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DIAGONAL COMMON QUADRATIC LYAPUNOV FUNCTIONS FOR SETS OF POSITIVE LTI SYSTEMS

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Abstract. This paper focuses on the problems of a diagonal common quadratic Lyapunov function (DCQLF) existence for sets of stable positive linear time-invariant (LTI) systems. We derive the equivalent algebraic conditions to verify the existence of a DCQLF, namely that the finite number Hurwitz Mezler matrices at least have a common diagonal Stein solution. Finally some reduced cases are considered.

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1. INTRODUCTION

In dynamical systems, an additional frequent and inherent constraint is the nonnegativity of the states. Many physical systems in the real world involve variables that have nonnegative sign: population levels, absolute temperature, concentration of substances, and so on. Such systems are referred to as positive systems which means that any trajectory of the system starting at an initial state in the positive orthant remains forever (see [4, 9]). This feature makes analysis and synthesis of positive systems a challenging and interesting job (see, for example, [3, 5, 6, 12] and references therein).

In this paper we are interested in the general stability properties of switched positive systems. Our objective in this paper is on the diagonal quadratic Lyapunov function existence for such systems with stable subsystems. Diagonal quadratic Lyapunov functions play a central role in the study of positive linear time-invariant(LTI) systems. For general LTI systems $\Sigma_A : \dot{x}(t) = Ax(t)$ with $A \in \mathbb{R}^{n \times n}$ and the corresponding Stein matrix inequalities

$$A^T P + PA < 0,$$

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where $P \in \mathbb{R}^{n\times n}$ is positive definite symmetric matrix, we can find a P satisfying the Stein matrix inequalities if and only if A is stable, i.e., each eigenvalue of A has negative real part. We call such a positive definite matrix Stein solution for A. If we can find a Stein solution $D = P$ in the set of diagonal matrices, we say A is diagonally stable, and $V(x) = x^T Dx$ is called a diagonal quadratic Lyapunov function for system Σ_A. We also say a set of matrices is simultaneously diagonal stable if there exists a diagonal matrix that is a Stein solution D for all matrices belonging such a set. Here $V(x) = x^T Dx$ is called a DCQLF for the LTI systems with these matrices. In practice, the diagonal Lyapunov functions are required to assure the stability of quantized systems if a finite precision arithmetic is used to calculate the states of systems[14]. The basic results about the diagonal Lyapunov functions were presented by [15]. This question has attracted a great deal of attention in the past [1, 2, 10, 11, 13, 16–20]. In general, the solution approaches on diagonal stability problem follows three distinct lines of enquiry. One line can be concluded from the work of [11, 20] and others. By checking the negative diagonal entry existence of the matrix AX for all positive semi-definite X, the authors attempted to derive algebraic conditions to verify the existence of a DCQLF. The second line is mainly based on the structure of several convex cones that arise in the study of DCQLF[13]. The authors considered a pair of positive LTI systems and demonstrated how this structure can be used to derive conditions for the existence of such functions. The authors of [17–19] provided the third approach. Through making use of both A and A^{-1}, the author derived the diagonal stability conditions. In particular, conditions for diagonal stability are given in his paper in terms of a common quadratic Lyapunov function existence problem for a pair of lower dimensional systems constructed from A and A^{-1}.

The purpose of this paper is to present some results about the simultaneous diagonal stability of Hurwitz Metzler matrices. By showing a common diagonal Stein solution existence for a finite number Hurwitz Metzler matrices, we object to determine some tractable conditions for the existence of a DCQLF for sets of positive LTI systems. The organization of this paper is as follows. The preparations will be made in Section 2. Section 3 is dedicated to derive some checkable necessary and sufficient conditions on the problem of DCQLF existence for a finite number of positive LTI systems. Some concluding remarks are presented in Section 4.

2. Preliminaries

Throughout, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^n$ stands for the n-dimensional real vector space and $\mathbb{R}^{n\times n}$ is the space of $n\times n$ matrices. For A in $\mathbb{R}^{n\times n}$, a_{kl} denotes the element in the (k,l) position of A. $A \succeq 0 (\preceq 0)$ means that all elements of matrix A are nonnegative (nonpositive) and $A \succ 0 (\prec 0)$ means that all elements of matrix A are positive (negative). $A \succeq B (A \succ B)$ implies that all elements of matrix $A - B$ are nonnegative (positive). The notion $A > 0 (< 0)$ means that A is a symmetric positive (negative) definite matrix and $A \geq 0 (\leq 0)$ means that matrix A is symmetrical positive (negative) semi-definite. $A = 0$ implies that all elements of A are zero and $A \neq 0$ stands for non-zero matrix. A^T (A^{-1}) represents the transpose (inverse) of matrix A. Let $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$. $\lambda(A)$ represents the eigenvalue of A and $\mu(A)$ denotes the real part of the eigenvalue $\lambda(A)$ of A. Also, when referring to the linear switched systems in this paper, stability shall be used to denote uniform asymptotic stability under arbitrary switching signals. Before proceeding, we recall some facts which are relevant for the work of this paper.

Firstly, Perron-Frobenius Theorem pointed in the following guarantees that the eigen-space of irreducible matrix[7] A corresponding to $\lambda(A)$ is one dimensional.

Lemma 1. *Given an irreducible matrix $A \succeq 0$ in $\mathbb{R}^{n\times n}$, then*

(1) *$\lambda(A)$ is an eigenvalue of A with algebraic multiplicity one.*
(2) *There exists a vector $x \succ 0$ such that $Ax = \lambda(A)x$.*

In [6], Farina and Rinaldi described the basic theory and several applications of positive linear systems. Such systems relate to Metzler and Hurwitz matrices. A matrix A is called a Metzler matrix if and only if $a_{kl} \geq 0$ for $k \neq l$, and Hurwitz matrix if and only if the real part $\mu(A)$ of the eigenvalue $\lambda(A)$ satisfies $\mu(A) < 0$. Now we collect some facts to introduce positive systems.

Theorem 1. *For $A \in \mathbb{R}^{n\times n}$, consider the associated LTI system $\Sigma_A : x(t) = Ax(t)$ with the initial condition $x_0 = x(0)$, then the following statements are equivalent.*

(1) *The LTI system Σ_A is positive.*
(2) *For all $t \geq 0$, the condition $x_0 \succeq 0$ implies that $x(t) \succeq 0$.*
(3) *A is a Metzler matrix.*

Theorem 1 actually gives the definition of positive systems. Furthermore, a classic result guarantees the stability of positive LTI systems.

Theorem 2. *The positive LTI system Σ_A is stable if and only if $A \in \mathbb{R}^{n\times n}$ is a Hurwitz Metzler matrix.*

Therefore, from Theorem 1 and 2, when referring to Hurwitz Metzler matrices in the sequel, this is equivalent to the stable positive LTI systems. Applying the properties of Hurwitz Metzler matrix, some results follows straightforward.

Lemma 2 ([13])**.** *Assume $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix such that $a_{kl} \neq 0$ for $1 \leq k, l \leq n$, and a diagonal matrix $D \in \mathbb{R}^{n\times n}$ satisfies $D \geq 0$ and $D \neq 0$. Then*

(1) *The matrix $A^T D + DA$ is Metzler.*
(2) *The matrix $A^T D + DA$ is irreducible.*

Here if replacing $a_{kl} \neq 0$ with irreducible and Hurwitz properties of A, we can also derive that $A^T D + DA$ are Metzler and irreducible.

Lemma 3. *Let matrices A, B in $\mathbb{R}^{n\times n}$ be Hurwitz Metzler, for any diagonal matrix $D \in \mathbb{R}^{n\times n}$, then $A^T D + DA = 0$ if and only if $B^T D + DB = 0$.*

Lemma 4 ([8])**.** *Assume that A, B in $\mathbb{R}^{n\times n}$ are Metzler matrices with $a_{kl}, b_{kl} \neq 0$ for $1 \le k, l \le n$, respectively. If $A \succeq B$ and A is Hurwitz, then B is also Hurwitz.*

Paper [8] pointed that, for two Hurwitz Metzler matrices A, B and any positive number δ, the Hurwitz of matrix pencil $A + \delta B$ is equivalent to its non-singular property. This fact can be easy to extend the finite numbers case as follows.

Lemma 5. *Suppose that $A_1, \cdots, A_m$ are Hurwitz Metzler in $\mathbb{R}^{n\times n}$, for all positive real numbers $\delta_1, \cdots, \delta_m$, then the following statements are equivalent.*

(1) *$\sum_{j=1}^{m} \delta_j A_j$ is a Hurwitz matrix.*
(2) *The determination* $\det \sum_{j=1}^{m} \delta_j A_j \neq 0$.

As stable LTI systems have diagonal Lyapunov functions, it is natural to ask under what conditions families of such systems will possess a DCQLF. The next section shall focus on this problem.

3. Diagonal stability

In this section, a necessary and sufficient condition for the existence of a DCQLF will be derived for sets of stable positive LTI systems. To be more precise, assume that $A_1, \cdots, A_m$ in $\mathbb{R}^{n\times n}$ are Hurwitz Metzler matrices with no zero entries, Theorem 3 provides a equivalent condition for the existence of a positive definite diagonal matrix D in $\mathbb{R}^{n\times n}$ such that $A_j^T D + DA_j < 0$ for all $j = 1, \cdots, m$.

However, before stating that theorem, we need the following technical result.

Lemma 6. *Assume that $A_1, \cdots, A_m \in \mathbb{R}^{n\times n}$ are Hurwitz Metzler matrices satisfying $a_{jkl} \neq 0$ for $1 \le k, l \le n$ and $1 \le j \le m$, then there exist some diagonal matrices $D_j > 0$ in $\mathbb{R}^{n\times n}$ such that*

$$\sum_{j=1}^{m} \det D_j A_j D_j = 0 \tag{3.1}$$

holds if the following conditions are satisfied:

(1) *The positive LTI systems $\Sigma_{A_1}, \cdots, \Sigma_{A_m}$ have no DCQLF.*
(2) *There is a diagonal matrix $\tilde{D} \in \mathbb{R}^{n\times n}$ satisfying $\tilde{D} \ge 0$ and $\tilde{D} \neq 0$ such that*

$$A_j^T \tilde{D} + \tilde{D} A_j \le 0, \ \forall j = 1, \cdots, m. \tag{3.2}$$

Proof. The proof of this lemma is broken into three steps.

Step 1: We claim that there exist m vectors $y_j \succ 0$ in $\mathbb{R}^n$ such that

$$(A_j^T \tilde{D} + \tilde{D} A_j) y_j = 0, \ \forall j = 1, \cdots, m. \tag{3.3}$$

For simplicity, write $\tilde{R}_j = A_j^T\tilde{D} + \tilde{D}A_j$. In fact, for each j, since $A_j \in \mathbb{R}^{n\times n}$ with $a_{jkl} \neq 0$ $(1 \le k,l \le n)$ is Metzler matrices, and diagonal matrix $\tilde{D} \ge 0, \tilde{D} \neq 0$, it is easy to check from Lemma 2 that $\tilde{R}_j$ is also irreducible Metzler matrices. Then in this case, there is $\alpha > 0$ large enough such that $\tilde{R}_j + \alpha I \succeq 0$. Obviously, $\tilde{R}_j + \alpha I$ is also irreducible, together (3.2) and Lemma 1 yields that $\lambda(\tilde{R}_j + \alpha I) = \alpha$ is an eigenvalue of algebraic multiplicity one for $\tilde{R}_j + \alpha I$. This implies that $\lambda(\tilde{R}_j) = 0$ is an eigenvalue of algebraic multiplicity one for $\tilde{R}_j$, Therefore, the rank of $\tilde{R}_j$ is $n-1$. Finally, by Lemma 1, we can conclude that there exist m vectors $y_j \succ 0$ such that (3.3) holds for all j as the claim.

Step 2: We wish to show that there is no diagonal matrix $\hat{D} \in \mathbb{R}^{n\times n}$ satisfying

$$y_j^T(A_j^T\hat{D} + \hat{D}A_j)y_j < 0,\ \forall j = 1,\cdots,m, \tag{3.4}$$

where y_j defined by (3.3).

To this end, conversely, for any vectors $x_j \succ 0$, suppose that there are some diagonal $\hat{D}$ satisfying (3.4) for all j. We shall find a diagonal $D > 0$ in $\mathbb{R}^{n\times n}$ such that

$$A_j^T D + DA_j < 0 \tag{3.5}$$

simultaneously hold for all j, which would contradict with statement (1).

With this in mind, for all j, consider the sets

$$\Omega_j = \{y \in \mathbb{R}^n : y^T y = 1 \ \text{ and } \ y^T(A_j^T\hat{D} + \hat{D}A_j)y \ge 0\}. \tag{3.6}$$

Now distinguish two cases.

Case 1: $\Omega_j = \varnothing$ for all j.

Under this case, $y^T(A_j^T\hat{D} + \hat{D}A_j)y < 0$ with $y^T y = 1$, together with (3.2) yields that for any positive constant $\epsilon_j > 0$,

$$y^T\left(A_j^T(\tilde{D} + \epsilon_j\hat{D}) + (\tilde{D} + \epsilon_j\hat{D})A_j\right)y < 0. \tag{3.7}$$

Since the inequality (3.7) is unchange if we scale y by any non-zero real number. That is, each $\tilde{D} + \epsilon_j\hat{D} > 0$ results from the Hurwitz of A_j. (3.5) thus holds by setting $D = \tilde{D} + \epsilon\hat{D}$ with $\epsilon = \min_{j=1}^m\{\epsilon_j\}$. A contradiction occurs.

Case 2: $\Omega_j \neq \varnothing$ for all j.

For vector $y \in \mathbb{R}^n$ with $y^T y = 1$, note that (3.4) is true as assumption, then $y_j \in \mathbb{R}^n/\Omega_j$, it follows from (3.3) that for any $y \in \Omega_j$

$$y^T(A_j^T\tilde{D} + \tilde{D}A_j)y < 0. \tag{3.8}$$

Now consider two subcases: $y \in \Omega_j$ and $y \in \mathbb{R}^n/\Omega_j$.

On the one hand, if $y \in \Omega_j$, then

$$y^T(A_j^T\hat{D} + \hat{D}A_j)y \ge 0. \tag{3.9}$$

From (3.8) and (3.9), one can guarantee the following inequality

$$y^T\left(A_j^T(\tilde{D}+\epsilon_j^{'}\hat{D})+(\tilde{D}+\epsilon_j^{'}\hat{D})A_j\right)y<0 \tag{3.10}$$

holds, where $\epsilon_j^{'}<|\tilde{M}_j|/\hat{M}_j$ with

$$\hat{M}_j=\max\{y^T(A_j^T\hat{D}+\hat{D}A_j)y|y\in\Omega_j\},$$

$$\tilde{M}_j=\max\{y^T(A_j^T\tilde{D}+\tilde{D}A_j)y|y\in\Omega_j\}.$$

Then it is easy to see that (3.10) holds for any $y\in\mathbb{R}^n$. This means that $\tilde{D}+\epsilon_j^{'}\hat{D}>0$. Therefore, (3.5) holds by setting $D=\tilde{D}+\epsilon^{'}\hat{D}$ with $\epsilon^{'}=\min\{\epsilon_j^{'},j=1,\cdots,m\}$, which contradict with statement (1).

On the other hand, if $y\in\mathbb{R}^n/\Omega_j$, then $y^T(A_j^T\hat{D}+\hat{D}A_j)y<0$, taking (3.2) into account yields that for $\epsilon_j^{'}$

$$y^T\left(A_j^T(\tilde{D}+\epsilon_j^{'}\hat{D})+(\tilde{D}+\epsilon_j^{'}\hat{D})A_j\right)y<0. \tag{3.11}$$

With the same discussion as case $y\in\Omega_j$, a contradiction shall occur.

Now based on the argument above, we thus know that (3.4) is not true.

Step 3: Now it remains to show that there exist m diagonal matrices $D_j>0$ ($j=1,\ldots,m$) such that (3.1) holds.

In Step 2, we have shown that there is no diagonal solution to (3.4) for all j. Now for diagonal $\hat{D}$, Hurwitz Metzler matrices A_j and vector $y_j\succ 0$, it follows that

$$\begin{aligned} &y_{j_k}^T(A_{j_k}^T\hat{D}+\hat{D}A_{j_k})y_{j_k}<0, 1\le k\le l-1\\ \Leftrightarrow &y_{j_k}^T(A_{j_k}^T\hat{D}+\hat{D}A_{j_k})y_{j_k}>0, l\le k\le m \end{aligned} \tag{3.12}$$

with $j_k\in\{1,\cdots,m\}$ and $1\le l<m$. On the other hand, from Lemma 3 we can know

$$y_1^T(A_1^T\hat{D}+\hat{D}A_1)y_1=0\Leftrightarrow\cdots\Leftrightarrow y_m^T(A_m^T\hat{D}+\hat{D}A_m)y_m=0. \tag{3.13}$$

Furthermore, taking (3.13) and (3.12) into account, we can obtain that for a constant $c>0$

$$\sum_{k=1}^{l-1}y_{j_k}^T(A_{j_k}^T\hat{D}+\hat{D}A_{j_k})y_{j_k}=-c\sum_{k=l}^{m}y_{j_k}^T(A_{j_k}^T\hat{D}+\hat{D}A_{j_k})y_{j_k}. \tag{3.14}$$

From (3.14) we may replace y_{j_k} with $y_{j_k}/\sqrt{c}$, in this case, due to $y_j\succ 0$, now fix a vector y_{j_u} $(1\le u\le l-1)$, then for any vector y_j $(j\ne j_u)$, we can find a associated diagonal matrix $D_j>0$ for all j such that

$$D_{j_u}y_{j_u}=D_jy_j. \tag{3.15}$$

Moreover, substituting (3.15) into (3.14) yields that

$$y_{j_u}^T(A_{j_u}^T\hat{D}+\hat{D}A_{j_u})y_{j_u}+\sum_{k\neq u,k=1}^{l-1} y_{j_u}^T D_{j_u}D_{j_k}^{-1}(A_{j_k}^T\hat{D}+\hat{D}A_{j_k})D_{j_k}^{-1}D_{j_u}y_{j_u}$$
$$=-\sum_{k=l}^{m} y_{j_u}^T D_{j_u}D_{j_k}^{-1}(A_{j_k}^T\hat{D}+\hat{D}A_{j_k})D_{j_k}^{-1}D_{j_u}y_{j_u}. \quad (3.16)$$

By induction, (3.16) implies

$$\det\left[(A_{j_u}^T\hat{D}+\hat{D}A_{j_u})+\sum_{k\neq u,k=1}^{m} D_{j_u}D_{j_k}^{-1}(A_{j_k}^T\hat{D}+\hat{D}A_{j_k})D_{j_k}^{-1}D_{j_u}\right]=0. \quad (3.17)$$

Now pre- and post-multiplying (3.17) by $D_{j_u}^{-1}$, respectively, then (3.17) is equivalent to

$$\det\left[\sum_{j=1}^{m} D_j^{-1}(A_j^T\hat{D}+\hat{D}A_j)D_j^{-1}\right]=2\det\sum_{j=1}^{m} D_j^{-1}A_j^T\hat{D}D_j^{-1}=0. \quad (3.18)$$

As diagonal matrices are commutative, for all j, replacing D_j^{-1} in (3.18) by D_j, (3.1) follows immediately. This completes the proof. □

Theorem 3. *Given m Hurwitz Metzler matrices $A_1,\cdots,A_m\in\mathbb{R}^{n\times n}$ with $a_{jkl}\neq 0$ for $1\le k,l\le n$ and $j=1,\cdots,m$, then the following statements are equivalent:*

(1) *The positive LTI systems $\Sigma_{A_1},\cdots,\Sigma_{A_m}$ share a DCQLF.*
(2) *For any m diagonal matrices $D_j>0$ in $\mathbb{R}^{n\times n}$,*

$$\det\sum_{j=1}^{m} D_jA_jD_j\neq 0. \quad (3.19)$$

Proof. (1)⇒ (2) According to definition of DCQLFs, the statement implies that for all j and diagonal matrix $D>0$

$$A_j^TD+DA_j<0 \quad (3.20)$$

simultaneous hold. By (3.20), for any diagonal $D_j>0$, we have

$$(D_jA_jD_j)^TD+D(D_jA_jD_j)=D_j(A_j^TD+DA_j)D_j<0. \quad (3.21)$$

One can further get from (3.21) that

$$(\sum_{j=1}^{m} D_jA_jD_j)^TD+D(\sum_{j=1}^{m} D_jA_jD_j)=\sum_{j=1}^{m} D_j(A_j^TD+DA_j)D_j<0. \quad (3.22)$$

That is, $\sum_{j=1}^{m} D_jA_jD_j$ is a Hurwitz matrix and thus non-singular.

(2)⇒ (1). We shall prove this by contradiction. First of all, suppose that the statement (1) is not true, i.e. $\Sigma_{A_1},\cdots,\Sigma_{A_m}$ have no DCQLF. In this case, for all j, if choose sufficiently large $\xi_j > 0$, then there is always a DCQLF for $\Sigma_{A_1-\xi_1 I},\cdots,\Sigma_{A_m-\xi_m I}$. Now, for any $j = 1,\cdots,m$, define

$$\tilde{\xi}_j = \inf\left\{\xi_j > 0 | \Sigma_{A_1-\xi_1 I},\cdots,\Sigma_{A_m-\xi_m I} \text{ have a DCQLF}\right\},$$

and further let $\tilde{\xi} = \min\{\tilde{\xi}_j, j = 1,\cdots,m\}$. Then according to Lemma 3, one can check that the matrices $A_1-\tilde{\xi}I,\cdots,A_m-\tilde{\xi}I$ satisfy the conditions of Lemma 6. Then there exist m diagonal matrices $D_j > 0$ such that

$$\det\sum_{j=1}^{m} D_j(A_j-\tilde{\xi}I)D_j = 0. \tag{3.23}$$

Moreover, from (3.23), matrix $\sum_{j=1}^{m} D_j A_j D_j$ has a positive real eigenvalue and thus is not Hurwitz. Note that all $D_j A_j D_j$ are Hurwitz Metzler matrices, it follows from Lemma 5 that there exist positive real numbers $\delta_1,\cdots,\delta_m$ such that

$$\det\sum_{j=1}^{m} \delta_j D_j A_j D_j = 0. \tag{3.24}$$

Finally, for all j, replacing $\sqrt{\delta_j}D_j$ in (3.20) by D_j, it turn out that the statement (2) is not true in this case. This completes the proof. $\square$

The above result follows from the problem of determining whether a DCQLF exists for sets of positive LTI systems. We have presented a equivalent condition for such problem through algebraic argument. Obviously, if sets of positive LTI systems share a DCQLF, each individual LTI system must have a DCQLF.

Note that the comments below Lemma 2, in Theorem 3, if we replace the condition of $a_{jkl} \neq 0$ with the irreducibility of $A_1,\cdots,A_m$, then the necessary and sufficient condition between statements (1) and (2) also holds. Next we shall consider some reduced cases. These results are straightforward from Theorem 3.

Corollary 1. *If $A_1,\cdots,A_m \in \mathbb{R}^{n\times n}$ are Hurwitz diagonal matrices, then LTI systems $\Sigma_{A_1},\cdots,\Sigma_{A_m}$ are positive and thus share a DCQLF.*

Under diagonal matrices, Corollary 1 just restricts the Hurwitz properties of system matrices. In fact, as Hurwitz diagonal, the positivity and irreducibility are naturally satisfied. Next we consider upper and lower triangular matrices.

Corollary 2. *Suppose that $A_1,\cdots,A_m \in \mathbb{R}^{n\times n}$ with $a_{jkl} \neq 0$ for $1 \leq k,l \leq n$ are Hurwitz Metzler matrix. If $A_1,\cdots,A_m \in \mathbb{R}^{n\times n}$ are upper (lower) triangular, then positive LTI systems $\Sigma_{A_1},\cdots,\Sigma_{A_m}$ have a DCQLF.*

Applying Lemma 4, Corollary 3 and 4 follow immediately.

Corollary 3. *For $i, j = 1, \cdots, m$, given m Metzler matrices $A_1, \cdots, A_m \in \mathbb{R}^{n\times n}$ with $a_{jkl} \neq 0$ for $1 \leq k, l \leq n$. If there is a matrix A_j such that A_j is Hurwitz and satisfies $A_j \succeq A_i$ for $i \neq j$, then positive LTI systems $\Sigma_{A_1}, \cdots, \Sigma_{A_m}$ have a DCQLF.*

Corollary 4. *For $i, j = 1, \cdots, m$, given m Metzler matrices $A_1, \cdots, A_m \in \mathbb{R}^{n\times n}$ with $a_{jkl} \neq 0$ for $1 \leq k, l \leq n$, and some diagonal $D_j > 0$. If there is a matrix A_j such that A_j is Hurwitz and satisfies $D_j A_j D_j \succeq A_i$ for $i \neq j$, then positive LTI systems $\Sigma_{A_1}, \cdots, \Sigma_{A_m}$ have a DCQLF.*

For 2×2 Hurwitz matrices with negative diagonal entries, this is equivalent to the determinants being positive. We thus obtain the following result.

Corollary 5. *If Hurwitz Metzler matrices $A_1, \cdots, A_m$ with $a_{jkl} \neq 0$ $(1 \leq k, l \leq n)$ are in $\mathbb{R}^{2\times 2}$, then positive LTI systems $\Sigma_{A_1}, \cdots, \Sigma_{A_m}$ share a DCQLF.*

4. Conclusions

In this paper, a necessary and sufficient condition for the existence of a DCQLF has been derived for families of positive LTI systems. By algebraic argument, we answered that under what conditions a finite number Hurwitz Metzler matrices have a common diagonal Stein solution. In other words, the simultaneous diagonal stability can be achieved for system matrices of positive LTI systems. Furthermore some results for reduced cases followed from such a result.

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