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ESSENTIAL G-RADICAL SUPPLEMENTED MODULES

CELIL NEBIYEV AND HASAN HÜSEYIN ÖKTEN

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Abstract. Let M be an R-module. If every essential submodule of M has a g-radical supplement in M, then M is called an essential g-radical supplemented (or briefly eg-radical supplemented) module. In this work, some properties of these modules are investigated. It is proved that every factor module and every homomorphic image of an eg-radical supplemented module are eg-radical supplemented. Let M be an eg-radical supplemented module. Then every finitely M-generated R-module is eg-radical supplemented.

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1. INTRODUCTION

Throughout this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an R-module. We denote a submodule N of M by $N \leq M$. A module M is said to be *simple* if M have no submodules with distinct from 0 and M. The sum of all simple submodules of a module M is called the *socle* of M and denoted by $\mathrm{Soc}(M)$. M is called a *semisimple* module, if M is a direct sum of simple modules (it is equivalent to $\mathrm{Soc}(M) = M$). Let M be an R-module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*) submodule of M and denoted by $N \ll M$. A submodule N of an R-module M is called an *essential* submodule, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. Let M be an R-module and K be a submodule of M. K is called a *generalized small* (briefly *g-small*) submodule of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, we denote this by $K \ll_g M$ (in [13], it is called an *e-small submodule* of M and denoted by $K \ll_e M$). Let M be an R-module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. M is said to be *essential supplemented* (briefly *e-supplemented*) if every essential submodule of M has a supplement in M. Let M be an R-module and $U,V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. M is said to be *essential g-supplemented* if every essential submodule of M has a g-supplement in M. The intersection of all maximal submodules of an R-module M is called the *radical* of M and denoted by $\mathrm{Rad}(M)$. If M have no maximal submodules, then we denote $\mathrm{Rad}(M) = M$. Let M be an R-module and $U,V \leq M$. If $M = U + V$ and $U \cap V \leq \mathrm{Rad}(V)$, then V is called a *generalized (Radical) supplement* (briefly *Rad-supplement*) of U in M. M is said to be *generalized (Radical) supplemented* (briefly *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. The intersection of all essential maximal submodules of an R-module M is called the *generalized radical* (briefly *g-radical*) of M and denoted by $\mathrm{Rad}_g M$ (in [13] it is denoted by $\mathrm{Rad}_e M$). If M have no essential maximal submodules, then we denote $\mathrm{Rad}_g M = M$. Let M be an R-module and $U,V \leq M$. If $M = U + V$ and $U \cap V \leq \mathrm{Rad}_g V$, then V is called a *g-radical supplement* of U in M. M is said to be *g-radical supplemented* if every submodule of M has a g-radical supplement in M. Let M be an R-module and $K \leq V \leq M$. We say V *lies above* K in M if $V/K \ll M/K$.

More details about supplemented modules are in [2,11]. More details about essential supplemented modules are in [7,8]. More informations about g-small submodules and g-supplemented modules are in [3,4]. The definition of essential g-supplemented modules and some properties of them are in [5]. More details about generalized (Radical) supplemented modules are in [10,12]. The definition of g-radical supplemented modules and some properties of them are in [4].

Lemma 1 ([4, Lemma 3]). *The following assertions are hold.*

(1) *If $N \leq M$, then* $\mathrm{Rad}_g N \leq \mathrm{Rad}_g M$.
(2) *If $K,L \leq M$, then* $\mathrm{Rad}_g K + \mathrm{Rad}_g L \leq \mathrm{Rad}_g (K+L)$.
(3) *If $f: M \longrightarrow N$ is an R-module homomorphism, then* $f(\mathrm{Rad}_g M) \leq \mathrm{Rad}_g N$.

2. ESSENTIAL G-RADICAL SUPPLEMENTED MODULES

Definition 1 ([6, Definition 1]). Let M be an R-module. If every essential submodule of M has a g-radical supplement in M, then M is called an essential g-radical supplemented (or briefly eg-radical supplemented) module.

Clearly we can see that every essential g-supplemented module is eg-radical supplemented. But the converse is not true in general (see Example 1 and Example 2). Every g-radical supplemented module is eg-radical supplemented.

Proposition 1. *Let M be an eg-radical supplemented R-module. If every nonzero submodule of M is essential in M, then M is g-radical supplemented.*

Proof. Clear from definitions. □

Lemma 2. *Let M be an eg-radical supplemented module. Then $M/\mathrm{Rad}_g M$ have no proper essential submodules.*

Proof. Let $U/\mathrm{Rad}_g M \trianglelefteq M/\mathrm{Rad}_g M$. Then $U \trianglelefteq M$ and since M is eg-radical supplemented, U has a g-radical supplement V in M. Here $M = U + V$ and $U \cap V \leq \mathrm{Rad}_g V \leq \mathrm{Rad}_g M$. Then $M/\mathrm{Rad}_g M = (U+V)/\mathrm{Rad}_g M = U/\mathrm{Rad}_g M + (V + \mathrm{Rad}_g M)/\mathrm{Rad}_g M$ and $U/\mathrm{Rad}_g M \cap (V+\mathrm{Rad}_g M)/\mathrm{Rad}_g M = (U \cap V + \mathrm{Rad}_g M)/\mathrm{Rad}_g M = \mathrm{Rad}_g M/\mathrm{Rad}_g M = 0$. Hence $M/\mathrm{Rad}_g M = U/\mathrm{Rad}_g M \oplus (V+\mathrm{Rad}_g M)/\mathrm{Rad}_g M$ and since $U/\mathrm{Rad}_g M \trianglelefteq M/\mathrm{Rad}_g M$, $U/\mathrm{Rad}_g M = M/\mathrm{Rad}_g M$. Thus $M/\mathrm{Rad}_g M$ have no proper essential submodules. □

Corollary 1. *Let M be an eg-radical supplemented module. Then $M/\mathrm{Rad}_g M$ is semisimple.*

Proof. Since M is eg-radical supplemented, by Lemma 2, $M/\mathrm{Rad}_g M$ have no proper essential submodules. Then by [11, Section 21.1], $\mathrm{Soc}(M/\mathrm{Rad}_g M) = M/\mathrm{Rad}_g M$ and $M/\mathrm{Rad}_g M$ is semisimple. □

Corollary 2. *Let M be an essential g-supplemented module. Then $M/\mathrm{Rad}_g M$ is semisimple.*

Proof. Clear from Corollary 1. □

Lemma 3. *Let M be an R-module, $U \trianglelefteq M$ and $N \leq M$. If $U+N$ has a g-radical supplement in M and N is eg-radical supplemented, then U has a g-radical supplement in M.*

Proof. Let X be a g-radical supplement of $U+N$ in M. Since $U \trianglelefteq M$, $U+X \trianglelefteq M$ and $(U+X)\cap N \trianglelefteq N$. Since N is eg-radical supplemented, $(U+X)\cap N$ has a g-radical supplement Y in N. Since X is a g-radical supplement of $U+N$ in M, $M = U+N+X$ and $(U+N)\cap X \leq \mathrm{Rad}_g X$. Since Y is a g-radical supplement of $(U+X)\cap N$ in N, $N = (U+X)\cap N + Y$ and $(U+X)\cap Y = (U+X)\cap N \cap Y \leq \mathrm{Rad}_g Y$. Then $M = U+N+X = U+(U+X)\cap N+Y+X = U+X+Y$ and, by Lemma 1, $U\cap(X+Y) \leq (U+X)\cap Y + (U+Y)\cap X \leq \mathrm{Rad}_g Y + (U+N)\cap X \leq \mathrm{Rad}_g Y + \mathrm{Rad}_g X \leq \mathrm{Rad}_g(X+Y)$. Hence $X+Y$ is a g-radical supplement of U in M. □

Lemma 4. *Let $M = M_1 + M_2$. If M_1 and M_2 are eg-radical supplemented, then M is also eg-radical supplemented.*

Proof. Let $U \trianglelefteq M$. Then $U + M_1 \trianglelefteq M$ and since $U + M_1 + M_2$ has a trivial g-radical supplement 0 in M and M_2 is eg-radical supplemented, by Lemma 3, $U+M_1$ has a g-radical supplement in M. Since M_1 is eg-radical supplemented and $U \trianglelefteq M$, by Lemma 3 again, U has a g-radical supplement in M. Hence M is eg-radical supplemented. □

Corollary 3. *Let $M = M_1 + M_2 + \cdots + M_n$. If M_i is eg-radical supplemented for every $i = 1,2,\ldots,n$, then M is also eg-radical supplemented.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let $f\colon M \to N$ be an R-module epimorphism, $U, V \leq M$ and* $\operatorname{Ker}(f) \leq U$. *If V is a g-radical supplement of U in M, then $f(V)$ is a g-radical supplement of $f(U)$ in N.*

Proof. Since V is a g-radical supplement of U in M, $M = U + V$ and $U \cap V \leq \operatorname{Rad}_g V$. Then $N = f(M) = f(U+V) = f(U) + f(V)$. Let $x \in f(U) \cap f(V)$. Then there exist $u \in U$ and $v \in V$ with $x = f(u) = f(v)$. Here $f(v-u) = f(v) - f(u) = 0$ and $v - u \in \operatorname{Ker}(f) \leq U$. Then $v = v - u + u \in U$ and since $v \in V$, $v \in U \cap V$. Hence $x = f(v) \in f(U \cap V)$ and $f(U) \cap f(V) \leq f(U \cap V)$. Here clearly we can see that $f(U \cap V) \leq f(U) \cap f(V)$ and $f(U) \cap f(V) = f(U \cap V)$. Since $U \cap V \leq \operatorname{Rad}_g V$, by Lemma 1, $f(U) \cap f(V) = f(U \cap V) \leq f(\operatorname{Rad}_g V) \leq \operatorname{Rad}_g f(V)$. Hence $f(V)$ is a g-radical supplement of $f(U)$ in N, as desired. □

Lemma 6. *Every homomorphic image of an eg-radical supplemented module is eg-radical supplemented.*

Proof. Let M be an eg-radical supplemented R-module and $f\colon M \to N$ be an R-module epimorphism. Let $U \trianglelefteq N$. By [11, Section 17.3 (3)], $f^{-1}(U) \trianglelefteq M$ and since M is eg-radical supplemented, $f^{-1}(U)$ has a g-radical supplement V in M. Since $\ker(f) \leq f^{-1}(U)$, by Lemma 5, $f(V)$ is a g-radical supplement of $f\left(f^{-1}(U)\right) = U$ in N. Hence N is eg-radical supplemented, as desired. □

Corollary 4. *Every factor module of an eg-radical supplemented module is eg-radical supplemented.*

Proof. Clear from Lemma 6. □

Lemma 7. *Let M be an eg-radical supplemented R-module. Then every finitely M-generated R-module is eg-radical supplemented.*

Proof. Let N be a finitely M-generated R-module. Then there exist a finite index set Λ and an R-module epimorphism $f\colon M^{(\Lambda)} \to N$. Since M is eg-radical supplemented, by Corollary 3, $M^{(\Lambda)}$ is eg-radical supplemented. Then by Lemma 6, N is eg-radical supplemented, as desired. □

Proposition 2. *Let R be a ring. Then the R-module ${}_RR$ is eg-radical supplemented if and only if every finitely generated R-module is eg-radical supplemented.*

Proof. ($\Longrightarrow$) Clear from Lemma 7.
($\Longleftarrow$) Clear, since ${}_RR$ is finitely generated. □

Definition 2. Let M be an R-module and $X \leq M$. If X is a g-radical supplement of an essential submodule of M, then X is called an eg-radical supplement submodule in M.

Let M be an R-module. It is defined the relation β^* on the set of submodules of an R-module M by $X\beta^*Y$ if and only if $Y+K=M$ for every $K\leq M$ such that $X+K=M$ and $X+T=M$ for every $T\leq M$ such that $Y+T=M$ (see [1]). It is defined the relation β_g^* on the set of submodules of an R-module M by $X\beta_g^*Y$ if and only if $Y+K=M$ for every $K\trianglelefteq M$ such that $X+K=M$ and $X+T=M$ for every $T\trianglelefteq M$ such that $Y+T=M$ (see [9]).

Lemma 8. *Let M be an R-module. If every essential submodule of M is β_g^* equivalent to an eg-radical supplement submodule in M, then M is eg-radical supplemented.*

Proof. Let $X\trianglelefteq M$. By hypothesis, there exists an eg-radical supplement submodule V in M with $X\beta_g^*V$. Let V be a g-radical supplement of an essential submodule U in M. Then $M=U+V$ and $U\cap V\leq \mathrm{Rad}_gV$. Since $U\trianglelefteq M$, by hypothesis, there exists an eg-radical supplement submodule Y in M with $U\beta_g^*Y$. Let Y be a g-radical supplement of S in M and $S\trianglelefteq M$. Then $M=S+Y$ and $S\cap Y\leq \mathrm{Rad}_gY$. Since $X\beta_g^*V$ and $M=U+V$, $M=X+U$ and since $U\beta_g^*Y$ and $X\trianglelefteq M$, $M=X+Y$. Assume $X\cap Y\not\leq \mathrm{Rad}_gY$. Then there exists an essential maximal submodule T of Y such that $X\cap Y+T=Y$. By using [2, Lemma 1.24], we can see that $M=S+Y=X\cap Y+S+T=Y+X\cap(S+T)=U+X\cap(S+T)$ $=X+U\cap(S+T)=V+U\cap(S+T)=U\cap V+S+T$. Since T is an essential maximal submodule of Y, by $\frac{M}{S+T}=\frac{Y+S+T}{S+T}\cong\frac{Y}{Y\cap(S+T)}=\frac{Y}{S\cap Y+T}=\frac{Y}{T}$ and $S+T\trianglelefteq M$, $S+T$ is an essential maximal submodule of M and hence $U\cap V\leq \mathrm{Rad}_gV\leq S+T$. Then $M=U\cap V+S+T=S+T$. This is a contradiction. Hence $X\cap Y\leq \mathrm{Rad}_gY$ and Y is a g-radical supplement of X in M. Thus M is eg-radical supplemented. □

Corollary 5. *Let M be an R-module. If every essential submodule of M is β^* equivalent to an eg-radical supplement submodule in M, then M is eg-radical supplemented.*

Proof. Clear from Lemma 8. □

Corollary 6. *Let M be an R-module. If every essential submodule of M lies above an eg-radical supplement submodule in M, then M is eg-radical supplemented.*

Proof. Clear from Corollary 5. □

Corollary 7. *Let M be an R-module. If every essential submodule of M is an eg-radical supplement submodule in M, then M is eg-radical supplemented.*

Proof. Clear from Corollary 6. □

Lemma 9. *Let M be an R-module. If every submodule of M is β^* equivalent to an eg-radical supplement submodule in M, then M is g-radical supplemented.*

Proof. Let $X\leq M$. By hypothesis, there exists an eg-radical supplement submodule V in M with $X\beta^*V$. Let V be a g-radical supplement of an essential submodule U

in M. Then $M = U + V$ and $U \cap V \leq \mathrm{Rad}_g V$. By hypothesis, there exists an eg-radical supplement submodule Y in M with $U\beta^* Y$. Let Y be a g-radical supplement of an essential submodule S in M. Then $M = S + Y$ and $S \cap Y \leq \mathrm{Rad}_g Y$. Since $X\beta^*\ V$ and $M = U + V$, $M = X + U$ and since $U\beta^* Y$, $M = X + Y$. Assume $X \cap Y \not\leq \mathrm{Rad}_g Y$. Then there exists an essential maximal submodule T of Y such that $X \cap Y + T = Y$. By using [2, Lemma 1.24], we can see that $M = S + Y = X \cap Y + S + T = Y + X \cap (S + T) = U + X \cap (S + T) = X + U \cap (S + T) = V + U \cap (S + T) = U \cap V + S + T$. Since T is an essential maximal submodule of Y, by $\frac{M}{S+T} = \frac{Y+S+T}{S+T} \cong \frac{Y}{Y\cap(S+T)} = \frac{Y}{S\cap Y+T} = \frac{Y}{T}$ and $S + T \trianglelefteq M$, $S + T$ is an essential maximal submodule of M and hence $U \cap V \leq \mathrm{Rad}_g V \leq S + T$. Then $M = U \cap V + S + T = S + T$. This is a contradiction. Hence $X \cap Y \leq \mathrm{Rad}_g Y$ and Y is a g-radical supplement of X in M. Thus M is g-radical supplemented. □

Corollary 8. *Let M be an R-module. If every submodule of M lies above an eg-radical supplement submodule in M, then M is g-radical supplemented.*

Proof. Clear from Lemma 9. □

Corollary 9. *Let M be an R-module. If every submodule of M is an eg-radical supplement submodule in M, then M is g-radical supplemented.*

Proof. Clear from Lemma 9. □

Example 1. Consider the $\mathbb{Z}$-module $\mathbb{Q}$. Since $\mathrm{Rad}_g \mathbb{Q} = \mathrm{Rad}(\mathbb{Q}) = \mathbb{Q}$, ${}_{\mathbb{Z}}\mathbb{Q}$ is eg-radical supplemented. But, since ${}_{\mathbb{Z}}\mathbb{Q}$ is not supplemented and every nonzero submodule of ${}_{\mathbb{Z}}\mathbb{Q}$ is essential in ${}_{\mathbb{Z}}\mathbb{Q}$, ${}_{\mathbb{Z}}\mathbb{Q}$ is not essential g-supplemented.

Example 2. Consider the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ for a prime p. It is easy to check that $\mathrm{Rad}_g \mathbb{Z}_{p^2} \neq \mathbb{Z}_{p^2}$. By [4, Lemma 4], $\mathrm{Rad}_g \left(\mathbb{Q} \oplus \mathbb{Z}_{p^2}\right) = \mathrm{Rad}_g \mathbb{Q} \oplus \mathrm{Rad}_g \mathbb{Z}_{p^2} \neq \mathbb{Q} \oplus \mathbb{Z}_{p^2}$. Since $\mathbb{Q}$ and $\mathbb{Z}_{p^2}$ are eg-radical supplemented, by Lemma 4, $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ is eg-radical supplemented. But $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ is not essential g-supplemented.

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Authors' addresses

Celil Nebiyev
(**Corresponding author**) Ondokuz Mayıs University, Department of Mathematics, Atakum, Samsun, Turkey
E-mail address: cnebiyev@omu.edu.tr

Hasan Hüseyin Ökten
Amasya University, Technical Sciences Vocational School, Amasya, Turkey
E-mail address: hokten@gmail.com