\overline{a}$ is not n-fine in R/I. Suppose that $\overline{a}$ is n-fine in R/I. Then there are $\overline{u_1}, \ldots, \overline{u_n} \in U(R/I)$ (with inverses $\overline{v_1}, \ldots, \overline{v_n}$ respectively) and $\overline{t} \in Nil(R/I)$ such that $\overline{a} = \overline{t} + \overline{u_1} + \ldots + \overline{u_n}$. Note that $t^m \in \bigcup_{\alpha \in A} I_\alpha$ for some positive integer $m \geq 2$ and $u_i v_i - 1$, $v_i u_i - 1 \in \bigcup_{\alpha \in A} I_\alpha$, hence $t^m \in I_{\alpha_0}$, $u_i v_i \in I_{\alpha_i}$ and $v_i u_i \in I_{\alpha'_i}$ for α_0, α_i and $\alpha'_i \in A$. Now, we can use Zorn's lemma to pick an ideal I_0 of R maximal with respect to the property that a is not n-fine in R/I. Then R/I is decomposable as a ring by (3): $R/I_0 = R/I_1 \bigoplus R/I_2$, where both the ideals I_1, I_2 strictly contain I_0 and so by the choice of I_0, $\overline{a}$ is n-fine in R/I_1 and R/I_2. But then $\overline{a}$ is n-fine in R/I_0 by Lemma 2(2), a contradiction.

□

3. n-FINENESS OF MATRIX RINGS

The purpose of this section is to investigate n-fine property of matrices over any arbitrary ring and of the endomorphism ring of a free R-module of rank at least 2. First we give the following interesting decomposition.

Theorem 2. *Over any ring, the* 2×2 *and* 3×3 *matrices are 2-fine.*

Proof. Let R be a ring and let $M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(R) \setminus \{0\}$. Put $N = \begin{pmatrix} a_{11}-1 & 1-a_{11} \\ a_{11}-1 & 1-a_{11} \end{pmatrix}$. It is checked easily that then $N^2 = 0$. Thus we have

$$M - N = \begin{pmatrix} 1 & a_{12}+a_{11}-1 \\ a_{21}-a_{11}+1 & a_{22}-a_{11}-1 \end{pmatrix}.$$

Now there exist invertible matrices P and Q such that

$$P(M-N)Q = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & c \end{pmatrix}$$

for an appropriate c and thus is a sum of two units. Hence M is 2-fine. Now, let

$$M = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

be 3×3 matrix over $R \setminus \{0\}$. We first construct an nilpotent in order to show 2-fineness of M. Set

$$N = \begin{pmatrix} b_{11}-1 & b_{22}-1 & 2-b_{11}-b_{22} \\ b_{11}-1 & b_{22}-1 & 2-b_{11}-b_{22} \\ b_{11}-1 & b_{22}-1 & 2-b_{11}-b_{22} \end{pmatrix}.$$

It may be directly verified that $N^2 = 0$. Thus

$$M - N = \begin{pmatrix} 1 & b_{12}-b_{22}+1 & b_{13}+b_{11}+b_{22}-2 \\ b_{21}-b_{11}+1 & 1 & b_{23}+b_{11}+b_{22}-2 \\ b_{31}-b_{11}+1 & b_{32}-b_{22}+1 & b_{33}+b_{11}+b_{22}-2 \end{pmatrix}.$$

We only need to show that $M-N$ is 2-good. Now there exist invertible matrices P and Q such that

$$P(M-N)Q = \begin{pmatrix} c_1 & 0 & c_2 \\ c_3 & 1 & 0 \\ 0 & c_4 & c_5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & c_2 \\ 0 & 0 & 1 \\ 1 & c_4 & c_5 \end{pmatrix} + \begin{pmatrix} c_1 & -1 & 0 \\ c_3 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

for an appropriate c_i where $i \in \{1, \ldots, 5\}$ and thus is a sum of two units. Hence M is 2-fine. This completes the proof. □

Remark 1.

(1) For any ring R, R can be embedded in the 2×2 matrix ring $M_2(R)$. That is, all rings can be embedded in a 2-fine ring by Theorem 2.

(2) It is known that fine rings are 2-fine rings. However, the converse is not true. For example, taking $R = M_2(\mathbb{Z})$, then R is a 2-fine ring by Theorem 2. Let $M = \begin{pmatrix} 3 & 5 \\ 0 & 0 \end{pmatrix}$, then M is not fine in R by [2, Corollary 5.4], that is, R is not a fine ring.

To facilitate the proof of Theorem 3, we isolate the following lemma, which is of some independent interest.

Lemma 3. *Let R be a ring, $n,m \geq 1$ and $k \geq 2$. If the matrix rings $M_n(R)$ and $M_m(R)$ are both k-fine, then so is the matrix ring $M_{n+m}(R)$.*

Proof. Let $M \in M_{n+m}(R)$ which we will write in the block decomposition form

$$M = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} \in M_n(R)$, $A_{22} \in M_m(R)$ and A_{12}, A_{21} are appropriately sized rectangular matrices. By hypothesis, there exist invertible $n \times n$, $m \times m$ matrices $U_1, U_2, \ldots, U_k$ and $V_1, V_2, \ldots, V_k$, and nilpotent matrices N_1, N_2 such that $A_{11} = N_1 + U_1 + U_2 + \ldots + U_k$ and $A_{22} = N_2 + V_1 + V_2 + \ldots + V_k$. Thus the decomposition

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} U_1 & A_{12} \\ 0 & V_1 \end{pmatrix} + \begin{pmatrix} U_2 & 0 \\ A_{21} & V_2 \end{pmatrix} + \ldots + \begin{pmatrix} U_k & 0 \\ 0 & V_k \end{pmatrix} + \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$$

shows that M is k-fine. □

We now present the titular result.

Theorem 3. *Over any ring R, the matrix ring $M_m(R)$ is 2-fine for any positive integer $m \geq 2$.*

Proof. The result follows directly by combining Theorem 2 and Lemma 3. □

Remark 2. Commenting on Theorem 3, [2, Proposition 2.12] ensures that over any ring S, the ring $R = M_n(S)$ where $n \geq 2$ is 3-fine. Indeed, if M is a matrix of R, then $M - I_n = T_1 + T_2$ where T_1 and T_2 are two fine matrices. Then, there are $N_1, N_2 \in Nil(R)$ and $U_1, U_2 \in U(R)$ such that $M - I_n = (N_1 + U_1) + (N_2 + U_2)$. Hence, $M = N_1 + (N_2 + I_n) + U_1 + U_2$ where $N_2 + I_n \in U(R)$, this implies that M is 3-fine.

Again, combining Proposition 1 and Theorem 3 we recover a result due to Henriksen [3, Theorem 3].

Corollary 1. *If R is any nonzero ring with identity, and $m \geq 2$, then every matrix in $M_m(R)$ is the sum of three invertible matrices in $M_m(R)$.*

Theorem 4. *Let $n \geq 1$. If R is a n-fine ring, then so is the matrix ring $M_m(R)$ for any positive integer m.*

Proof. For $n = 1$, the result follows from [2, Theorem 3.1]. For $n \geq 2$, it is clear by induction and by Lemma 3. □

We will conclude this section by considering the 2-fineness of the endomorphism ring of a free R-module of rank at least 2.

Proposition 3. *Let R be a ring and let the free R-module F be (isomorphic to) the direct sum of $\alpha \geq 2$ copies of R where α is a cardinal number. Then the ring of endomorphisms E of F is 2-fine.*

Proof. Assume first that $\alpha \geq 2$ is finite, so $E \cong M_\alpha(R)$. Then E is 2-fine for $\alpha = 2, 3$ by Theorem 2 and the values $\alpha < \omega$ for which E is 2-fine are closed under addition by Theorem 3. So E is 2-fine for all finite α.
Assume now that α is infinite. Then from $F \cong F \oplus F$, $E \cong M_2(E)$, and so E is 2-fine by Theorem 2. □

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