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ON THE DIOPHANTINE EQUATIONS $z^2 = f(x)^2 \pm f(y)^2$ INVOLVING LAURENT POLYNOMIALS II.

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Abstract. By the theory of Pell's equation, we give conditions for $f(x) = b + \frac{c}{x}$ with $b, c \in \mathbb{Z}\setminus\{0\}$ such that the Diophantine equations $z^2 = f(x)^2 \pm f(y)^2$ have infinitely many solutions $x, y \in \mathbb{Z}$ and $z \in \mathbb{Q}$, which gives a positive answer to Question 3.2 of Zhang and Shamsi Zargar [16]. By the theory of elliptic curve, we study the non-trivial rational solutions of the above Diophantine equations for Laurent polynomials $f(x) = \frac{\prod_{t=0}^{n}(x+k^t)}{x}$, $\frac{\prod_{t=0}^{n}(x-k^t)(x+k^t)}{x}$, $n \geq 1$, $k \in \mathbb{Z}\setminus\{0, \pm 1\}$, and give a positive answer to Question 3.1 of Zhang and Shamsi Zargar [16].

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1. INTRODUCTION

Let $f(x) \in \mathbb{Q}[x]$ be a polynomial without multiple roots and $\deg f \geq 2$. Many authors considered the non-trivial integer and rational (parametric) solutions of the Diophantine equations

$$z^2 = f(x)^2 + f(y)^2 \tag{1.1}$$

and

$$z^2 = f(x)^2 - f(y)^2 \tag{1.2}$$

for different polynomials $f(x)$. Let us recall that a triple (x, y, z) is a non-trivial solution of Eq. (1.1) (respectively, Eq. (1.2)) if $f(x)f(y) \neq 0$ (respectively, $f(x)^2 \neq f(y)^2$, $f(y) \neq 0$).

In 1962, W. Sierpiński [5] obtained infinitely many non-trivial positive integer solutions of Eq. (1.1) for $f(x) = \frac{x(x+1)}{2}$. In 2010, M. Ulas and A. Togbé [9] studied the non-trivial rational solutions of Eqs. (1.1) and (1.2) for $f(x)$ being quadratic and cubic

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polynomials. At the same year, B. He, A. Togbé and M. Ulas [4] further investigated the non-trivial integer solutions of Eqs. (1.1) and (1.2) for some special polynomials $f(x)$. In 2018, Y. Zhang and A. Shamsi Zargar [15] proved that Eq. (1.1) has infinitely many non-trivial rational solutions for the quartic polynomials $f(x)=x(x-1)(x+1)(x+\frac{1-k^2}{2k})$, $k\in\mathbb{Z}\backslash\{0,\pm1\}$ and Eq. (1.2) has infinitely many rational solutions for the quartic polynomials $f(x)=x(x-1)(x+1)(x-\frac{2k}{k^2+1})$, $k\in\mathbb{Z}\backslash\{0,\pm1\}$, which gave a positive answer to **Question 4.3** of [9] for quartic polynomials. In 2019, A. E. A. Youmbai and D. Behloul [11] extended the results of Zhang-Shamsi Zargar [15] to the polynomials $x(\prod_{t=0}^{n}(x-k^t)(x+k^t))$ of degree $2n+3$ and gave a positive answer to **Question 4.3** of [9] for the polynomials $x(\prod_{t=0}^{n}(x+k^t))$ of degree $n+2$.

As a generalization of Eqs. (1.1) and (1.2), in 2017, Sz. Tengely and M. Ulas [8] investigated the existence of the non-trivial integer solutions of the Diophantine equations $z^2=f(x)^2\pm g(y)^2$ and proved similar results for some special higher degree polynomials as well.

A Laurent polynomial with coefficients in a field $\mathbb{F}$ is an expression of the form

$$f(x)=\sum_k a_k x^k,\quad a_k\in\mathbb{F},$$

where x is a formal variable, the summation index k is an integer (not necessarily positive) and only finitely many coefficients a_k are non-zero.

In 2018, Y. Zhang [13] studied the non-trivial rational parametric solutions of the Diophantine equation

$$f(x)f(y)=f(z)^n \tag{1.3}$$

for $n=1,2$, involving the Laurent polynomials $f(x)=ax+b+c/x$. In 2019, Y. Zhang and A. Shamsi Zargar [14] further investigated the non-trivial rational (parametric) solutions of Eq. (1.3), where $f(x)=x^k+ax^{k-1}+b/x, k\geq 2,\ x^2+a/x+b/x^2$ for $n=1$, and $f(x)=x^2+ax+b+a^3/(27x),\ x^2+ax+b+a^3/(16x)+a^4/(256x^2)$ for $n=2$.

In 1783, L. Euler [2, p. 167] studied the rational solutions of Eq. (1.1) for $f(x)=x+\frac{1}{x}$. In 2020, Y. Zhang and A. Shamsi Zargar [16] continued the study of Euler and considered the non-trivial rational (parametric) solutions of Eqs. (1.1) and (1.2) for some simple Laurent polynomials, such as

$$f(x)=x+b+\frac{c}{x},\quad \frac{(x+1)(x+b)(x+c)}{x}$$

with $b,c\in\mathbb{Z}\backslash\{0\}$.

By the theory of Pell's equation and the same method of Zhang [12, Theorem 1.1], we give a positive answer to **Question 3.2** of Zhang and Shamsi Zargar [16] in Theorem 1.

Theorem 1. *1) For*

$$f(x)=b+\frac{c}{x}$$

with $b, c \in \mathbb{Z}\backslash\{0\}$, *if there exists an integer* y_0 *such that* $(2b^2y_0^2 + 2bcy_0 + c^2)x^2 + 2bcy_0^2x + c^2y_0^2 = v^2$ *has a non-zero integer solution* (x_0, v_0) *and* $2b^2y_0^2 + 2bcy_0 + c^2 > 0$ *is not a perfect square, then Eq. (1.1) has infinitely many non-trivial solutions* (x, y_0, z) *with* $x, y_0 \in \mathbb{Z}$ *and* $z \in \mathbb{Q}$.

2) For

$$f(x) = b + \frac{c}{x}$$

with $b, c \in \mathbb{Z}\backslash\{0\}$, *if there is an integer* x_0 *such that* $c(2bx_0 + c)y^2 - 2bcx_0^2y - c^2x_0^2 = v^2$ *has a non-zero integer solution* (y_0, v_0) *and* $c(2bx_0 + c) > 0$ *is not a perfect square, then Eq. (1.2) has infinitely many non-trivial solutions* (x_0, y, z) *with* $x_0, y \in \mathbb{Z}$ *and* $z \in \mathbb{Q}$.

Following the methods of Youmbai and Behloul [11, Theorems 1.1-1.4] and Zhang and Shamsi Zargar [15, Theorems 1.1-1.2], we give a positive answer to **Question 3.1** of Zhang and Shamsi Zargar [16] in Theorems 2 and 3.

Theorem 2. *For*

$$f(x) = \frac{\prod_{t=0}^{n}(x + k^t)}{x}, \quad n \geq 1$$

with $k \in \mathbb{Z}\backslash\{0, \pm 1\}$, *Eqs. (1.1) and (1.2) have a rational parametric solution.*

Theorem 3. *When*

$$f(x) = \frac{\prod_{t=0}^{n}(x - k^t)(x + k^t)}{x}, \quad n \geq 1,$$

for all but finitely many $k \in \mathbb{Z}\backslash\{0, \pm 1\}$, *Eqs. (1.1) and (1.2) have infinitely many non-trivial rational solutions.*

By the map $x \mapsto 1/x$, we have

Proposition 1. *For*

$$f(x) = \frac{\prod_{t=0}^{n}(k^t x + 1)}{x^n}, \quad n \geq 1$$

with $k \in \mathbb{Z}\backslash\{0, \pm 1\}$, *Eqs. (1.1) and (1.2) have a rational parametric solution.*

When

$$f(x) = \frac{\prod_{t=0}^{n}(1 - k^t x)(1 + k^t x)}{x^{2n+1}}, \quad n \geq 1,$$

for all but finitely many $k \in \mathbb{Z}\backslash\{0, \pm 1\}$, *Eqs. (1.1) and (1.2) have infinitely many non-trivial rational solutions.*

2. PRELIMINARIES

To prove Theorem 1, we need the following lemma about the integer solutions of Pell's equation.

Lemma 1 (Corollary in [3]). *Let m_1, m_2, D be positive integers, D is not a square, and $a^2 - Db^2 = M$, then there are infinitely many integer solutions (u,v) of the Pell's equation $u^2 - Dv^2 = M$ with*

$$u \equiv a \pmod{m_1} \quad \text{and} \quad v \equiv b \pmod{m_2}.$$

To simplify the proof of Theorem 3, we state the following useful Lemma.

Lemma 2. *The quartic curve $C:\ V^2 = aU^4 + bU^3 + cU^2 + dU + e^2$ is birationally equivalent to an elliptic curve with Weierstrass equation*

$$E:\ Y^2 = X^3 - 27(12ae^2 - 3bd + c^2)X + 27(2c^3 - 72ace^2 + 27b^2e^2 + 27ad^2 - 9bcd),$$

by the map $\varphi : C \ni (U,V) \mapsto (X,Y) \in E$

$$X = \frac{3(cU^2 + 3dU + 6e^2 + 6eV)}{U^2},$$
$$Y = \frac{27(beU^3 + 2ceU^2 + 3deU + dUV + 4e^3 + 4e^2V)}{U^3},$$

and its inverse map φ^{-1} is

$$U = -\frac{3(24ce^2 + 4e^2X - 9d^2)}{54be^2 - 9cd + 3dX - 2eY},$$
$$V = -\frac{N(V)}{(54be^2 - 9cd + 3dX - 2eY)^2},$$

where

$$N(V) = -8e^3X^3 - 9e(8ce^2 - 3d^2)X^2 + 4e^3Y^2 + (-216be^4 + 108cde^2 - 27d^3)Y + 27e(108b^2e^4 - 108bcde^2 + 32c^3e^2 + 27bd^3 - 9c^2d^2).$$

Proof. This is a modified version in [10, p. 37, Theorem 2.17]. □

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. 1) For

$$f(x) = b + \frac{c}{x}$$

with $b, c \in \mathbb{Z}\backslash\{0\}$, Eq. (1.1) equals

$$z^2 = \frac{(2b^2y^2 + 2bcy + c^2)x^2 + 2bcy^2x + c^2y^2}{x^2y^2}.$$

To get integral values of x and y, let us consider the integer solutions (x,v) of the quadratic equation

$$(2b^2y^2+2bcy+c^2)x^2+2bcy^2x+c^2y^2=v^2.$$

Put $U=(2b^2y^2+2bcy+c^2)x+bcy^2,\ \ V=v$, then we get the Pell's equation

$$U^2-(2b^2y^2+2bcy+c^2)V^2=-c^2y^2(by+c)^2.$$

Take $y=y_0$, then $U=(2b^2y_0^2+2bcy_0+c^2)x+bcy_0^2$ and

$$U^2-(2b^2y_0^2+2bcy_0+c^2)V^2=-c^2y_0^2(by_0+c)^2. \tag{3.1}$$

Note that if $2b^2y_0^2+2bcy_0+c^2>0$ is not a perfect square, then the Pell's equation

$$U^2-(2b^2y_0^2+2bcy_0+c^2)V^2=1$$

has infinitely many integer solutions. If there exists an integer y_0 such that $(2b^2y_0^2+2bcy_0+c^2)x^2+2bcy_0^2x+c^2y_0^2=v^2$ has a non-zero integer solution (x_0,v_0), then

$$(U_0,V_0)=((2b^2y_0^2+2bcy_0+c^2)x_0+bcy_0^2,\ v_0)$$

is an integer solution of Eq. (3.1). So there are infinitely many integer solutions of Eq. (3.1).

Note that $U_0=(2b^2y_0^2+2bcy_0+c^2)x_0+bcy_0^2$ satisfies

$$U_0\equiv bcy_0^2 \pmod{2b^2y_0^2+2bcy_0+c^2}.$$

By Lemma 1, Eq. (3.1) has infinitely many integer solutions U satisfying the above condition. Therefore, there are infinitely many

$$x=\frac{U-bcy_0^2}{2b^2y_0^2+2bcy_0+c^2}\in\mathbb{Z}.$$

Thus, for

$$f(x)=b+\frac{c}{x}$$

with $b,c\in\mathbb{Z}\backslash\{0\}$, Eq. (1.1) has infinitely many non-trivial solutions (x,y_0,z) with $x,y_0\in\mathbb{Z}$ and $z\in\mathbb{Q}$.

2) For

$$f(x)=b+\frac{c}{x}$$

with $b,c\in\mathbb{Z}\backslash\{0\}$, Eq. (1.2) becomes

$$z^2=\frac{c(2bx+c)y^2-2bcx^2y-c^2x^2}{x^2y^2}.$$

To get integral values of x and y, let us consider the integer solutions (y,v) of the quadratic equation

$$c(2bx+c)y^2-2bcx^2y-c^2x^2=v^2.$$

Take $U=c(2bx+c)y-bcx^2,\ \ V=v$, then we obtain the Pell's equation

$$U^2-c(2bx+c)V^2=-c^2x^2(bx+c)^2.$$

By the same method as 1), we can give the proof of 2). □

Example 1. When $b=c=1$, $f(x)=1+\frac{1}{x}$, Eq. (1.1) reduces to

$$z^2=\frac{(2y^2+2y+1)x^2+2xy^2+y^2}{x^2y^2}.$$

Consider $(2y^2+2y+1)x^2+2xy^2+y^2=v^2$. If $y_0=1$, we have

$$5x^2+2x+1=v^2.$$

It is easy to check that $(x_0,v_0)=(2,5)$ is a solution of the above equation, and $2y_0^2+2y_0+1=5>0$ is not a perfect square. By the theory of Pell's equation, $5x^2+2x+1=v^2$ has infinitely many integer solutions (x,v). Then Eq. (1.1) has infinitely many non-trivial solutions $(x,1,z)$ with $x\in\mathbb{Z}$ and $z\in\mathbb{Q}$.

Remark 1. In fact, we can use the transformation $x=T,\ \ y=cT$ to study Eqs. (1.1) and (1.2). But we also need some conditions about b,c to get infinitely many non-trivial solutions. Here, we give two simple cases to display this method.

1) When $b=1$, $c\in\mathbb{Z}\backslash\{0,1\}$, $f(x)=1+\frac{c}{x}$. Let

$$x=T,\quad y=cT.$$

Then Eq. (1.1) equals

$$z^2=\frac{2T^2+(2c+2)T+c^2+1}{T^2}.$$

To get integral values of x and y, let us consider the integer solutions (T,S) of the quadratic equation

$$2T^2+(2c+2)T+c^2+1=S^2.$$

Let $U=2T+c+1,\ \ V=S$, then we get the Pell's equation

$$U^2-2V^2=-(c-1)^2.$$

Note that $(U,V)=(c-1,c-1)$ is an integer solution of the above Pell's equation, and $(U,V)=(3,2)$ is an integer solution of the Pell's equation $U^2-2V^2=1$. Thus, an infinity of integer solutions of $U^2-2V^2=-(c-1)^2$ are given by

$$U_m+V_m\sqrt{2}=(3+2\sqrt{2})^m(c-1+(c-1)\sqrt{2}),\ m\geq 0.$$

Then

$$\begin{cases} U_m = 6U_{m-1} - U_{m-2}, & U_0 = c-1, \ U_1 = 7(c-1); \\ V_m = 6V_{m-1} - V_{m-2}, & V_0 = c-1, \ V_1 = 5(c-1). \end{cases}$$

It is easy to prove that

$$U_m \equiv c+1 \pmod 2, \quad V_m = S_m \in \mathbb{Z}.$$

Then we have

$$T_m = \frac{U_m - (c+1)}{2} \in \mathbb{Z}, \ \ m \geq 0.$$

So

$$x_m = T_m \in \mathbb{Z}, \quad y_m = cT_m \in \mathbb{Z}, \quad z_m = \frac{S_m}{T_m} \in \mathbb{Q}, \ \ m \geq 0.$$

2) For

$$f(x) = 1 + \frac{2c}{x}$$

with $c \in \mathbb{Z} \backslash \{0,1\}$, let

$$x = T, \quad y = cT.$$

Then Eq. (1.2) reduces to

$$z^2 = \frac{4(c-1)(c+T+1)}{T^2}.$$

To get integral values of x and y, let us consider the integer solutions (T,S) of the following equation

$$(c-1)(c+T+1) = S^2.$$

Solve it for T, then we have

$$T = \frac{S^2}{c-1} - c - 1.$$

Put $S = (c-1)u$, then

$$T = (c-1)u^2 - c - 1.$$

So

$$x = (c-1)u^2 - c - 1 \in \mathbb{Z}, \quad y = cx \in \mathbb{Z}, \quad z = \frac{2u(c-1)}{(c-1)u^2 - c - 1} \in \mathbb{Q},$$

where u is an integer parameter.

Proof of Theorem 2. 1) For

$$f(x)=\frac{\prod_{t=0}^{n}(x+k^t)}{x},\quad n\geq 1$$

with $k\in\mathbb{Z}\backslash\{0,\pm 1\}$, let

$$x=T,\quad y=kT.$$

Then Eq. (1.1) equals

$$z^2=\left(\frac{\prod_{t=0}^{n-1}(x+k^t)}{T}\right)^2\left((k^{2n}+1)T^2+2k^n(k^{n-1}+1)T+k^{2n-2}(k^2+1)\right).$$

Consider the conic section

$$\mathcal{C}_1:\ S^2=(k^{2n}+1)T^2+2k^n(k^{n-1}+1)T+k^{2n-2}(k^2+1).$$

Take $U=T+\frac{1}{k},\ \ V=kS$, then we obtain

$$\mathcal{C}_{1,k}:\ V^2=k^2(k^{2n}+1)U^2+2k(k^{n+1}-1)U+(k^{n+1}-1)^2,$$

which can be parametrized by

$$U=\frac{2(1-k^{n+1})(t-k)}{t^2-k^2(k^{2n}+1)},$$
$$V=\frac{(1-k^{n+1})(t^2-2kt+k^2(k^{2n}+1))}{t^2-k^2(k^{2n}+1)},$$

where t is a rational parameter. Then Eq. (1.1) has a rational parametric solution

$$(x,y,z)=\left(U-\frac{1}{k},\ kU-1,\ \frac{\prod_{t=0}^{n-1}(U-\frac{1}{k}+k^t)}{kU-1}V\right),$$

where U,V are given in above.

2) For

$$f(x)=\frac{\prod_{t=0}^{n}(x+k^t)}{x},\quad n\geq 1$$

with $k\in\mathbb{Z}\backslash\{0,\pm 1\}$, let

$$x=T,\quad y=kT.$$

Then Eq. (1.2) becomes

$$z^2=\left(\frac{\prod_{t=0}^{n-1}(x+k^t)}{T}\right)^2\left((-k^{2n}+1)T^2+2k^n(-k^{n-1}+1)T+k^{2n-2}(k^2-1)\right).$$

Consider

$$\mathcal{C}_2:\ S^2=(1-k^{2n})T^2+2k^n(1-k^{n-1})T+k^{2n-2}(k^2-1).$$

Put $U = T + \frac{1}{k},\ \ V = kS$, then we have

$$\mathcal{C}_{2,k}:\ V^2 = k^2(1-k^{2n})U^2 + 2k(k^{n+1}-1)U + (k^{n+1}-1)^2.$$

The remainder of the proof is similar as 1), we omit it. □

Proof of Theorem 3. 1) For

$$f(x) = \frac{\prod_{t=0}^{n}(x-k^t)(x+k^t)}{x},\ \ n \geq 1$$

with $k \in \mathbb{Z}\backslash\{0,\pm 1\}$, let

$$x = T, \quad y = kT.$$

Then Eq. (1.1) reduces to

$$z^2 = \left(\frac{\prod_{t=0}^{n-1}(x^2-k^{2t})}{T}\right)^2 \left((k^{4n+2}+1)T^4 - 2k^{2n}(k^{2n}+1)T^2 + k^{4n-2}(k^2+1)\right).$$

Consider the quartic curve

$$\mathcal{C}_3:\ S^2 = (k^{4n+2}+1)T^4 - 2k^{2n}(k^{2n}+1)T^2 + k^{4n-2}(k^2+1).$$

Take $U = T - \frac{1}{k},\ \ V = k^2 S$, then we get

$$\begin{aligned}\mathcal{C}_3(n,k):\ V^2 = {} & k^4(k^{4n+2}+1)U^4 + 4k^3(k^{4n+2}+1)U^3 + 2k^2(2k^{4n+2}-k^{2n+2}+3)U^2 \\ & - 4k(k^{2n+2}-1)U + (k^{2n+2}-1)^2.\end{aligned}$$

The discriminant of $\mathcal{C}_3(n,k)$ is

$$256k^{12n+18}(k^2+1)(k^{4n+2}+1)(k^{2n+2}-1)^4.$$

So $\mathcal{C}_3(n,k)$ is smooth, when $k \neq 0, \pm 1$. By Lemma 2, the corresponding elliptic curve $\mathcal{E}_3(n,k)$ of $\mathcal{C}_3(n,k)$ is

$$\begin{aligned}\mathcal{E}_3(n,k):\ Y^2 = {} & X^3 - 108k^{4n+6}(3k^{4n+4}+4k^{4n+2}+2k^{2n+2}+4k^2+3)X \\ & + 432k^{6n+10}(k^{2n}+1)(9k^{4n+4}+8k^{4n+2}-2k^{2n+2}+8k^2+9).\end{aligned}$$

In order to prove Theorem 3, it needs to show that the curve $\mathcal{C}_3(n,k)$ has infinitely many rational points, equivalently, $\mathcal{E}_3(n,k)$ has a rational point of infinite order. It is easy to see that the rational points $(U,V) = (0, \pm(k^{2n+2}-1))$ on $\mathcal{C}_3(n,k)$ lead to rational points of order 2 on $\mathcal{E}_3(n,k)$. Since $f(k^n) = 0$, Eq. (1.1) has a trivial solution $(k^n, k^{n+1}, f(k^{n+1}))$. From this observation, we find another rational point on $\mathcal{C}_3(n,k)$:

$$P = \left(k^n - \frac{1}{k}, k^{2n+1}(k^{2n+2}-1)\right).$$

By the map $\varphi_1 : \mathcal{C}_3(n,k) \ni (U,V) \mapsto (X,Y) \in \mathcal{E}_3(n,k)$, we can obtain the point

$$Q = \varphi_1(P) = (6k^{2n+3}(3k^{2n+2}+2k^{2n+1}+6k^{n+1}+2k+3), 108k^{3n+5}k^{n+1}(k^{n+1}+1)^2)$$

on $\mathcal{E}_3(n,k)$. By the group law, we have

$$[2]Q = \left(\frac{3k^{2n+2}(3k^{4n+4}+4k^{4n+2}+8k^{3n+2}+14k^{2n+2}+8k^{n+2}+4k^2+3)}{(k^n+1)^2}, \right.$$
$$\left. -\frac{27k^{3n+3}(k^{2n+2}+1)(k^{4n+4}-4k^{3n+2}-6k^{2n+2}-4k^{n+2}+1)}{(k^n+1)^3} \right).$$

Let $\mathcal{E}_3(n,2)$ be the specialization of $\mathcal{E}_3(n,k)$ at $k=2$, and the specialization of $[2]Q$ at $k=2$ is

$$[2]Q_2 = \left(\frac{3\times 2^{2n+2}(2^{4n+6}+2^{3n+5}+7\times 2^{2n+3}+2^{n+5}+19)}{(2^n+1)^2}, \right.$$
$$\left. -\frac{27\times 2^{3n+3}(2^{2n+2}+1)(2^{4n+4}-2^{3n+5}-3\times 2^{2n+3}-2^{n+4}+1)}{(2^n+1)^3} \right).$$

A quick calculation reveals that the remainder of the division of the numerator of the X-coordinate of the point $[2]Q_2$ by its denominator equals

$$|r| = 18(2^{n+8}+2^{n+2}+210),$$

which is non-zero. So the X-coordinate of $[2]Q_2$ is not a polynomial. For $1\le n\le 12$, one can check that

$$\frac{|r|}{(2^n+1)^2}$$

is not an integer except for $n=1,2$, and that it is nonzero and is less than 1 in modulus for $n>12$. Then for integers $n\neq 1,2$, the point $[2]Q_2$ has non-integral X-coordinate and hence, by Nagell-Lutz Theorem (see [7, p.56]), is of infinite order. Thus, $\mathcal{E}_3(n,k)$ has a positive rank in the field $\mathbb{Q}(k)$. By the Specialization Theorem of Silverman (see [6, p.457, Theorem 20.3]), when $n\neq 1,2$, for all but finitely many $k\in\mathbb{Z}\backslash\{0,\pm1\}$, $\mathcal{E}_3(n,k)$ has a positive rank and infinitely many rational points.

When $n=1,2$, we have

$$\mathcal{E}_3(1,2):\ Y^2 = X^3 - 118886400X + 399900672000,$$
$$\mathcal{E}_3(2,2):\ Y^2 = X^3 - 29251141632X + 1385178693894144.$$

Using the package of Magma [1], the ranks of the above two elliptic curves are 2, hence, they have infinitely many rational points. Therefore, when $n\ge 1$, for all but finitely many $k\in\mathbb{Z}\backslash\{0,\pm1\}$, $\mathcal{E}_3(n,k)$ has infinitely many rational points, i.e., the curve $\mathcal{C}_3(n,k)$ has infinitely many rational points.

Thus, when

$$f(x) = \frac{\prod_{t=0}^{n}(x-k^t)(x+k^t)}{x},\ \ n\ge 1,$$

for all but finitely many $k\in\mathbb{Z}\backslash\{0,\pm1\}$, Eq. (1.1) has infinitely many non-trivial rational solutions.

2) For

$$f(x) = \frac{\prod_{t=0}^{n}(x-k^t)(x+k^t)}{x}, \quad n \geq 1$$

with $k \in \mathbb{Z}\backslash\{0, \pm 1\}$, let

$$x = T, \quad y = kT.$$

Then Eq. (1.2) equals

$$z^2 = \left(\frac{\prod_{t=0}^{n-1}(x^2-k^{2t})}{T}\right)^2 \left((-k^{4n+2}+1)T^4 + 2k^{2n}(k^{2n}-1)T^2 + k^{4n-2}(k^2-1)\right).$$

Consider the quartic curve

$$\mathcal{C}_4: \ S^2 = (-k^{4n+2}+1)T^4 + 2k^{2n}(k^{2n}-1)T^2 + k^{4n-2}(k^2-1).$$

Put $U = T - \frac{1}{k}, \ V = k^2 S$, then we have

$$\begin{aligned}\mathcal{C}_4(n,k): \ V^2 = &\ k^4(1-k^{4n+2})U^4 + 4k^3(1-k^{4n+2})U^3 - 2k^2(2k^{4n+2}+k^{2n+2}-3)U^2 \\ &- 4k(k^{2n+2}-1)U + (k^{2n+2}-1)^2.\end{aligned}$$

The discriminant of $\mathcal{C}_4(n,k)$ is

$$-256k^{12n+18}(k^2-1)(k^{4n+2}-1)(k^{2n+2}-1)^4.$$

So $\mathcal{C}_4(n,k)$ is smooth, when $k \neq 0, \pm 1$. By Lemma 2, the corresponding elliptic curve $\mathcal{E}_4(n,k)$ of $\mathcal{C}_4(n,k)$ is

$$\begin{aligned}\mathcal{E}_4(n,k): \ Y^2 = &\ X^3 + 108k^{4n+6}(3k^{4n+4} - 4k^{4n+2} + 2k^{2n+2} - 4k^2 + 3)X \\ &+ 432k^{6n+10}(k^{2n}-1)(9k^{4n+4} - 8k^{4n+2} - 2k^{2n+2} - 8k^2 + 9).\end{aligned}$$

By the method of Fermat [2, p. 639], from the point $P_0 = (0, k^{2n+2}-1)$, we can get another point

$$P' = \left(-\frac{4(k^{4n+4}-1)}{k(k^{4n+4}+2k^{2n+2}+4k^{4n+2}-3)}, \ \frac{(k^{2n+2}-1)N(k)}{(k^{4n+4}+2k^{2n+2}+4k^{4n+2}-3)^2}\right)$$

on $\mathcal{E}_4(n,k)$, where

$$\begin{aligned}N(k) = &\ k^{8n+8} - 24k^{8n+6} + 16k^{8n+4} - 4k^{6n+6} - 16k^{6n+4} + 6k^{4n+4} - 24k^{4n+2} \\ &- 4k^{2n+2} + 1.\end{aligned}$$

By the map $\varphi_2 : \mathcal{C}_4(n,k) \ni (U,V) \mapsto (X,Y) \in \mathcal{E}_4(n,k)$, we get the rational point

$$\begin{aligned}Q' &= \varphi_2(P') \\ &= \Big(3(3k^{8n+8} - 40k^{8n+6} + 48k^{8n+4} + 4k^{6n+6} + 16k^{6n+4} + 50k^{4n+4} - 40k^{4n+2} \\ &\quad + 4k^{2n+2} + 3)/(4(k^{2n+2}+1)^2), \\ &\quad -27(k^{4n+4} - 4k^{4n+2} - 2k^{2n+2} + 1)(k^{8n+8} + 16k^{8n+6} - 16k^{8n+4} + 4k^{6n+6}\end{aligned}$$

$$- 10k^{4n+4} + 16k^{4n+2} + 4k^{2n+2} + 1)/(8(k^{2n+2}+1)^3)\Big)$$

lying on $\mathcal{E}_4(n,k)$.

Using Nagell-Lutz Theorem and the Specialization Theorem of Silverman, we obtain the result in a similar way like in 1). □

Example 2. When $n = 1$, $k \in \mathbb{Z}\backslash\{0,\pm 1\}$,

$$f(x) = \frac{(x-1)(x+1)(x-k)(x+k)}{x},$$

we have

$$\mathcal{C}_3(1,k):\ V^2 = k^4(k^6+1)U^4 + 4k^3(k^6+1)U^3 + 2k^2(2k^6-k^4+3)U^2 - 4k(k^4-1)U + (k^4-1)^2$$

and

$$\mathcal{E}_3(1,k):\ Y^2 = X^3 - 108k^{10}(3k^4-2k^2+3)(k^2+1)^2X + 432k^{16}(k^2+1)^3(9k^4-10k^2+9).$$

From Theorem 3, $\mathcal{E}_3(1,k)$ has rational points:

$$Q(1) = (6k^5(k^2+1)(3k^2+2k+3), 108k^8(k+1)(k^2+1)^2),$$

$$[2]Q(1) = \left(\frac{3k^4(3k^8+4k^6+8k^5+14k^4+8k^3+4k^2+3)}{(k+1)^2}, -\frac{27k^6(k^8-4k^5-6k^4-4k^3+1)(k^4+1)}{(k+1)^3}\right).$$

By the group law, we get

$$[4]Q(1) = (X([4]Q(1)), Y([4]Q(1))),$$

where

$$\begin{aligned}X([4]Q(1)) = {} & 3k^4(3k^{32}+40k^{30}+128k^{29}+248k^{28}+384k^{27}+1016k^{26}+2176k^{25}\\ & +2964k^{24}+1664k^{23}-2136k^{22}-7040k^{21}-8504k^{20}-2688k^{19}\\ & +10296k^{18}+23808k^{17}+29778k^{16}+23808k^{15}+10296k^{14}-2688k^{13}\\ & -8504k^{12}-7040k^{11}-2136k^{10}+1664k^9+2964k^8+2176k^7+1016k^6\\ & +384k^5+248k^4+128k^3+40k^2+3)/(4(k^8-4k^5-6k^4-4k^3+1)^2\\ & \times (k^4+1)^2(k+1)^2)\end{aligned}$$

and

$$\begin{aligned}Y([4]Q(1) = {} & -(27k^6(k^{16}+4k^{14}+16k^{13}+28k^{12}+16k^{11}-4k^{10}-32k^9-42k^8\\ & -32k^7-4k^6+16k^5+28k^4+16k^3+4k^2+1)(k^{32}-16k^{30}-80k^{29}\end{aligned}$$

$$
\begin{aligned}
&-136k^{28}-144k^{27}-80k^{26}-32k^{25}-68k^{24}-544k^{23}-1712k^{22}\\
&-3312k^{21}-3896k^{20}-2096k^{19}+2320k^{18}+7232k^{17}+9478k^{16}\\
&+7232k^{15}+2320k^{14}-2096k^{13}-3896k^{12}-3312k^{11}-1712k^{10}\\
&-544k^{9}-68k^{8}-32k^{7}-80k^{6}-144k^{5}-136k^{4}-80k^{3}-16k^{2}+1))\\
&/(8(k+1)^3(k^8-4k^5-6k^4-4k^3+1)^3(k^4+1)^3).
\end{aligned}
$$

Using the method of Zhang and Shamsi Zargar [15, Theorem 1.1], we can show that, for $k \in \mathbb{Z}\backslash\{0,\pm 1\}$, $[4]Q(1)$ is a rational point of infinite order. Then $\mathcal{E}_3(1,k)$ and $\mathcal{C}_3(1,k)$ have infinitely many rational points. Thus, Eq. (1.1) has infinitely many non-trivial rational solutions.

From the rational point $[2]Q(1)$ on $\mathcal{E}_3(1,k)$, we can get the rational point

$$
\begin{aligned}
(U,V) = \big(&2(k-1)(k^2+1)(k^8+4k^6+8k^5+6k^4+1)(k+1)^2/(kD(k)),\\
&k^2(k^4-1)(k^{24}+24k^{22}+64k^{21}+82k^{20}+24k^{19}-92k^{18}-192k^{17}\\
&-81k^{16}+288k^{15}+704k^{14}+768k^{13}+492k^{12}+144k^{11}-72k^{10}-192k^{9}\\
&-177k^{8}-96k^{7}+8k^{6}+64k^{5}+66k^{4}+24k^{3}+4k^{2}+1)/(D(k)^2)\big)
\end{aligned}
$$

on $\mathcal{C}_3(1,k)$, where

$$D(k) = k^{13}-4k^{11}-16k^{10}-15k^{9}-6k^{8}+4k^{7}+8k^{6}+11k^{5}+4k^{4}+3k+2.$$

Therefore, for

$$f(x) = \frac{(x-1)(x+1)(x-k)(x+k)}{x}, \quad k \in \mathbb{Z}\backslash\{0,\pm 1\},$$

Eq. (1.1) has a rational solution

$$(x,y,z) = \left(U+\frac{1}{k},\ kU+1,\ \frac{(kU+1-k)(kU+1+k)V}{k^3(kU+1)}\right),$$

where U,V are given in above.

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