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NEW INEQUALITIES OF WIRTINGER TYPE FOR DIFFERENT KINDS OF CONVEX FUNCTIONS

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Abstract. T.Z. Mirković [14] obtained new inequalities of Wirtinger type by using some classical inequalities and special means for convex function. So in this paper, we obtain some inequalities of Wirtinger type for s-convex function, m-convex function, (α,m)-convex function, quasi-convex function and P-function. Also several special cases are discussed, which can be deduced from our main results.

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1. INTRODUCTION

W.Wirtinger proved the following Theorem 1 regarding periodic functions.The proof of Wirtinger was published in 1916 in the book [5] by W. Blaschke.

Theorem 1. *Let f be a periodic function with period 2π and let $f^{'} \in L^2$. Then, if $\int_0^{2\pi} f(x)dx = 0$, the following inequality holds*

$$\int_0^{2\pi} f^2(x)dx \leq \int_0^{2\pi} (f')^2(x)dx \tag{1.1}$$

with equality if and only if $f(x) = A\cos x + B\sin x$, where A and B are constants.

Inequality (1.1) is known in the literature as Wirtinger's inequality. Wirtinger's inequality compares the integral of a square of a function with that of a square of its first derivative. Last years, a large number of papers which generalize and extend Wirtinger's inequality have been appeared in the literature (see [1], [4], [13] [15], [3], [12], [17], [19]).

In 1905, E.Almansi proved the following theorem [2].

Theorem 2. *Let f and f' are continuous the interval (a,b), that $f(a)=f(b)$ and that $\int_a^b f(x)dx=0$ then the following inequality holds*

$$\int_a^b f^2(x)dx \le \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx. \tag{1.2}$$

We recall some previously known definitions of different type of convexity.

Definition 1. The function $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ is said to be convex if the following inequality holds

$$f(\lambda x+(1-\lambda)y)\le \lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

Definition 2. (see [8],[16]) Let $0<s\le 1$. A function $f:[0,\infty)\to\mathbb{R}$ is said to be s-Orlicz convex or s-convex in the first sense, if for every $x,y\in[0,\infty)$ and $\alpha,\beta\ge 0$ with $\alpha^s+\beta^s=1$, we have:

$$f(\alpha x+\beta y)\le \alpha^s f(x)+\beta^s f(y). \tag{1.3}$$

We denote the set of all s-convex functions in the first sense by K_s^1.

Definition 3. (see [6],[11]) Let $0<s\le 1$. A function $f:[0,\infty)\to\mathbb{R}$, is said to be s-Breckner convex or s-convex in the second sense, if for every $x,y\in[0,\infty)$ and $\alpha,\beta\ge 0$ with $\alpha+\beta=1$, we have:

$$f(\alpha x+\beta y)\le \alpha^s f(x)+\beta^s f(y). \tag{1.4}$$

The set of all s-convex functions in the second sense is denoted by K_s^2.

Definition 4. ([7]) A function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is P-function or that f belongs to the class of $P(I)$, if it is nonnegative and, for all $x,y\in I$ and $\lambda\in[0,1]$, satisfies the following inequality;

$$f\big(\lambda x+(1-\lambda)y\big)\le f(x)+f(y). \tag{1.5}$$

Definition 5. (see, e.g., [9]) The function $f:[0,b]\to\mathbb{R}$ is said to be m-convex, where $m\in[0,1]$, if for every $x,y\in[0,b]$ and $t\in[0,1]$ we have:

$$f(tx+m(1-t)y)\le tf(x)+m(1-t)f(y).$$

Denote by $K_m(b)$ the set of the m-convex functions on $[0,b]$ for which $f(0)\le 0$.

Definition 6. (see, e.g., [9]) The function $f:[0,b]\to\mathbb{R}$ is said to be (α,m)-convex, where $(\alpha,m)\in[0,1]^2$, if for every $x,y\in[0,b]$ and $t\in[0,1]$ we have:

$$f(tx+m(1-t)y)\le t^\alpha f(x)+m(1-t^\alpha)f(y).$$

Denote by $K_m^\alpha(b)$ the set of the (α,m)-convex functions on $[0,b]$ for which $f(0)\le 0$.

Definition 7. [10] Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ for all $t \in [0,1]$ and all $x, y \in I$, if the following inequality

$$f(tx+(1-t)y) \leq max\{f(x), f(y)\}$$

holds, then f is called a quasi-convex function on I.

We recall definition of Beta function (see, e.g., [18])

Definition 8. Assume that $\Re(a) > 0$ and $\Re(b) > 0$, the Beta function is denoted by B(a,b) and defined as

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt.$$

An important property connecting the Gamma and Beta functions can be stated as following:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

T.Z. Mirković [14] proved the following theorems involving inequalities of Wirtinger type for convex functions.

Theorem 3. *Let f and f' are continuous on the interval (a,b), with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If $(f')^2$ is convex on $[a,b]$, then the following inequality holds*

$$\int_a^b [f(x)]^2 dx \leq \frac{(b-a)^3}{8\pi^2}\left([f'(a)]^2 + [f'(b)]^2\right). \tag{1.6}$$

Theorem 4. *Let f and f' are continuous on the interval (a,b), with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If f' is convex on $[a,b]$, then the following inequality holds*

$$\int_a^b [f(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2}\left(\frac{[f'(a)]^2 + [f'(a)][f'(b)] + [f'(b)]^2}{3}\right). \tag{1.7}$$

Theorem 5. *Let f and f' are continuous on the interval (a,b), with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If f' is positive, $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ are convex on $[a,b]$, then the following inequality holds*

$$\int_a^b [f(x)]^2 dx \tag{1.8}$$
$$\leq \alpha(b-a)^3 \frac{[f'(a)]^{\frac{1}{\alpha}} + [f'(b)]^{\frac{1}{\alpha}}}{8\pi^2} + \beta(b-a)^3 \frac{[f'(a)]^{\frac{1}{\beta}} + [f'(b)]^{\frac{1}{\beta}}}{8\pi^2}$$

where $\alpha, \beta > 0$ *and* $\alpha + \beta = 1$.

2. Main Results

In this section, we obtained some new inequalities of Wirtinger type for different kinds of convex functions.

Theorem 6. *Let f and f' are continuous on the interval (a,b), with $f(a)=f(b)$ and $\int_a^b f(x)dx=0$. If $(f')^2$ is P-function on $[a,b]$, then the following inequality holds:*

$$\int_a^b [f(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2}\left([f'(a)]^2+[f'(b)]^2\right). \tag{2.1}$$

Proof. Since $(f')^2$ is a P-function on $[a,b]$, therefore for $t\in[0,1]$ we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b [f'(x)]^2 dx &= \int_0^1 \left[f'(ta+(1-t)b)\right]^2 dt\\ &\leq \int_0^1 \left([f'(a)]^2+[f'(b)]^2\right) dt\\ &= [f'(a)]^2+[f'(b)]^2.\end{aligned}$$

Multiplying the both sides of above inequality by $\frac{(b-a)^3}{(2\pi)^2}$, we get

$$\left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2}\left([f'(a)]^2+[f'(b)]^2\right).$$

By using inequality (1.2), we get inequality (2.1) and the proof is completed. □

Theorem 7. *Let f and f' are continuous on the interval (a,b), $0\leq a<b$, with $f(a)=f(b)$ and $\int_a^b f(x)dx=0$. If $(f')^2$ is s-convex in the second sense on $[a,b]$, then the following inequality holds:*

$$\int_a^b [f(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2}\left(\frac{[f'(a)]^2+[f'(b)]^2}{s+1}\right) \qquad \textit{where } s\in(0,1]. \tag{2.2}$$

Proof. Since $(f')^2$ is a s-convex function on $[a,b]$, therefore for $t\in[0,1]$, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b [f'(x)]^2 dx &= \int_0^1 \left[f'(ta+(1-t)b)\right]^2 dt\\ &\leq \int_0^1 \left[t^s[f'(a)]^2+(1-t)^s[f'(b)]^2\right] dt\\ &= \frac{1}{s+1}\left[f'(a)\right]^2+\frac{1}{s+1}\left[f'(b)\right]^2\\ &= \frac{[f'(a)]^2+[f'(b)]^2}{s+1}.\end{aligned}$$

By multiplying by $\frac{(b-a)^3}{(2\pi)^2}$, the both sides of above inequality, we can write

$$\left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2} \frac{[f'(a)]^2 + [f'(b)]^2}{(s+1)}.$$

With the help of the inequality (1.2), we obtain (2.2) and the proof is completed. □

Remark 1. In Theorem 7, if we choose $s = 1$, the inequality (2.2) reduces to the inequality (1.6).

Theorem 8. *Let f and f' are continuous on the interval (a,b), with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If $(f')^2$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\int_a^b [f(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2} \max\left\{[f'(a)]^2, [f'(b)]^2\right\}. \tag{2.3}$$

Proof. Since $(f')^2$ is a quasi-convex on $[a,b]$, we have

$$\begin{aligned}
\frac{1}{b-a}\int_a^b [f'(x)]^2 dx &= \int_0^1 \left[f'(ta + (1-t)b)\right]^2 dt \\
&\leq \int_0^1 \max\left\{[f'(a)]^2, [f'(b)]^2\right\} dt \\
&= \max\left\{[f'(a)]^2, [f'(b)]^2\right\}
\end{aligned}$$

for $t \in [0,1]$. Multiplying the both sides of above inequality by $\frac{(b-a)^3}{(2\pi)^2}$, we get

$$\left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2} \max\left\{[f'(a)]^2, [f'(b)]^2\right\}.$$

By using inequality (1.2) for the resulting inequality, the proof is completed. □

Theorem 9. *Let f and f' are continuous on the interval (a,mb), $0 \leq a < mb$, with $f(a) = f(mb)$ and $\int_a^{mb} f(x)dx = 0$. If $(f')^2$ is m-convex function on $[a,b]$, then the following inequality holds*

$$\int_a^{mb} [f(x)]^2 dx \leq \frac{(mb-a)^3}{(2\pi)^2} \left[\frac{[f'(a)]^2 + m[f'(b)]^2}{2}\right] \tag{2.4}$$

for $m \in [0,1]$.

Proof. By using the m-convexity of $(f')^2$ on $[a,b]$, we have

$$\begin{aligned}
\frac{1}{mb-a}\int_a^{mb} [f'(x)]^2 dx &= \int_0^1 \left[f'(ta + m(1-t)b)\right]^2 dt \\
&\leq \int_0^1 \left(t[f'(a)]^2 + m(1-t)[f'(b)]^2\right) dt \\
&= \frac{[f'(a)]^2 + m[f'(b)]^2}{2}
\end{aligned}$$

for $t \in [0,1]$. Multiplying the both sides of above inequality by $\frac{(mb-a)^3}{(2\pi)^2}$, we get

$$\left(\frac{mb-a}{2\pi}\right)^2 \int_a^{mb} [f'(x)]^2 dx \leq \frac{(mb-a)^3}{(2\pi)^2} \left[\frac{[f'(a)]^2 + m[f'(b)]^2}{2}\right]$$

By using inequality (1.2), we conclude the desired result. □

Remark 2. In Theorem 9, if we set $m = 1$, the inequality (2.4) reduces to the inequality (1.6).

Theorem 10. *Let f and f' are continuous on the interval (a, mb), $0 \leq a < mb$, with $f(a) = f(mb)$ and $\int_a^{mb} f(x)dx = 0$. If $(f')^2$ is (α, m)-convex on $[a,b]$, then the following inequality holds:*

$$\int_a^{mb} [f(x)]^2 dx \leq \frac{(mb-a)^3}{(2\pi)^2} \left[\frac{[f'(a)]^2 + \alpha m[f'(b)]^2}{\alpha+1}\right] \tag{2.5}$$

where $(\alpha, m) \in [0,1]^2$.

Proof. Since $(f')^2$ is a (α, m)-convex on $[a,b]$, we have

$$\begin{aligned} \frac{1}{mb-a} \int_a^{mb} [f'(x)]^2 dx &= \int_0^1 \left[f'(ta + m(1-t)b)\right]^2 dt \\ &\leq \int_0^1 \left(t^\alpha [f'(a)]^2 + m(1-t^\alpha)[f'(b)]^2\right) dt \\ &= \frac{1}{\alpha+1} \left[f'(a)\right]^2 + m\frac{\alpha}{\alpha+1} \left[f'(b)\right]^2 \end{aligned}$$

for $t \in [0,1]$ and $(\alpha, m) \in [0,1]^2$. Multiplying the both sides of above inequality by $\frac{(mb-a)^3}{(2\pi)^2}$ and using inequality (1.2) for the resulting inequality, we get the required inequality. □

Remark 3. In Theorem 10, if we take $m = 1$ and $\alpha = 1$ the inequality (2.5) reduces to the inequality (1.6).

Theorem 11. *Let f and f' are continuous on the interval (a,b), with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If f' is P-function on $[a,b]$, then the following inequality holds:*

$$\int_a^b [f(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2} \left[f'(a) + f'(a)\right]^2. \tag{2.6}$$

Proof. By using the change of the variable, it is easy to see that

$$\begin{aligned} \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx &= \frac{(b-a)^3}{(2\pi)^2} \int_0^1 \left[f'(ta + (1-t)b)\right]^2 dt \\ &\leq \frac{(b-a)^3}{(2\pi)^2} \int_0^1 \left[f'(a) + f'(b)\right]^2 dt \end{aligned}$$

$$= \frac{(b-a)^3}{(2\pi)^2}\left[f'(a)+f'(b)\right]^2.$$

By using inequality (1.2), we get inequality (2.6) and the proof is completed. □

Theorem 12. *Let f and f' are continuous on the interval (a,b), with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If f' is quasi-convex function on $[a,b]$, then the following inequality holds:*

$$\int_a^b [f(x)]^2 dx \le \frac{(b-a)^3}{(2\pi)^2}\left[\max\left\{f'(a), f'(b)\right\}\right]^2. \tag{2.7}$$

Proof. From the definition of quasi-convex functions and by using the change of the variable, we have

$$\begin{aligned}\left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx &= \frac{(b-a)^3}{(2\pi)^2}\int_0^1 \left[f'(ta+(1-t)b)\right]^2 dt \\ &\le \frac{(b-a)^3}{(2\pi)^2}\int_0^1 \left[\max\left\{f'(a), f'(b)\right\}\right]^2 dt \\ &= \frac{(b-a)^3}{(2\pi)^2}\left[\max\left\{f'(a), f'(b)\right\}\right]^2.\end{aligned}$$

By using inequality (1.2) in the above inequality, the proof is completed. □

Theorem 13. *Let f and f' are continuous on the interval (a,mb), $0 \le a < mb$, with $f(a) = f(mb)$ and $\int_a^{mb} f(x)dx = 0$. If f' is m-convex function on $[a,b]$, then the following inequality holds:*

$$\int_a^{mb} [f(x)]^2 dx \le \frac{(mb-a)^3}{(2\pi)^2}\left[\frac{[f'(a)]^2 + m[f'(a)][f'(b)] + m^2[f'(b)]^2}{3}\right] \tag{2.8}$$

for $m \in [0,1]$.

Proof. By using the m-convexity of f' on $[a,b]$, we have

$$\begin{aligned}\left(\frac{mb-a}{2\pi}\right)^2 \int_a^{mb} [f'(x)]^2 dx &= \frac{(mb-a)^3}{(2\pi)^2}\int_0^1 \left[f'(ta+m(1-t)b)\right]^2 dt \\ &\le \frac{(mb-a)^3}{(2\pi)^2}\int_0^1 \left[tf'(a)+m(1-t)f'(b)\right]^2 dt \\ &= \frac{(mb-a)^3}{(2\pi)^2}\int_0^1 \Big[t^2[f'(a)]^2 \\ &+ (2mt-2mt^2)f'(a)f'(b) + (m^2-2m^2t+m^2t^2)[f'(b)]^2\Big] dt \\ &= \frac{(mb-a)^3}{(2\pi)^2}\left[\frac{[f'(a)]^2 + m[f'(a)][f'(b)] + m^2[f'(b)]^2}{3}\right].\end{aligned}$$

By using inequality (1.2) we get inequality (2.8) and the proof is complete. □

Remark 4. In Theorem 13, if we take $m = 1$, the inequality (2.8) reduces to the inequality (1.7)

Theorem 14. *Let f and f' are continuous on the interval (a,mb), $0 \leq a < mb$, with $f(a) = f(mb)$ and $\int_a^{mb} f(x)dx = 0$. If f' is (α,m)-convex function on $[a,b]$, then the following inequality holds:*

$$\int_a^{mb} [f(x)]^2 dx \leq \frac{(mb-a)^3}{(2\pi)^2} \left[\frac{(\alpha+1)[f'(a)]^2 + 2m\alpha[f'(a)][f'(b)] + 2\alpha^2 m^2 [f'(b)]^2}{(\alpha+1)(2\alpha+1)} \right] \tag{2.9}$$

$(\alpha,m) \in [0,1]^2$.

Proof. From the definition of f' and by using the change of the variable, we have

$$\begin{aligned}
\left(\frac{mb-a}{2\pi}\right)^2 \int_a^{mb} [f'(x)]^2 dx &= \frac{(mb-a)^3}{(2\pi)^2} \int_0^1 \left[f'(ta+m(1-t)b)\right]^2 dt \\
&\leq \frac{(mb-a)^3}{(2\pi)^2} \int_0^1 \left[t^\alpha f'(a) + m(1-t^\alpha) f'(b)\right]^2 dt \\
&= \frac{(mb-a)^3}{(2\pi)^2} \int_0^1 \Big[t^{2\alpha}[f'(a)]^2 \\
&+ (2mt^\alpha - 2mt^{2\alpha}) f'(a) f'(b) + (m^2 - 2m^2 t^\alpha + m^2 t^{2\alpha})[f'(b)]^2\Big] dt \\
&= \frac{(mb-a)^3}{(2\pi)^2} \left[\frac{(\alpha+1)[f'(a)]^2 + 2m\alpha[f'(a)][f'(b)] + 2\alpha^2 m^2 [f'(b)]^2}{(\alpha+1)(2\alpha+1)} \right].
\end{aligned}$$

By a similar argument to the proof of previous theorems, by using inequality (1.2), we get the desired result. □

Remark 5. In Theorem 14, if we take $m = 1$, $\alpha = 1$ the inequality (2.9) reduces to the inequality (1.7)

Theorem 15. *Let f and f' are continuous on the interval (a,b), $0 \leq a < b$, with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If f' is s-convex function in the second sense on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\int_a^b [f(x)]^2 dx \\
&\leq \frac{(b-a)^3}{(2\pi)^2} \left[\frac{1}{2s+1} [f'(a)]^2 + 2B(s+1,s+1) f'(a) f'(b) + \frac{1}{2s+1} [f'(b)]^2 \right]
\end{aligned} \tag{2.10}$$

for $s \in (0,1]$.

Proof. Since f' is s-convex function in the second sense, we can write

$$\left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx$$

$$
\begin{aligned}
&= \frac{(b-a)^3}{(2\pi)^2} \int_0^1 \left[f'(ta+(1-t)b)\right]^2 dt \\
&\le \frac{(b-a)^3}{(2\pi)^2} \int_0^1 \left[t^s f'(a)+(1-t)^s f'(b)\right]^2 dt \\
&= \frac{(b-a)^3}{(2\pi)^2} \int_0^1 \left[t^{2s}[f'(a)]^2+2t^s(1-t)^s f'(a)f'(b)+(1-t)^{2s}[f'(b)]^2\right] dt \\
&= \frac{(b-a)^3}{(2\pi)^2} \Bigg[[f'(a)]^2 \int_0^1 t^{2s}dt+2f'(a)f'(b)\int_0^1 t^s(1-t)^s dt \\
&\qquad +[f'(b)]^2 \int_0^1 (1-t)^{2s}dt\Bigg] \\
&= \frac{(b-a)^3}{(2\pi)^2} \left[\frac{1}{2s+1}[f'(a)]^2+2\,B(s+1,s+1)f'(a)f'(b)+\frac{1}{2s+1}[f'(b)]^2\right].
\end{aligned}
$$

By using inequality (1.2), we get inequality (2.10) and the proof is complete. □

Remark 6. In Theorem 15, if we take $s=1$, the inequality (2.10) reduces to the inequality (1.7)

Theorem 16. *Let f and f' are continuous on the interval (a,b), $0\le a<b$, with $f(a)=f(b)$ and $\int_a^b f(x)dx=0$. If f' is positive, $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ are s-convex in the second sense on $[a,b]$, then the following inequality holds:*

$$
\begin{aligned}
&\int_a^b [f(x)]^2 dx \qquad (2.11)\\
&\quad \le \frac{(b-a)^3}{(2\pi)^2}\left\{\frac{\alpha}{s+1}\left[(f'(a))^{\frac{1}{\alpha}}+(f'(b))^{\frac{1}{\alpha}}\right]+\frac{\beta}{s+1}\left[(f'(a))^{\frac{1}{\beta}}+(f'(b))^{\frac{1}{\beta}}\right]\right\}
\end{aligned}
$$

where $s\in(0,1]$, $\alpha,\beta>0$ and $\alpha+\beta=1$.

Proof. From the definition of $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ on $[a,b]$, by using inequality $cd\le \alpha c^{\frac{1}{\alpha}}+\beta d^{\frac{1}{\beta}}$ $\alpha,\beta,c,d>0$ and $\alpha+\beta=1$, we get

$$
\begin{aligned}
&\left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx \\
&\quad = \frac{(b-a)^3}{(2\pi)^2}\int_0^1 f'\big(ta+(1-t)b\big)\, f'\big(ta+(1-t)b\big)\,dt \\
&\quad \le \frac{(b-a)^3}{(2\pi)^2}\left\{\alpha\int_0^1 \left[f'\big(ta+(1-t)b\big)\right]^{\frac{1}{\alpha}}dt+\beta\int_0^1 \left[f'\big(ta+(1-t)b\big)\right]^{\frac{1}{\beta}}dt\right\} \\
&\quad \le \frac{(b-a)^3}{(2\pi)^2}\left\{\alpha\int_0^1 \left[t^s\big(f'(a)\big)^{\frac{1}{\alpha}}+(1-t)^s\big(f'(b)\big)^{\frac{1}{\alpha}}\right]dt\right.
\end{aligned}
$$

$$+\beta\int_0^1\left[t^s\left(f'(a)\right)^{\frac{1}{\beta}}+(1-t)^s\left(f'(b)\right)^{\frac{1}{\beta}}\right]dt\Bigg\}$$

$$=\frac{(b-a)^3}{(2\pi)^2}\left\{\alpha\left[\frac{1}{s+1}\left(f'(a)\right)^{\frac{1}{\alpha}}+\frac{1}{s+1}\left(f'(b)\right)^{\frac{1}{\alpha}}\right]\right.$$

$$\left.+\beta\left[\frac{1}{s+1}\left(f'(a)\right)^{\frac{1}{\beta}}+\frac{1}{s+1}\left(f'(b)\right)^{\frac{1}{\beta}}\right]\right\}$$

$$=\frac{(b-a)^3}{(2\pi)^2}\left\{\frac{\alpha}{s+1}\left[\left(f'(a)\right)^{\frac{1}{\alpha}}+\left(f'(b)\right)^{\frac{1}{\alpha}}\right]+\frac{\beta}{s+1}\left[\left(f'(a)\right)^{\frac{1}{\beta}}+\left(f'(b)\right)^{\frac{1}{\beta}}\right]\right\}.$$

By applying (1.2), we get required inequality (2.11) and the proof is complete. □

Remark 7. In Theorem 16, if we take $s=1$, the inequality (2.11) reduces to the inequality (1.8)

Theorem 17. *Let f and f' are continuous on the interval (a,b), with $f(a)=f(b)$ and $\int_a^b f(x)dx=0$. If f' is positive, $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ are P-function on $[a,b]$, then the following inequality holds:*

$$\int_a^b [f(x)]^2 dx \tag{2.12}$$

$$\leq \frac{(b-a)^3}{(2\pi)^2}\left\{\alpha\left[(f'(a))^{\frac{1}{\alpha}}+(f'(b))^{\frac{1}{\alpha}}\right]+\beta\left[(f'(a))^{\frac{1}{\beta}}+(f'(b))^{\frac{1}{\beta}}\right]\right\}$$

where $\alpha,\beta>0$ *and* $\alpha+\beta=1$.

Proof. By using the inequality $cd\leq\alpha c^{\frac{1}{\alpha}}+\beta d^{\frac{1}{\beta}}$ $\alpha,\beta,c,d>0$ and $\alpha+\beta=1$ we get

$$\left(\frac{b-a}{2\pi}\right)^2\int_a^b [f'(x)]^2 dx$$

$$=\frac{(b-a)^3}{(2\pi)^2}\int_0^1 f'\big(ta+(1-t)b\big)\,f'\big(ta+(1-t)b\big)\,dt$$

$$\leq\frac{(b-a)^3}{(2\pi)^2}\left\{\alpha\int_0^1\left[f'\big(ta+(1-t)b\big)\right]^{\frac{1}{\alpha}}dt+\beta\int_0^1\left[f'\big(ta+(1-t)b\big)\right]^{\frac{1}{\beta}}dt\right\}$$

$$\leq\frac{(b-a)^3}{(2\pi)^2}\left\{\alpha\int_0^1\left[\left(f'(a)\right)^{\frac{1}{\alpha}}+\left(f'(b)\right)^{\frac{1}{\alpha}}\right]dt\right.$$

$$\left.+\beta\int_0^1\left[\left(f'(a)\right)^{\frac{1}{\beta}}+\left(f'(b)\right)^{\frac{1}{\beta}}\right]dt\right\}$$

$$= \frac{(b-a)^3}{(2\pi)^2}\left\{\alpha\left[\left(f'(a)\right)^{\frac{1}{\alpha}} + \left(f'(b)\right)^{\frac{1}{\alpha}}\right] + \beta\left[\left(f'(a)\right)^{\frac{1}{\beta}} + \left(f'(b)\right)^{\frac{1}{\beta}}\right]\right\}.$$

By applying (1.2), we get required inequality (2.12) and the proof is complete. □

Theorem 18. *Let f and f' are continuous on the interval (a,b), with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If f' is positive, $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ are quasi-convex function on $[a,b]$, then the following inequality holds*

$$\int_a^b [f(x)]^2 dx \le \frac{(b-a)^3}{(2\pi)^2}\left\{\alpha\left[\max\left\{(f'(a))^{\frac{1}{\alpha}}, (f'(b))^{\frac{1}{\alpha}}\right\}\right] + \beta\left[\max\left\{(f'(a))^{\frac{1}{\beta}}, (f'(b))^{\frac{1}{\beta}}\right\}\right]\right\} \tag{2.13}$$

where $\alpha, \beta > 0$ and $\alpha + \beta = 1$.

Proof. By a similar way to the previous theorem, but now by using the quasi-convexity of $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ on $[a,b]$, we get

$$\begin{aligned}
\left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx
&= \frac{(b-a)^3}{(2\pi)^2}\int_0^1 f'\left(ta+(1-t)b\right) f'\left(ta+(1-t)b\right) dt \\
&\le \frac{(b-a)^3}{(2\pi)^2}\left\{\alpha\int_0^1 \left[f'\left(ta+(1-t)b\right)\right]^{\frac{1}{\alpha}} dt + \beta\int_0^1 \left[f'\left(ta+(1-t)b\right)\right]^{\frac{1}{\beta}} dt\right\} \\
&\le \frac{(b-a)^3}{(2\pi)^2}\left\{\alpha\int_0^1 \max\left\{\left(f'(a)\right)^{\frac{1}{\alpha}}, \left(f'(b)\right)^{\frac{1}{\alpha}}\right\} dt\right. \\
&\quad \left. + \beta\int_0^1 \max\left\{\left(f'(a)\right)^{\frac{1}{\beta}}, \left(f'(b)\right)^{\frac{1}{\beta}}\right\} dt\right\} \\
&= \frac{(b-a)^3}{(2\pi)^2}\left\{\alpha\left[\max\left\{(f'(a))^{\frac{1}{\alpha}}, (f'(b))^{\frac{1}{\alpha}}\right\}\right]\right. \\
&\quad \left. + \beta\left[\max\left\{(f'(a))^{\frac{1}{\beta}}, (f'(b))^{\frac{1}{\beta}}\right\}\right]\right\}.
\end{aligned}$$

By applying (1.2), we get required inequality (2.13) and the proof is complete. □

Theorem 19. *Let f and f' are continuous on the interval (a, mb), $0 \le a < mb$, with $f(a) = f(mb)$ and $\int_a^{mb} f(x)dx = 0$. If f' is positive, $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ are m-convex function on $[a,b]$, then the following inequality holds*

$$\int_a^{mb} [f(x)]^2 dx \tag{2.14}$$

$$\leq \alpha(mb-a)^3 \frac{[f'(a)]^{\frac{1}{\alpha}} + m[f'(b)]^{\frac{1}{\alpha}}}{8\pi^2} + \beta(mb-a)^3 \frac{[f'(a)]^{\frac{1}{\beta}} + m[f'(b)]^{\frac{1}{\beta}}}{8\pi^2}$$

where $m \in [0,1]$, $\alpha, \beta > 0$ *and* $\alpha + \beta = 1$.

Proof. By using the inequality $cd \leq \alpha c^{\frac{1}{\alpha}} + \beta d^{\frac{1}{\beta}}$ $\alpha, \beta, c, d > 0$ and $\alpha + \beta = 1$ and m-convexity of $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ on $[a,b]$, one can easily write

$$\begin{aligned}
&\left(\frac{mb-a}{2\pi}\right)^2 \int_a^{mb} [f'(x)]^2 dx \\
&= \frac{(mb-a)^3}{(2\pi)^2} \int_0^1 f'(ta+m(1-t)b)\, f'(ta+m(1-t)b)\, dt \\
&\leq \frac{(mb-a)^3}{(2\pi)^2} \left\{ \alpha \int_0^1 \left[f'(ta+m(1-t)b)\right]^{\frac{1}{\alpha}} dt \right. \\
&\qquad \left. + \beta \int_0^1 \left[f'(ta+m(1-t)b)\right]^{\frac{1}{\beta}} dt \right\} \\
&\leq \frac{(mb-a)^3}{(2\pi)^2} \left\{ \alpha \int_0^1 \left[t\left(f'(a)\right)^{\frac{1}{\alpha}} + m(1-t)\left(f'(b)\right)^{\frac{1}{\alpha}}\right] dt \right. \\
&\qquad \left. + \beta \int_0^1 \left[t\left(f'(a)\right)^{\frac{1}{\beta}} + m(1-t)\left(f'(b)\right)^{\frac{1}{\beta}}\right] dt \right. \\
&= \frac{(mb-a)^3}{(2\pi)^2} \left\{ \alpha \left(\frac{[f'(a)]^{\frac{1}{\alpha}} + m[f'(b)]^{\frac{1}{\alpha}}}{2} \right) \right. \\
&\qquad \left. + \beta \left(\frac{[f'(a)]^{\frac{1}{\beta}} + m[f'(b)]^{\frac{1}{\beta}}}{2} \right) \right\}.
\end{aligned}$$

By applying (1.2), we get required inequality (2.14) and the proof is complete. □

Remark 8. In Theorem 19, if we take $m = 1$, the inequality (2.14) reduces to the inequality (1.8)

Theorem 20. *Let f and f' are continuous on the interval (a, mb), $0 \leq a < mb$, with $f(a) = f(mb)$ and $\int_a^{mb} f(x)dx = 0$. If f' is positive, $(f')^{\frac{1}{\theta}}$ and $(f')^{\frac{1}{\beta}}$ are (α, m)-convex function on $[a,b]$, then the following inequality holds*

$$\begin{aligned}
&\int_a^{mb} [f(x)]^2 dx \qquad (2.15) \\
&\leq \frac{(mb-a)^3}{(2\pi)^2} \left\{ \theta \left(\frac{[f'(a)]^{\frac{1}{\theta}} + m\alpha[f'(b)]^{\frac{1}{\theta}}}{\alpha+1} \right) + \beta \left(\frac{[f'(a)]^{\frac{1}{\beta}} + m\alpha[f'(b)]^{\frac{1}{\beta}}}{\alpha+1} \right) \right\}
\end{aligned}$$

where $(\alpha, m) \in [0,1]^2, \theta, \beta > 0$ *and* $\theta + \beta = 1$.

Proof. By using the same inequality and the similar computations to the proof of the Theorem 19, we have

$$\begin{aligned}
&\left(\frac{mb-a}{2\pi}\right)^2 \int_a^{mb} [f'(x)]^2 dx \\
&= \frac{(mb-a)^3}{(2\pi)^2} \int_0^1 f'(ta+m(1-t)b)\, f'(ta+m(1-t)b)\, dt \\
&\leq \frac{(mb-a)^3}{(2\pi)^2} \left\{ \theta \int_0^1 \left[f'(ta+m(1-t)b)\right]^{\frac{1}{\theta}} dt \right. \\
&\qquad \left. + \beta \int_0^1 \left[f'(ta+m(1-t)b)\right]^{\frac{1}{\beta}} dt \right\} \\
&\leq \frac{(mb-a)^3}{(2\pi)^2} \left\{ \theta \int_0^1 \left[t^{\alpha}\left(f'(a)\right)^{\frac{1}{\theta}} + m(1-t^{\alpha})\left(f'(b)\right)^{\frac{1}{\theta}}\right] dt \right. \\
&\qquad \left. + \beta \int_0^1 \left[t^{\alpha}\left(f'(a)\right)^{\frac{1}{\beta}} + m(1-t^{\alpha})\left(f'(b)\right)^{\frac{1}{\beta}}\right] dt \right. \\
&= \frac{(mb-a)^3}{(2\pi)^2} \left\{ \theta \left(\frac{[f'(a)]^{\frac{1}{\theta}} + m\alpha [f'(b)]^{\frac{1}{\theta}}}{\alpha+1} \right) \right. \\
&\qquad \left. + \beta \left(\frac{[f'(a)]^{\frac{1}{\beta}} + m\alpha [f'(b)]^{\frac{1}{\beta}}}{\alpha+1} \right) \right\}.
\end{aligned}$$

By applying (1.2), this completes the proof. □

Remark 9. In Theorem 20, if we take $m = 1$, $\alpha = 1$ the inequality (2.15) reduces to the inequality (1.8).

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