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A NEW APPROACH TO HOMOTOPY THEORY VIA BEST PROXIMITY POINT

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Abstract. In this paper, we extend the result of Romaguera [21] with the aid of best proximity point theory on partial metric spaces by considering the approach of Haghi et al. [9], and so celebrated Boyd-Wong fixed point theorem [7]. We first introduce two concepts called generalized proximal BW-contraction and generalized best BW-contraction. Then, we obtain some best proximity point theorems for such mappings. To illustrate the effectiveness of our results, we provide some nontrivial and interesting examples. Finally, unlike homotopy applications existing in the literature, we present for the first time an application of the best proximity result to the homotopy theory.

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1. Introduction and preliminaries

It is well known that $\varkappa = \eta$ if and only if $d(\varkappa,\eta) = 0$ for each points $\varkappa,\eta$ in a metric space (Λ,d). However, motivated by the experience of computer science, there was a tendency to relax this equivalence. In this context, Matthews [14] introduced a new concept so called partial metric by weakening the mentioned condition as follows:

Definition 1 ([14])**.** Let Λ be a nonempty set and $\gamma : \Lambda\times\Lambda \to [0,\infty)$ be a function. Then, γ is said to be a partial metric on Λ if the following conditions hold:

p1) $\varkappa = \eta$ iff $\gamma(\varkappa,\varkappa) = \gamma(\varkappa,\eta) = \gamma(\eta,\eta)$,
p2) $\gamma(\varkappa,\varkappa) \le \gamma(\varkappa,\eta)$,
p3) $\gamma(\varkappa,\eta) = \gamma(\eta,\varkappa)$,
p4) $\gamma(\varkappa,\upsilon) \le \gamma(\varkappa,\eta) + \gamma(\eta,\upsilon) - \gamma(\eta,\eta)$

for all $\varkappa,\eta,\upsilon \in \Lambda$. In this case, the pair (Λ,γ) is called partial metric space.

It is clear that every metric space is a partial metric space but conversely may not be true. Indeed, $(\mathbb{R}^+ = [0,\infty),\gamma)$ is a nonmetric partial metric space, where $\gamma(\varkappa,\eta) = \max\{\varkappa,\eta\}$. For the sake of completeness, we recall the necessary and useful properties of partial metric spaces.

Let (Λ,γ) be a partial metric space. Then, γ generates a T_0-topology τ_γ on Λ which has as a base the family of open balls $\{B(\varkappa,\varepsilon):\varkappa\in\Lambda \text{ and } \varepsilon>0\}$ where

$$B(\varkappa,\varepsilon)=\{\eta\in\Lambda:\gamma(\varkappa,\eta)<\gamma(\varkappa,\varkappa)+\varepsilon\}.$$

Let $\{\varkappa_n\}$ be a sequence in Λ and $\varkappa\in\Lambda$. It is easy to see that the sequence $\{\varkappa_n\}$ converges to $\varkappa$ with respect to τ_γ if and only if

$$\lim_{n\to\infty}\gamma(\varkappa_n,\varkappa)=\gamma(\varkappa,\varkappa).$$

If $\lim_{n,m\to\infty}\gamma(\varkappa_n,\varkappa_m)$ exists and is finite, then $\{\varkappa_n\}$ is said to be a Cauchy sequence. If every Cauchy sequence $\{\varkappa_n\}$ converges to a point $\varkappa$ in Λ such that

$$\lim_{n,m\to\infty}\gamma(\varkappa_n,\varkappa_m)=\gamma(\varkappa,\varkappa)$$

then, (Λ,γ) is said to be a complete partial metric space.

If γ is a partial metric on Λ, then the mapping $d_\gamma:\Lambda\times\Lambda\to[0,\infty)$ defined by

$$d_\gamma(\varkappa,\eta)=2\gamma(\varkappa,\eta)-\gamma(\varkappa,\varkappa)-\gamma(\eta,\eta)$$

for all $\varkappa,\eta\in\Lambda$, is a metric on Λ.

The following lemma shows the relation between a partial metric γ and the induced ordinary metric d_γ.

Lemma 1. *Let (Λ,γ) be a partial metric space, $\{\varkappa_n\}$ be a sequence in Λ and $\varkappa\in\Lambda$. Then, the following ones hold:*

- (i) *$\{\varkappa_n\}$ is a Cauchy sequence (Λ,γ) iff it is a Cauchy sequence (Λ,d_γ).*
- (ii) *(Λ,γ) is a complete partial metric space iff (Λ,d_γ) is a complete metric space. Further,*

$$\lim_{n\to\infty}d_\gamma(\varkappa_n,\varkappa)=0\Longleftrightarrow\gamma(\varkappa,\varkappa)=\lim_{n\to\infty}\gamma(\varkappa_n,\varkappa)=\lim_{n,m\to\infty}\gamma(\varkappa_n,\varkappa_m).$$

Because of the usefulness of partial metric spaces especially in computer science many authors have studied fixed point theory on these spaces. In this sense, Aydi et al. initiated study of fixed point theory for multivalued mappings on partial metric spaces by introducing partial Hausdorff metric [4]. Considering a binary relation $\mathcal{R}$ on a partial metric space Perveen et al. introduced a new contraction, and then obtained a fixed point theorem for such mappings [19]. Bugajewski et al. also discussed the vital role of bottom sets in fixed point theory on these spaces [8]. Recently, Romaguera [21] obtained the following nice and interesting result on these settings which extends the famous Boyd-Wong fixed point theorem [7].

Theorem 1. *Let (Λ,γ) be a complete partial metric space and let $T:\Lambda\to\Lambda$ be a mapping such that*

$$\gamma(T\varkappa,T\eta)\leq\varphi(C_T(\varkappa,\eta))$$

for all $\varkappa, \eta \in \Lambda$, *where*

$$C_T(\varkappa, \eta) = \max\left\{\gamma(\varkappa, \eta), \gamma(\varkappa, T\varkappa), \gamma(\eta, T\eta), \frac{1}{2}\left[\gamma(\varkappa, T\eta) + \gamma(\eta, T\varkappa)\right]\right\}$$

and $\varphi : [0, \infty) \to [0, \infty)$ *is upper semicontinuous from the right such that* $\varphi(\lambda) < \lambda$ *for all* $\lambda > 0$. *Then* T *has a unique fixed point* $\varkappa \in \Lambda$. *Moreover,* $\gamma(\varkappa, \varkappa) = 0$.

We denote the family of all functions φ satisfying conditions in Theorem 1 by Φ.

On the other hand, very recently, fixed point theory has been improved different aspect from the results existing in the literature by using best proximity point theory. Let (Λ, d) be a metric space and $\varnothing \neq P, Q \subseteq \Lambda$. Then, the mapping $T : P \to Q$ cannot have a fixed point in case of $P \cap Q = \varnothing$. In this case, it is sensible to search the existence a point $\varkappa \in P$ such that $d(\varkappa, T\varkappa) = d(P, Q)$ which is called best proximity point of T. Note that, if $P = Q = \Lambda$, then a best proximity point becomes a fixed point. Therefore, every fixed point results are special cases of the corresponding best proximity point results. For this reason, this topic has been studied in various ways [2, 3, 5, 6, 20, 22]. Now, we state the related fundamental concepts and notations of best proximity point theory in realm of partial metric spaces.

Let (Λ, γ) be a partial metric space and $\varnothing \neq P, Q \subseteq \Lambda$. We will consider the subsets P_0 and Q_0 defined as

$$P_0 = \{\varkappa \in P : \gamma(\varkappa, \eta) = \gamma(P, Q) \text{ for some } \eta \in Q\}$$

and

$$Q_0 = \{\eta \in Q : \gamma(\varkappa, \eta) = \gamma(P, Q) \text{ for some } \varkappa \in P\},$$

where

$$\gamma(P, Q) = \inf\{\gamma(\varkappa, \eta) : \varkappa \in P \text{ and } \eta \in Q\}.$$

We will call Q is approximately compact with respect to P if and only if every sequence $\{\eta_n\}$ in Q satisfying $\gamma(\varkappa, \eta_n) \to \gamma(\varkappa, Q)$ for some $\varkappa \in P$ has a subsequence $\{\eta_{n_k}\}$ such that $d_\gamma(\eta_{n_k}, \eta) \to 0$ as $k \to \infty$ for some $\eta \in Q$.

Definition 2. Let (Λ, γ) be a partial metric space and $\varnothing \neq P, Q \subseteq \Lambda$. Then, the pair (P, Q) is said to have the P-Property if and only if it is satisfied

$$\left.\begin{array}{l}\gamma(\varkappa_1, \eta_1) = \gamma(P, Q)\\ \gamma(\varkappa_2, \eta_2) = \gamma(P, Q)\end{array}\right\} \Longrightarrow \gamma(\varkappa_1, \varkappa_2) = \gamma(\eta_1, \eta_2)$$

for all $\varkappa_1, \varkappa_2 \in P_0$ and $\eta_1, \eta_2 \in Q_0$.

Remark 1. Recently, taking into account the metric d_γ defined by $d_\gamma(\varkappa, \eta) = 0$ if $\varkappa = \eta$ and $d_\gamma(\varkappa, \eta) = \gamma(\varkappa, \eta)$ if $\varkappa \neq \eta$ with the help of partial metric γ in [10], Haghi et al. [9] showed that some fixed point results for the certain operators on partial metric spaces can be obtained directly from their standard metric counterparts. However, even if it can be shown that a mapping T has a fixed point $\varkappa$ by using this approach, it cannot be obtained that $\gamma(\varkappa, \varkappa) = 0$ which is very useful for computer sciences (for more details, see [12, 13, 15, 16]). Moreover, the results obtained in this article are even new in the settings of metric spaces.

In this paper, we first introduce two concepts called generalized proximal BW-contraction and generalized best BW-contraction. Then, we obtain some best proximity point theorems for such mappings on partial metric spaces. Hence, many results in the literature are extended and improved. To illustrate the effectiveness of our results, we provide some nontrivial and interesting examples. Finally, unlike homotopy applications existing in the literature, we present for the first time an application of the best proximity result to the homotopy theory.

2. Generalized proximal BW-contractions

In this section, we first introduce the following new concept, the so called generalized proximal BW-contraction and obtain some best proximity point results for such mappings.

Definition 3. Let (Λ,γ) be a partial metric spaces, $\varnothing \neq P,Q \subseteq \Lambda$ and $T : P \to Q$ be a mapping. If there exists a $\varphi \in \Phi$ such that

$$\left.\begin{array}{l}\gamma(u_1,T\varkappa_1)=\gamma(P,Q)\\ \gamma(u_2,T\varkappa_2)=\gamma(P,Q)\end{array}\right\} \Longrightarrow \gamma(u_1,u_2) \leq \varphi(R_T(\varkappa_1,\varkappa_2)) \tag{2.1}$$

for all $u_1,u_2,\varkappa_1,\varkappa_2 \in P$, where

$$R_T(\varkappa_1,\varkappa_2) = \max\{\gamma(\varkappa_1,\varkappa_2), S_T(\varkappa_1,\varkappa_2) - \gamma(P,Q)\}$$

and

$$S_T(\varkappa_1,\varkappa_2) = \max\left\{\gamma(\varkappa_1,T\varkappa_1),\gamma(\varkappa_2,T\varkappa_2),\frac{\gamma(\varkappa_1,T\varkappa_2)+\gamma(\varkappa_2,T\varkappa_1)}{2}\right\},$$

then T is called a generalized proximal BW-contraction mapping.

Before the main result of this section, we give the following lemma which is useful in our results.

Lemma 2. *Let (Λ,γ) be a partial metric space, $\varnothing \neq P,Q \subseteq \Lambda$ and $T : P \to Q$ be a mapping. Then,*

$$R_T(\varkappa,\eta) = \max\{\gamma(\varkappa,\eta),\gamma(\eta,T\eta) - \gamma(P,Q)\}$$

for all $\varkappa,\eta \in P$ satisfying $\gamma(\eta,T\varkappa) = \gamma(P,Q)$.

Proof. Let $\varkappa,\eta \in \Lambda$ satisfying $\gamma(\eta,T\varkappa) = \gamma(P,Q)$. Then, we have

$$\gamma(\varkappa,T\varkappa) \leq \gamma(\varkappa,\eta)+\gamma(\eta,T\varkappa),$$

and so,

$$\gamma(\varkappa,T\varkappa) - \gamma(P,Q) \leq \gamma(\varkappa,\eta). \tag{2.2}$$

Further, using triangle inequality and (2.4), we obtain

$$\begin{aligned}\frac{\gamma(\varkappa,T\eta)+\gamma(\eta,T\varkappa)}{2}-\gamma(P,Q) &\leq \frac{\gamma(\varkappa,\eta)+\gamma(\eta,T\eta)}{2}+\frac{\gamma(P,Q)}{2}-\gamma(P,Q)\\ &= \frac{\gamma(\varkappa,\eta)+\gamma(\eta,T\eta)}{2}-\frac{\gamma(P,Q)}{2}\\ &= \frac{\gamma(\varkappa,\eta)}{2}+\frac{\gamma(\eta,T\eta)-\gamma(P,Q)}{2}\\ &\leq \max\{\gamma(\varkappa,\eta),\gamma(\eta,T\eta)-\gamma(P,Q)\}. \end{aligned} \tag{2.3}$$

Thus, from (2.2) and (2.3), we get

$$\begin{aligned} R_T(\varkappa,\eta) &= \max\{\gamma(\varkappa,\eta),S_T(\varkappa,\eta)-\gamma(P,Q)\}\\ &= \max\left\{\gamma(\varkappa,\eta),\max\left\{\gamma(\varkappa,T\varkappa),\gamma(\eta,T\eta),\frac{\gamma(\varkappa,T\eta)+\gamma(\eta,T\varkappa)}{2}\right\}-\gamma(P,Q)\right\}\\ &\leq \max\{\gamma(\varkappa,\eta),\gamma(\eta,T\eta)-\gamma(P,Q)\}\leq R_T(\varkappa,\eta)\end{aligned}$$

and so

$$R_T(\varkappa,\eta)=\max\{\gamma(\varkappa,\eta),\gamma(\eta,T\eta)-\gamma(P,Q)\}.$$

□

Theorem 2. *Let (Λ,γ) be a partial metric space, $\varnothing\neq P,Q\subseteq\Lambda$ with $P_0\neq\varnothing$ and $T:P\to Q$ be a generalized proximal BW-contraction mapping satisfying $T(P_0)\subseteq Q_0$. If (P_0,γ) is complete, then the mapping T has a best proximity point $\varkappa^*$ in P_0. Moreover, $\gamma(\varkappa^*,\varkappa^*)=0$.*

Proof. Let $\varkappa_0\in P_0$ be an arbitrary point. Since $T\varkappa_0\in T(P_0)\subseteq Q_0$, there exists $\varkappa_1\in P_0$ such that

$$\gamma(\varkappa_1,T\varkappa_0)=\gamma(P,Q).$$

Similarly, since $T\varkappa_1\in T(P_0)\subseteq Q_0$, there exists $\varkappa_2\in P_0$ such that

$$\gamma(\varkappa_2,T\varkappa_1)=\gamma(P,Q).$$

Hence, by the generalized proximal BW-contractivity of T, we have

$$\gamma(\varkappa_1,\varkappa_2)\leq\varphi(R_T(\varkappa_0,\varkappa_1)).$$

Continuing this process, we can construct a sequence $\{\varkappa_n\}$ in P_0 such that

$$\gamma(\varkappa_{n+1},T\varkappa_n)=\gamma(P,Q) \tag{2.4}$$

and

$$\gamma(\varkappa_n,\varkappa_{n+1})\leq\varphi(R_T(\varkappa_{n-1},\varkappa_n)) \tag{2.5}$$

for all $n\geq 1$. If there exists $n_0\geq 1$ such that $\gamma(\varkappa_{n_0},\varkappa_{n_0+1})=0$, then $\varkappa_{n_0}$ is a best proximity point of T. Moreover $\gamma(\varkappa_{n_0},\varkappa_{n_0})=0$. Assume that $\gamma(\varkappa_n,\varkappa_{n+1})>0$ for all $n\geq 1$. Thus, from (2.4) and Lemma 2, we have

$$R_T(\varkappa_{n-1},\varkappa_n)=\max\{\gamma(\varkappa_{n-1},\varkappa_n),\gamma(\varkappa_n,T\varkappa_n)-\gamma(P,Q)\}$$

for all $n \geq 1$. Now from (2.5), we get

$$\gamma(\varkappa_n, \varkappa_{n+1}) \leq \varphi\left(\max\left\{\gamma(\varkappa_{n-1}, \varkappa_n), \gamma(\varkappa_n, T\varkappa_n) - \gamma(P,Q)\right\}\right) \tag{2.6}$$

for all $n \geq 1$. If there exists $n_0 \geq 1$ such that

$$\gamma(\varkappa_{n_0}, T\varkappa_{n_0}) - \gamma(P,Q) \geq \gamma(\varkappa_{n_0-1}, \varkappa_{n_0})$$

then, from (2.4) and (2.6), we have

$$\begin{aligned}\gamma(\varkappa_{n_0}, \varkappa_{n_0+1}) &\leq \varphi(\gamma(\varkappa_{n_0}, T\varkappa_{n_0}) - \gamma(P,Q)) \\ &< \gamma(\varkappa_{n_0}, T\varkappa_{n_0}) - \gamma(P,Q) \\ &\leq \gamma(\varkappa_{n_0}, \varkappa_{n_0+1}) + \gamma(\varkappa_{n_0+1}, T\varkappa_{n_0}) - \gamma(P,Q) \\ &= \gamma(\varkappa_{n_0}, \varkappa_{n_0+1}),\end{aligned}$$

which is a contradiction. Therefore, we have

$$\gamma(\varkappa_n, T\varkappa_n) - \gamma(P,Q) < \gamma(\varkappa_{n-1}, \varkappa_n)$$

for all $n \geq 1$ and so, we get

$$\gamma(\varkappa_n, \varkappa_{n+1}) \leq \varphi(\gamma(\varkappa_{n-1}, \varkappa_n)) < \gamma(\varkappa_{n-1}, \varkappa_n) \tag{2.7}$$

for all $n \geq 1$. Hence $\{\gamma(\varkappa_n, \varkappa_{n+1})\}$ is a decreasing sequences in $[0,\infty)$ and so it is convergent. Then, there exists $u \geq 0$ such that

$$\lim_{n\to\infty} \gamma(\varkappa_n, \varkappa_{n+1}) = u.$$

We claim that $u = 0$. Assume that $u > 0$. Then, using equation (2.7), we have

$$u = \lim_{n\to\infty} \gamma(\varkappa_n, \varkappa_{n+1}) \leq \lim_{n\to\infty} \varphi(\gamma(\varkappa_{n-1}, \varkappa_n)) = \lim_{n\to\infty} \sup \varphi(\gamma(\varkappa_{n-1}, \varkappa_n)) \leq \varphi(u) < u,$$

which is a contradiction. Therefore, $\lim_{n\to\infty} \gamma(\varkappa_n, \varkappa_{n+1}) = 0$. Now, we shall show that $\lim_{n,m\to\infty} \gamma(\varkappa_n, \varkappa_m) = 0$. Assume the contrary. Then, there exist $\varepsilon > 0$ and two sequences $\{\varkappa_{n_k}\}$ and $\{\varkappa_{m_k}\}$ of $\{\varkappa_n\}$ such that

$$\gamma(\varkappa_{n_k}, \varkappa_{m_k}) \geq \varepsilon \text{ and } \gamma(\varkappa_{n_k}, \varkappa_{m_k-1}) < \varepsilon \tag{2.8}$$

for all $m_k > n_k \geq k$, where m_k is the smallest natural number satisfying (2.8) corresponding to n_k. Then, we have

$$\varepsilon \leq \gamma(\varkappa_{n_k}, \varkappa_{m_k}) \leq \gamma(\varkappa_{n_k}, \varkappa_{m_k-1}) + \gamma(\varkappa_{m_k-1}, \varkappa_{m_k}) < \varepsilon + \gamma(\varkappa_{m_k-1}, \varkappa_{m_k}). \tag{2.9}$$

Letting $k \to \infty$ in inequality (2.9), we get $\lim_{k\to\infty} \gamma(\varkappa_{n_k}, \varkappa_{m_k}) = \varepsilon$. Also, we have

$$\begin{aligned}&\frac{\gamma(\varkappa_{n_k}, T\varkappa_{m_k}) + \gamma(\varkappa_{m_k}, T\varkappa_{n_k})}{2} - \gamma(P,Q) \\ &\leq \frac{\gamma(\varkappa_{n_k}, \varkappa_{m_k}) + \gamma(\varkappa_{m_k}, T\varkappa_{m_k}) - \gamma(P,Q)}{2} + \frac{\gamma(\varkappa_{n_k}, \varkappa_{m_k}) + \gamma(\varkappa_{n_k}, T\varkappa_{n_k}) - \gamma(P,Q)}{2} \\ &= \gamma(\varkappa_{n_k}, \varkappa_{m_k}) + \frac{\gamma(\varkappa_{n_k}, T\varkappa_{n_k}) - \gamma(P,Q)}{2} + \frac{\gamma(\varkappa_{m_k}, T\varkappa_{m_k}) - \gamma(P,Q)}{2}\end{aligned} \tag{2.10}$$

for all $k \geq 1$. Then, taking into account inequalities (2.4) and (2.10), we obtain

$$\begin{aligned}\gamma(\varkappa_{n_k}, \varkappa_{m_k}) &\leq R_T(\varkappa_{n_k}, \varkappa_{m_k}) \\ &= \max\{\gamma(\varkappa_{n_k}, \varkappa_{m_k}), S_T(\varkappa_{n_k}, \varkappa_{m_k}) - \gamma(P,Q)\} \\ &\leq \gamma(\varkappa_{n_k}, \varkappa_{m_k}) + \gamma(\varkappa_{n_k}, T\varkappa_{n_k}) - \gamma(P,Q) + \gamma(\varkappa_{m_k}, T\varkappa_{m_k}) - \gamma(P,Q) \\ &\leq \gamma(\varkappa_{n_k}, \varkappa_{m_k}) + \gamma(\varkappa_{n_k}, \varkappa_{n_k+1}) + \gamma(\varkappa_{m_k}, \varkappa_{m_k+1})\end{aligned}$$

which implies that

$$\lim_{k\to\infty} R_T(\varkappa_{n_k}, \varkappa_{m_k}) = \varepsilon.$$

Since $R_T(\varkappa_{n_k}, \varkappa_{m_k}) \geq \gamma(\varkappa_{n_k}, \varkappa_{m_k}) \geq \varepsilon$ for all $k \geq 1$, we have

$$\lim_{k\to\infty} \sup \varphi(R_T(\varkappa_{n_k}, \varkappa_{m_k})) \leq \varphi(\varepsilon).$$

On the other hand, for all $k \geq 1$, we obtain

$$\begin{aligned}\varepsilon &\leq \gamma(\varkappa_{n_k}, \varkappa_{m_k}) \\ &\leq \gamma(\varkappa_{n_k}, \varkappa_{n_k+1}) + \gamma(\varkappa_{n_k+1}, \varkappa_{m_k+1}) + \gamma(\varkappa_{m_k}, \varkappa_{m_k+1}) \\ &\leq \gamma(\varkappa_{n_k}, \varkappa_{n_k+1}) + \varphi(R_T(\varkappa_{n_k}, \varkappa_{m_k})) + \gamma(\varkappa_{m_k}, \varkappa_{m_k+1})\end{aligned}$$

and so, taking limit supremum in last inequality, we have

$$\varepsilon \leq \varphi(\varepsilon) < \varepsilon,$$

which is a contradiction. Hence, $\{\varkappa_n\}$ is a Cauchy sequence in P_0. Since P_0 is a complete partial metric space, there exist $\varkappa^* \in P_0$ such that

$$\lim_{n\to\infty} \gamma(\varkappa_n, \varkappa^*) = \gamma(\varkappa^*, \varkappa^*) = 0 = \lim_{n,m\to\infty} \gamma(\varkappa_n, \varkappa_m).$$

In this case, since $T\varkappa^* \in T(P_0) \subseteq Q_0$, there exists $\upsilon \in P_0$ such that

$$\gamma(\upsilon, T\varkappa^*) = \gamma(P,Q). \tag{2.11}$$

Now, we have

$$\begin{aligned}\gamma(\varkappa^*, T\varkappa^*) - \gamma(P,Q) &\leq R_T(\varkappa_n, \varkappa^*) \\ &\leq \gamma(\varkappa_n, \varkappa^*) + [\gamma(\varkappa_n, T\varkappa_n) - \gamma(P,Q)] + [\gamma(\varkappa^*, T\varkappa^*) - \gamma(P,Q)] \\ &\leq \gamma(\varkappa_n, \varkappa^*) + \gamma(\varkappa_n, \varkappa_{n+1}) + [\gamma(\varkappa^*, T\varkappa^*) - \gamma(P,Q)]\end{aligned}$$

and so

$$\lim_{n\to\infty} R_T(\varkappa_n, \varkappa^*) = \gamma(\varkappa^*, T\varkappa^*) - \gamma(P,Q).$$

Then, from (2.4), (2.11) and the generalized proximal BW-contractivity of T, we get

$$\gamma(\varkappa^*, \upsilon) = \lim_{n\to\infty} \gamma(\varkappa_{n+1}, \upsilon) \leq \lim_{n\to\infty} \varphi(R_T(\varkappa_n, \varkappa^*))9 \leq \varphi(\gamma(\varkappa^*, T\varkappa^*) - \gamma(P,Q)).$$

Now, assume $\gamma(\varkappa^*, T\varkappa^*) > \gamma(P,Q)$. In this case, from the last inequality, we have

$$\gamma(\varkappa^*, \upsilon) < \gamma(\varkappa^*, T\varkappa^*) - \gamma(P,Q) \leq \gamma(\varkappa^*, \upsilon) + \gamma(\upsilon, T\varkappa^*) - \gamma(P,Q) = \gamma(\varkappa^*, \upsilon),$$

which is a contradiction. Therefore, $\gamma(\varkappa^*, T\varkappa^*) = \gamma(P,Q)$, that is, $\varkappa^*$ is a best proximity point of T. □

Remark 2. Let (Λ,γ) be a complete partial metric space, $\varnothing \neq P,Q \subseteq \Lambda$ where P is closed and Q is approximately compact with respect to P. Then, (P_0,γ) is complete. Indeed, let $\{\varkappa_n\}$ be a Cauchy sequence in P_0. Then, $\{\varkappa_n\}$ is a Cauchy sequence in Λ. Since (Λ,γ) is complete, there exists $\varkappa^* \in \Lambda$ such that

$$\lim_{n\to\infty} \gamma(\varkappa_n, \varkappa^*) = \gamma(\varkappa^*, \varkappa^*) = \lim_{n,m\to\infty} \gamma(\varkappa_n, \varkappa_m)$$

Because of Lemma 1, $\{\varkappa_n\}$ converges to $\varkappa^*$ with respect to ordinary metric d_γ, that is,

$$\lim_{n\to\infty} d_\gamma(\varkappa_n, \varkappa^*) = 0. \tag{2.12}$$

Moreover, since $\varkappa_n \in P_0$ for all $n \geq 1$, there exists a sequence $\{\eta_n\}$ in Q_0 such that

$$\gamma(\varkappa_n, \eta_n) = \gamma(P,Q) \tag{2.13}$$

for all $n \geq 1$. On the other hand, since

$$\begin{aligned}
\gamma(\varkappa^*, Q) &\leq \gamma(\varkappa^*, \eta_n) \\
&\leq \gamma(\varkappa^*, \varkappa_n) + \gamma(\varkappa_n, \eta_n) - \gamma(\varkappa_n, \varkappa_n) \\
&= \gamma(\varkappa^*, \varkappa_n) - \gamma(\varkappa_n, \varkappa_n) + \gamma(P,Q) \\
&\leq \gamma(\varkappa^*, \varkappa_n) - \gamma(\varkappa_n, \varkappa_n) + \gamma(\varkappa^*, Q),
\end{aligned}$$

we have $\gamma(\varkappa^*, \eta_n) \to \gamma(\varkappa^*, Q)$ as $n \to \infty$. Since Q is an approximately compact with respect to P, we get that there exists a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ such that $\lim_{k\to\infty} d_\gamma(\eta_{n_k}, \eta^*) = 0$ for some $\eta^* \in Q$. Hence, from (2.12) and (2.13), we have $\gamma(\varkappa^*, \eta^*) = \gamma(P,Q)$ and so, $\varkappa^* \in P_0$.

Taking into account Remark 2, we can obtain the following corollary.

Corollary 1. *Let (Λ,γ) be a complete partial metric space, $\varnothing \neq P,Q \subseteq \Lambda$ where P is closed and Q is approximately compact with respect to P. Assume that $T : P \to Q$ is a generalized proximal BW-contraction with $T(P_0) \subseteq Q_0$ and $P_0 \neq \varnothing$. Then T has a best proximity point $\varkappa^*$ in Λ. Moreover, $\gamma(\varkappa^*, \varkappa^*) = 0$.*

Now, we present an example to illustrate both the effectiveness of Theorem 2 and the meaningfulness of Remark 2.

Example 1. Let $\Lambda = (\{0\} \cup [1,\infty)) \times [0,\infty)$ endowed with the partial metric γ defined by

$$\gamma(\varkappa, \eta) = \begin{cases} \frac{\varkappa_1}{2} &, \quad \varkappa = \eta \\ \varkappa_1 + \eta_1 + |\varkappa_2 - \eta_2| &, \quad \varkappa \neq \eta \end{cases}$$

for all $\varkappa = (\varkappa_1, \varkappa_2), \eta = (\eta_1, \eta_2) \in \Lambda$. Consider the subsets P and Q as

$$P = (\{0\} \cup [2,\infty)) \times [0,\infty)$$

and

$$Q = [1,2) \times [0,\infty).$$

Then, $\gamma(P,Q) = 1$, $P_0 = \{0\} \times [0,\infty)$ and $Q_0 = \{1\} \times [0,\infty)$ Further, we have (P_0,γ) is complete. Now, we define mappings $T : P \to Q$ and $\varphi : [0,\infty) \to [0,\infty)$ by

$$T(\varkappa_1,\varkappa_2) = \begin{cases} (1, \frac{\varkappa_2}{1+\varkappa_2}) & , \quad \varkappa_1 = 0 \\ (2 - \frac{1}{\varkappa_1}, \varkappa_2) & , \quad \text{otherwise} \end{cases}$$

and

$$\varphi(\lambda) = \begin{cases} \frac{1}{2} & , \quad \lambda = 0 \\ \frac{\lambda}{1+\lambda} & , \quad \lambda \neq 0 \end{cases}.$$

It can be seen that $T(P_0) \subseteq Q_0$ and T is a generalized proximal BW-contraction. Then, all hypotheses of Theorem 2 are satisfied. Therefore, the mapping T has a best proximity point $\varkappa^*$ in P. Moreover $\gamma(\varkappa^*,\varkappa^*) = 0$.

Note that Q is not approximately compact with respect to P. Indeed, let us consider the sequence $\eta_n = (1 + \frac{1}{2n}, 0)$ for all $n \geq 1$. In this case, $\gamma((0,0),\eta_n) \to \gamma((0,0),Q)$ as $n \to \infty$. However, the sequence $\{\eta_n\}$ does not have a d_γ-convergent subsequence in Q. Therefore Corollary 1 can not be applied to this example.

3. Generalized best BW-contractions

We start to this section by giving the definition of generalized best BW-contraction.

Definition 4. Let (Λ,γ) be a partial metric space, $\varnothing \neq P,Q \subseteq \Lambda$ and $T : P \to Q$ be a mapping. Then, T is called generalized best BW-contraction if there exists $\varphi \in \Phi$ such that

$$\gamma(T\varkappa, T\eta) \leq \varphi(R_T(\varkappa,\eta)) \tag{3.1}$$

for all $\varkappa,\eta \in P$.

Theorem 3. *Let (Λ,γ) be a complete partial metric space, P,Q be nonempty closed subsets of Λ and $T : P \to Q$ be a generalized best BW-contraction. Assume that the pair (P,Q) has the P-Property, $P_0 \neq \varnothing$ and $T(P_0) \subseteq Q_0$. Then, T has a best proximity point $\varkappa^*$ in P. Moreover, $\gamma(\varkappa^*,\varkappa^*) = 0$.*

Proof. Let $\varkappa_0 \in P_0$ be an arbitrary point. Since $T\varkappa_0 \in T(P_0) \subseteq Q_0$, there exists $\varkappa_1 \in P_0$ such that

$$\gamma(\varkappa_1, T\varkappa_0) = \gamma(P,Q). \tag{3.2}$$

Similarly, since $T\varkappa_1 \in T(P_0) \subseteq Q_0$ there exists $\varkappa_2 \in P_0$ such that

$$\gamma(\varkappa_2, T\varkappa_1) = \gamma(P,Q). \tag{3.3}$$

Considering (3.2), (3.3) and P-Property, we have

$$\gamma(\varkappa_1,\varkappa_2) = \gamma(T\varkappa_0, T\varkappa_1).$$

Repeating this process, we construct a sequence $\{\varkappa_n\}$ such that

$$\gamma(\varkappa_{n+1}, T\varkappa_n) = \gamma(P,Q) \tag{3.4}$$

and

$$\gamma(\varkappa_n,\varkappa_{n+1})=\gamma(T\varkappa_{n-1},T\varkappa_n) \tag{3.5}$$

for all $n\geq 1$. If there exists $n_0\geq 1$ such that $\gamma(\varkappa_{n_0},\varkappa_{n_0+1})=0$, then $\varkappa_{n_0}$ is a best proximity point of T. Moreover, $\gamma(\varkappa_{n_0},\varkappa_{n_0})=0$. Therefore, assume $\gamma(\varkappa_n,\varkappa_{n+1})>0$ for all $n\geq 1$. In equation (3.1), taking $\varkappa=\varkappa_n$ and $\eta=\varkappa_{n+1}$, we have

$$\gamma(\varkappa_{n+1},\varkappa_{n+2})=\gamma(T\varkappa_n,T\varkappa_{n+1})\leq\varphi(R_T(\varkappa_n,\varkappa_{n+1})). \tag{3.6}$$

Thus, from (3.4) and Lemma 2, we get

$$\gamma(\varkappa_{n+1},\varkappa_{n+2})\leq\varphi\left(\max\{\gamma(\varkappa_n,\varkappa_{n+1}),\gamma(\varkappa_{n+1},T\varkappa_{n+1})-\gamma(P,Q)\}\right) \tag{3.7}$$

for all $n\geq 1$. Now, if there exists $n_0\geq 1$ such that

$$\gamma(\varkappa_{n_0},\varkappa_{n_0+1})\leq\gamma(\varkappa_{n_0+1},T\varkappa_{n_0+1})-\gamma(P,Q)$$

then, we have

$$\begin{aligned}\gamma(\varkappa_{n_0+1},\varkappa_{n_0+2})&\leq\varphi(\gamma(\varkappa_{n_0+1},T\varkappa_{n_0+1})-\gamma(P,Q))\\&<\gamma(\varkappa_{n_0+1},T\varkappa_{n_0+1})-\gamma(P,Q)\\&\leq\gamma(\varkappa_{n_0+1},\varkappa_{n_0+2})+\gamma(\varkappa_{n_0+2},T\varkappa_{n_0+1})-\gamma(P,Q)\\&=\gamma(\varkappa_{n_0+1},\varkappa_{n_0+2})\end{aligned}$$

which is a contradiction. Therefore, we obtain

$$\gamma(\varkappa_n,\varkappa_{n+1})>\gamma(\varkappa_{n+1},T\varkappa_{n+1})-\gamma(P,Q)$$

for all $n\geq 1$ and so, from (3.7) we get

$$\gamma(\varkappa_{n+1},\varkappa_{n+2})=\gamma(T\varkappa_n,T\varkappa_{n+1})\leq\varphi(\gamma(\varkappa_n,\varkappa_{n+1}))<\gamma(\varkappa_n,\varkappa_{n+1}) \tag{3.8}$$

for all $n\geq 1$. Hence $\{\gamma(\varkappa_n,\varkappa_{n+1})\}$ is a decreasing sequences in $[0,\infty)$ and so it is convergent. Then, there exists $u\geq 0$ such that

$$\lim_{n\to\infty}\gamma(\varkappa_n,\varkappa_{n+1})=u.$$

We claim that $u=0$. Assume that $u>0$. Then, using equations (3.5) and (3.8), we have

$$\begin{aligned}u&=\lim_{n\to\infty}\gamma(\varkappa_n,\varkappa_{n+1})=\lim_{n\to\infty}\gamma(T\varkappa_{n-1},T\varkappa_n)\leq\lim_{n\to\infty}\varphi(\gamma(\varkappa_{n-1},\varkappa_n))\\&=\lim_{n\to\infty}\sup\varphi(\gamma(\varkappa_{n-1},\varkappa_n))\leq\varphi(u)<u,\end{aligned}$$

which is a contradiction. Therefore, $\lim_{n\to\infty}\gamma(\varkappa_n,\varkappa_{n+1})=0$. Now, we shall show that $\lim_{n,m\to\infty}\gamma(\varkappa_n,\varkappa_m)=0$. Assume the contrary, that is, there exist $\varepsilon>0$ and two sequences $\{\varkappa_{n_k}\}$ and $\{\varkappa_{m_k}\}$ of $\{\varkappa_n\}$ such that

$$\gamma(\varkappa_{n_k},\varkappa_{m_k})\geq\varepsilon \text{ and } \gamma(\varkappa_{n_k},\varkappa_{m_k-1})<\varepsilon \tag{3.9}$$

for all $m_k > n_k \geq k$ where m_k is the smallest natural number satisfying (3.9) corresponding to n_k. Then, we have

$$\varepsilon \leq \gamma(\varkappa_{n_k}, \varkappa_{m_k}) \leq \gamma(\varkappa_{n_k}, \varkappa_{m_k-1}) + \gamma(\varkappa_{m_k-1}, \varkappa_{m_k}) < \varepsilon + \gamma(\varkappa_{m_k-1}, \varkappa_{m_k}) \tag{3.10}$$

Letting $k \to \infty$ in inequality (3.10), we get

$$\lim_{k\to\infty} \gamma(\varkappa_{n_k}, \varkappa_{m_k}) = \varepsilon.$$

Then, we obtain

$$\begin{aligned}
&\gamma(\varkappa_{n_k}, \varkappa_{m_k}) \leq R_T(\varkappa_{n_k}, \varkappa_{m_k}) \\
&\quad = \max\{\gamma(\varkappa_{n_k}, \varkappa_{m_k}), S_T(\varkappa_{n_k}, \varkappa_{m_k}) - \gamma(P,Q)\} \\
&\quad \leq \gamma(\varkappa_{n_k}, \varkappa_{m_k}) + \gamma(\varkappa_{n_k}, \varkappa_{n_k+1}) + \gamma(\varkappa_{m_k}, \varkappa_{m_k+1}) + \frac{\gamma(\varkappa_{n_k}, \varkappa_{n_k+1})}{2} + \frac{\gamma(\varkappa_{m_k}, \varkappa_{m_k+1})}{2}
\end{aligned}$$

which implies that

$$\lim_{k\to\infty} R_T(\varkappa_{n_k}, \varkappa_{m_k}) = \varepsilon$$

Since $R_T(\varkappa_{n_k}, \varkappa_{m_k}) \geq \varepsilon$ for all $k \geq 1$, we have

$$\limsup_{k\to\infty} \varphi(R_T(\varkappa_{n_k}, \varkappa_{m_k})) \leq \varphi(\varepsilon).$$

On the other hand, for all $k \geq 1$, we obtain

$$\begin{aligned}
\varepsilon &\leq \gamma(\varkappa_{n_k}, \varkappa_{m_k}) \\
&\leq \gamma(\varkappa_{n_k}, \varkappa_{n_k+1}) + \gamma(\varkappa_{n_k+1}, \varkappa_{m_k+1}) + \gamma(\varkappa_{m_k}, \varkappa_{m_k+1}) \\
&\leq \gamma(\varkappa_{n_k}, \varkappa_{n_k+1}) + \varphi(R_T(\varkappa_{n_k}, \varkappa_{m_k})) + \gamma(\varkappa_{m_k}, \varkappa_{m_k+1})
\end{aligned}$$

and taking limit supremum we have

$$\varepsilon \leq \varphi(\varepsilon) < \varepsilon,$$

which is a contradiction. Hence, $\{\varkappa_n\}$ is a Cauchy sequence in P. From (3.5), we have $\{T\varkappa_n\}$ is a Cauchy sequence in Q. Since (Λ, γ) is complete partial metric space and P, Q are closed subsets of Λ, there exist $\varkappa^* \in P$ and $\eta^* \in Q$ such that

$$\lim_{n\to\infty} \gamma(\varkappa_n, \varkappa^*) = \gamma(\varkappa^*, \varkappa^*) = 0 = \lim_{n,m\to\infty} \gamma(\varkappa_n, \varkappa_m)$$

and

$$\lim_{n\to\infty} \gamma(T\varkappa_n, \eta^*) = \gamma(\eta^*, \eta^*) = 0 = \lim_{n,m\to\infty} \gamma(T\varkappa_n, T\varkappa_m).$$

Letting $n \to \infty$ in (3.4), we have

$$\gamma(\varkappa^*, \eta^*) = \gamma(P,Q).$$

From (3.1), we have

$$\begin{aligned}
&\gamma(\varkappa^*, T\varkappa^*) - \gamma(P,Q) \leq R_T(\varkappa_n, \varkappa^*) \\
&\qquad \leq \gamma(\varkappa_n, \varkappa^*) + \max\left\{ \begin{array}{c} \gamma(\varkappa_n, T\varkappa_n), \gamma(\varkappa^*, T\varkappa^*) \\ , \frac{\gamma(\varkappa^*, T\varkappa_n) + \gamma(\varkappa_n, T\varkappa^*)}{2} \end{array} \right\} - \gamma(P,Q)
\end{aligned}$$

and so taking limit $n \to \infty$ we have

$$\gamma(\varkappa^*, T\varkappa^*) - \gamma(P,Q) \le \lim_{n\to\infty} R_T(\varkappa_n, \varkappa^*) \le \gamma(\varkappa^*, T\varkappa^*) - \gamma(P,Q).$$

Therefore we have

$$\lim_{n\to\infty} R_T(\varkappa_n, \varkappa^*) = \gamma(\varkappa^*, T\varkappa^*) - \gamma(P,Q)$$

and then

$$\begin{aligned}\gamma(\eta^*, T\varkappa^*) &= \lim_{n\to\infty} \gamma(T\varkappa_n, T\varkappa^*)\\ &\le \lim_{n\to\infty} \sup \varphi(R_T(\varkappa_n, \varkappa^*))\\ &\le \varphi(\gamma(\varkappa^*, T\varkappa^*) - \gamma(P,Q)).\end{aligned}$$

Now, assume $\gamma(\varkappa^*, T\varkappa^*) > \gamma(P,Q)$. In this case, from the last inequality, we have

$$\gamma(\eta^*, T\varkappa^*) < \gamma(\varkappa^*, T\varkappa^*) - \gamma(P,Q).$$

Therefore we have

$$\begin{aligned}\gamma(\varkappa^*, T\varkappa^*) - \gamma(P,Q) &\le \gamma(\varkappa^*, \eta^*) + \gamma(\eta^*, T\varkappa^*) - \gamma(P,Q)\\ &= \gamma(\eta^*, T\varkappa^*)\\ &< \gamma(\varkappa^*, T\varkappa^*) - \gamma(P,Q),\end{aligned}$$

which is a contradiction. Therefore, $\gamma(\varkappa^*, T\varkappa^*) = \gamma(P,Q)$, that is, $\varkappa^*$ is a best proximity point of T. □

We aim to show the importance of Theorem 3 with the nontrivial following example.

Example 2. Let $\Lambda = [0,\infty) \times [0,\infty)$ and $\gamma : \Lambda \times \Lambda \to [0,\infty)$ be a function defined by

$$\gamma(\varkappa, \eta) = \max\{\varkappa_1, \eta_1\} + |\varkappa_2 - \eta_2|$$

for all $\varkappa = (\varkappa_1, \varkappa_2), \eta = (\eta_1, \eta_2) \in \Lambda$. Then, (Λ, γ) is a complete partial metric space. Consider the closed subsets P,Q of Λ as follows:

$$P = [0,\infty) \times \{0\}$$

and

$$Q = [0,\infty) \times \{1\}.$$

Then, $\gamma(P,Q) = 1$, $P_0 = \{(0,0)\}$ and $Q_0 = \{(0,1)\}$. Also, it can be seen that (P,Q) has P-property. Define the mappings $T : P \to Q$ and $\varphi : [0,\infty) \to [0,\infty)$ by

$$T(\varkappa, 0) = \begin{cases} (\frac{\varkappa}{6}, 1) & , \quad \varkappa \in [0, e^2) \\ (\arctan \varkappa, 1) & , \quad \varkappa \in [e^2, \infty) \end{cases}$$

for all $\varkappa \in [0,\infty)$ and

$$\varphi(\lambda) = \begin{cases} \frac{2\lambda}{e^2} & , \quad \lambda \in [0, e^2) \\ \ln \lambda & , \quad \lambda \in [e^2, \infty) \end{cases}$$

for all $\lambda \in [0,\infty)$. Then, we have $T(P_0) \subseteq Q_0$ and $\varphi \in \Phi$. Finally, considering the following cases, we shall show that T is a generalized best BW-contraction:
Case 1. Let $\varkappa = (\varkappa_1, 0)$ and $\eta = (\eta_1, 0)$ with $\varkappa_1, \eta_1 \in [0, e^2)$. Then, we have

$$\gamma(T\varkappa, T\eta) = \max\{\frac{\varkappa_1}{6}, \frac{\eta_1}{6}\} \leq \frac{2\max\{\varkappa_1, \eta_1\}}{e^2} = \varphi(\gamma(\varkappa, \eta)) \leq \varphi(R_T(\varkappa, \eta)).$$

Case 2. Let $\varkappa = (\varkappa_1, 0)$ and $\eta = (\eta_1, 0)$ with $\varkappa_1 \in [0, e^2)$ and $\eta_1 \in [e^2, \infty)$. Then, we have

$$\begin{aligned}\gamma(T\varkappa, T\eta) &= \max\{\frac{\varkappa_1}{6}, \arctan\eta_1\} = \arctan\eta_1 \\ &\leq \ln\eta_1 = \varphi(\gamma(\varkappa, \eta)) = 9\varphi(R_T(\varkappa, \eta)).\end{aligned}$$

Case 3: Let $\varkappa = (\varkappa_1, 0)$ and $\eta = (\eta_1, 0)$ with $\varkappa_1, \eta_1 \in [e^2, \infty)$. Then, we have

$$\begin{aligned}\gamma(T\varkappa, T\eta) &= \max\{\arctan\varkappa_1, \arctan\eta_1\} = \arctan(\max\{\varkappa_1, \eta_1\}) \\ &\leq \ln(\max\{\varkappa_1, \eta_1\}) = \varphi(\gamma(\varkappa, \eta)) = \varphi(R_T(\varkappa, \eta))\end{aligned}$$

The related inequalities can be seen in Figure 1 and Figure 2. Therefore, all hypotheses of Theorem 3 are satisfied and so the mapping T has a best proximity point $\varkappa^*$ in P. Moreover $\gamma(\varkappa^*, \varkappa^*) = 0$.

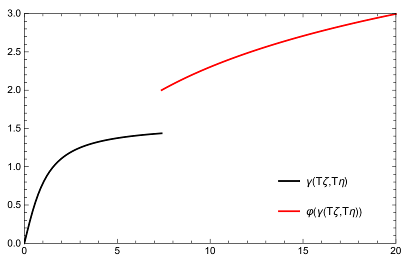

FIGURE 1. $\gamma(T\varkappa, T\eta)$ and $\varphi(\gamma(\varkappa, \eta))$ in Case 2.

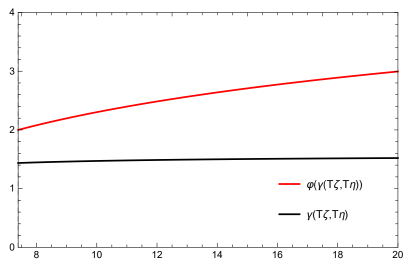

FIGURE 2. $\gamma(T\varkappa, T\eta)$ and $\varphi(\gamma(\varkappa, \eta))$ in Case 3.

From Theorem 3, we get the following corollary.

Corollary 2. *Let (Λ, γ) be a complete partial metric space, P, Q be nonempty closed subsets of Λ and $T : P \to Q$ be a mapping satisfying*

$$\gamma(T\varkappa, T\eta) \leq \varphi(\gamma(\varkappa, \eta))$$

for all $\varkappa, \eta \in P$, where $\varphi \in \Phi$. Assume that the pair (P, Q) has the P-Property, $P_0 \neq \varnothing$ and $T(P_0) \subseteq Q_0$. Then T has a best proximity point $\varkappa^*$ in P. Moreover, $\gamma(\varkappa^*, \varkappa^*) = 0$.

Taking $P = Q = \Lambda$ in both Theorem 2 and Theorem 3, we obtain Theorem 1 which is the main result of [21]. Moreover, in a similar way, from Corollary 2 we deduce the following fixed point result.

Corollary 3. *Let* (Λ,γ) *be a complete partial metric space and* $T:\Lambda\to\Lambda$ *be a mapping. Assume that the mapping* T *satisfies*

$$\gamma(T\varkappa,T\eta)\le\varphi(\gamma(\varkappa,\eta))$$

for all $\varkappa,\eta\in\Lambda$, *where* $\varphi\in\Phi$. *Then,* T *has a fixed point* $\varkappa^*$ *in* Λ. *Moreover,* $\gamma(\varkappa^*,\varkappa^*)=0$.

4. Homotopy result

Homotopy theory is one of the important parts of algebraic topology. Recently, the relationship of this topic with other branches of mathematics has revealed and has attracted the attention of many authors. Although there are many applications of fixed point results to homotopy theory, until now there is no any application of best proximity point results to homotopy theory [1, 11, 18]. In this section, we present an application of our new best proximity point result to homotopy theory. So, we investigate that if a mapping T satisfies all hypotheses of Corollary 2, then we show that all mappings which are homotopic to T also have a best proximity point. We begin this section by recalling the definition of homotopy.

Definition 5. Let (Λ_1,τ_1) and (Λ_2,τ_2) be topological spaces, $T,F:\Lambda_1\to\Lambda_2$ be continuous mappings. If there exists continuous function $H:\Lambda_1\times[0,1]\to\Lambda_2$ such that $H(\varkappa,0)=T\varkappa$ and $H(\varkappa,1)=F\varkappa$ for all $\varkappa\in\Lambda_1$, then it is said to be that T and F are homotopic mappings. Also, the mapping H is called homotopy.

In the rest of the paper, we denote the family of all functions in Φ satisfying the following implication by Φ_H as in [17]:

$$\lim_{n\to\infty}\{s_n-\varphi(s_n)\}=0\Rightarrow\lim_{n\to\infty}s_n=0 \tag{4.1}$$

for all sequence in $\{s_n\}\subseteq[0,\infty)$. The following example is important to show that the family Φ_H is nonempty set. Let's consider the sequence $(s_n)=\left(\frac{1}{n}\right)_{n\ge1}$ and the mapping $\varphi:[0,\infty)\to[0,\infty)$ defined by

$$\varphi(\lambda)=\begin{cases}0 & , \quad \lambda<1\\ \frac{1}{2\lambda} & , \quad \lambda\ge1\end{cases}.$$

Then, it can be easily seen that $\varphi\in\Phi$ and the implication (4.1) is satisfied. Hence, we have $\varphi\in\Phi_H$.

Now, we can present the main result of this section.

Theorem 4. *Let* (Λ,γ) *be a complete partial metric space,* P,Q *be nonempty closed subsets of* Λ *and* $\varnothing\ne U\subseteq P$. *Assume that the pair* (P,Q) *has P-Property and* $H:P\times[0,1]\to Q$ *is a mapping satisfying*

(*i*) $\gamma(\varkappa,H(\varkappa,\lambda))>\gamma(P,Q)$ *for all* $\varkappa\in P\backslash U$ *and* $\lambda\in[0,1]$,

(ii) *there exists* $\varphi \in \Phi_H$ *such that*

$$\gamma(H(\varkappa,\lambda),H(\eta,\lambda)) \le \varphi(\gamma(\varkappa,\eta)) \tag{4.2}$$

for all $\varkappa,\eta \in P$ *and* $\lambda \in [0,1]$,

(iii) *there exists a continuous function* $\eta : [0,1] \to [0,\infty)$ *such that*

$$\gamma(H(\varkappa,\lambda),H(\varkappa,s)) \le |\eta(\lambda)-\eta(s)|$$

for all $\varkappa \in P$ *and* $\lambda, s \in [0,1]$,

(iv) *for all* $\lambda \in [0,1]$ *satisfying* $\gamma(\varkappa,H(\varkappa,\lambda)) = \gamma(P,Q)$ *for some* $\varkappa \in U$, *there exists* $\varepsilon_\lambda > 0$ *such that* $H(P_0,\lambda^*) \subseteq Q_0$ *for all* $\lambda^* \in (\lambda-\varepsilon_\lambda,\lambda+\varepsilon_\lambda)$.

If $H(\cdot,0)$ *has a best proximity point in* P, *then* $H(\cdot,1)$ *has a best proximity point in* P.

Proof. Define a set

$$K = \{\lambda \in [0,1] : \gamma(\varkappa,H(\varkappa,\lambda)) = \gamma(P,Q) \text{ for some } \varkappa \in U\}.$$

Since $H(\cdot,0)$ has a best proximity point in P and (i) holds, then $0 \in K$. Hence, K is a nonempty set. We shall show that K is both open and closed in $[0,1]$ and hence by connectedness of $[0,1]$, we have that $K = [0,1]$. We first show that K is closed. For this, let $\{\lambda_n\}$ be a sequence in K with $\lambda_n \to \lambda^* \in [0,1]$ as $n \to \infty$. Using definition of K, there exists $\varkappa_n \in U$ such that

$$\gamma(\varkappa_n,H(\varkappa_n,\lambda_n)) = \gamma(P,Q) \tag{4.3}$$

for all $n \ge 1$. Then, from P-Property and (ii), we have

$$\begin{aligned}\gamma(\varkappa_n,\varkappa_m) &= \gamma(H(\varkappa_n,\lambda_n),H(\varkappa_m,\lambda_m))\\ &\le \gamma(H(\varkappa_n,\lambda_n),H(\varkappa_n,\lambda_m)) + \gamma(H(\varkappa_n,\lambda_m),H(\varkappa_m,\lambda_m))\\ &\le |\eta(\lambda_n)-\eta(\lambda_m)| + \varphi(\gamma(\varkappa_n,\varkappa_m))\end{aligned}$$

for all $n,m \ge 1$. Since η is a continuous function and the sequence $\{\lambda_n\}$ is convergent, we have

$$\lim_{n,m\to\infty} \{\gamma(\varkappa_n,\varkappa_m) - \varphi(\gamma(\varkappa_n,\varkappa_m))\} = 0 \text{ as } n,m \to \infty.$$

From (4.1), we have $\lim_{n,m\to\infty} \gamma(\varkappa_n,\varkappa_m) = 0$. Therefore, $\{\varkappa_n\}$ is a Cauchy sequence. Since (Λ,γ) is complete and P is a closed subset of Λ, there exists $\varkappa^* \in P$ such that

$$\lim_{n,m\to\infty} \gamma(\varkappa_n,\varkappa_m) = \lim_{n\to\infty} \gamma(\varkappa_n,\varkappa^*) = \gamma(\varkappa^*,\varkappa^*) = 0.$$

Now, we have to consider the following two cases:

(a) Assume that there exists $n_0 \ge 1$ such that $\varkappa_n = \varkappa^*$ for all $n \ge n_0$. In this case, from (4.3), we have

$$\gamma(\varkappa^*,H(\varkappa^*,\lambda_n)) = \gamma(P,Q) \tag{4.4}$$

for all $n \ge n_0$. Also, from (iii), we obtain

$$\gamma(H(\varkappa^*,\lambda_n),H(\varkappa^*,\lambda^*)) \le |\eta(\lambda_n)-\eta(\lambda^*)|$$

for all $n \geq n_0$. Then, we have $\lim_{n\to\infty} \gamma(H(\varkappa^*,\lambda_n),H(\varkappa^*,\lambda^*)) = 0$ and so, from (4.4), we get

$$\gamma(\varkappa^*,H(\varkappa^*,\lambda^*)) = \lim_{n\to\infty} \gamma(\varkappa^*,H(\varkappa^*,\lambda_n)) = \gamma(P,Q).$$

(b) Now, assume the contrary to (a), that is, there exists a subsequence $\{\varkappa_{n_k}\}$ of $\{\varkappa_n\}$ such that $\varkappa_{n_k} \neq \varkappa^*$ for all $k \geq 1$. Then, from (4.3), we have

$$\begin{aligned}
\gamma(P,Q) &\leq \gamma(\varkappa_{n_k},H(\varkappa^*,\lambda^*)) \\
&\leq \gamma(\varkappa_{n_k},H(\varkappa_{n_k},\lambda_{n_k})) + \gamma(H(\varkappa_{n_k},\lambda_{n_k}),H(\varkappa_{n_k},\lambda^*)) + \gamma(H(\varkappa_{n_k},\lambda^*),H(\varkappa^*,\lambda^*)) \\
&\leq \gamma(P,Q) + |\eta(\lambda_{n_k}) - \eta(\lambda^*)| + \varphi(\gamma(\varkappa_{n_k},\varkappa^*)) \\
&< \gamma(P,Q) + |\eta(\lambda_{n_k}) - \eta(\lambda^*)| + \gamma(\varkappa_{n_k},\varkappa^*)
\end{aligned}$$

and so

$$\gamma(\varkappa^*,H(\varkappa^*,\lambda^*)) = \lim_{k\to\infty} \gamma(\varkappa_{n_k},H(\varkappa^*,\lambda^*)) = \gamma(P,Q).$$

Thus, in both cases, we obtain $\lambda^* \in K$ and so K is closed in $[0,1]$.

Now, we shall show that K is open. Let $\lambda_0 \in K$. Then, there exists $\varkappa_0 \in U$ such that $\gamma(\varkappa_0,H(\varkappa_0,\lambda_0)) = \gamma(P,Q)$. From (iv), for $\lambda_0 \in [0,1]$, there exists $\varepsilon_{\lambda_0} > 0$ such that $H(P_0,\lambda^*) \subseteq Q_0$ for all $\lambda^* \in (\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0})$. If we consider the mappings $H(\cdot,\lambda^*) : P \to Q$ for all $\lambda^* \in (\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0})$, then, from (ii), the mappings $H(\cdot,\lambda^*)$ are BW-contraction. Therefore, all hypotheses of Corollary 2 are satisfied. Hence, for all $\lambda^* \in (\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0})$, $H(\cdot,\lambda^*)$ has a best proximity point $\varkappa^*_{\lambda^*}$ in P. Moreover, $\gamma(\varkappa^*_{\lambda^*},\varkappa^*_{\lambda^*}) = 0$. From (i), $\varkappa^*_{\lambda^*} \in U$ for all $\lambda^* \in (\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0})$. Hence, we have $(\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0}) \subseteq K$, that is, K is open in $[0,1]$. □

Taking $Q = \Lambda$ in Theorem 4, we obtain the following corollary.

Theorem 5. *Let (Λ,γ) be a complete partial metric space, P be a nonempty closed subset of Λ and $\varnothing \neq U \subseteq P$. Assume that $H : P \times [0,1] \to \Lambda$ is a mapping satisfying*

(i) *$\gamma(\varkappa,H(\varkappa,\lambda)) > 0$ for all $\varkappa \in P \backslash U$ and $\lambda \in [0,1]$,*

(ii) *there exists a $\varphi \in \Phi_H$ such that*

$$\gamma(H(\varkappa,\lambda),H(\eta,\lambda)) \leq \varphi(\gamma(\varkappa,\eta))$$

for all $\varkappa,\eta \in P$ and $\lambda \in [0,1]$,

(iii) *there exists a continuous function $\eta : [0,1] \to [0,\infty)$ such that*

$$\gamma(H(\varkappa,\lambda),H(\varkappa,s)) \leq |\eta(\lambda) - \eta(s)|$$

for all $\varkappa \in P$ and $\lambda, s \in [0,1]$,

(iv) *for all $\lambda \in [0,1]$ satisfying $\varkappa = H(\varkappa,\lambda)$ for some $\varkappa \in U$, there exists $\varepsilon_\lambda > 0$ such that $H(P,\lambda^*) \subseteq P$ for all $\lambda^* \in (\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda)$.*

If $H(\cdot,0)$ has a fixed point in P, then $H(\cdot,1)$ has a fixed point P.

Proof. Assume that $H(\cdot,0)$ has a fixed point $\varkappa$ in P. In this case, since $\varkappa = H(\varkappa,0)$, we have

$$\gamma(\varkappa,\varkappa) \leq \gamma(H(\varkappa,0),H(\varkappa,0)) = 0.$$

Hence, $\gamma(\varkappa,\varkappa) = 0$ and so $\gamma(P,\Lambda) = 0$. Hence the conditions (*i*), (*ii*) and (*iii*) of Theorem 4 are hold. Further, we have $P_0 = Q_0 = P$ whenever $Q = \Lambda$, and so condition (*iv*) is also hold. Therefore, there exists $\varkappa^* \in P$ such that

$$\gamma(\varkappa^*,H(\varkappa^*,1)) = \gamma(P,\Lambda) = 0.$$

This shows that $\varkappa^*$ is a fixed point of $H(\cdot,1)$. □

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