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HERSTEIN'S THEOREM FOR PAIR OF GENERALIZED DERIVATIONS ON PRIME RINGS WITH INVOLUTION

MOHAMMAD SALAHUDDIN KHAN

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Abstract. The aim of this paper is to study the $*$-identities with a pair of generalized derivations on $*$-ideals of prime rings with involution. In particular, we prove that if a noncommutative prime $*$-ring admit two generalized derivations $\mathcal{F}$ and G such that $[\mathcal{F}(x), \mathcal{G}(x^*)] = 0$ for all $x \in I$, where I is a nonzero $*$-ideal of $\mathcal{R}$, then there exists $\lambda \in C$ such that $\mathcal{F} = \lambda\mathcal{G}$. Finally, we provide an example which shows that the primeness of $\mathcal{R}$ is crucial in our results.

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1. Notations and introduction

This research has been motivated by the work of Ali et al. [2]. In all that follows, unless specially stated, $\mathcal{R}$ always denotes an associative ring with center $\mathcal{Z}(\mathcal{R})$. As usual the symbols $s \circ t$ and $[s,t]$ will denote the anti-commutator $st + ts$ and commutator $st - ts$, respectively. Given an integer $n \geq 2$, a ring $\mathcal{R}$ is said to be n-torsion free if $nx = 0$ (where $x \in \mathcal{R}$) implies that $x = 0$. A ring $\mathcal{R}$ is called prime if $a\mathcal{R}b = (0)$ (where $a, b \in \mathcal{R}$) implies $a = 0$ or $b = 0$. We denote by Q_r , Q_l , Q_s and C right, left, symmetric Martindale ring of quotients and extended centroid of a semiprime ring $\mathcal{R}$, respectively (see [5, Chapter 2]). An additive map $x \mapsto x^*$ of $\mathcal{R}$ into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ hold for all $x, y \in \mathcal{R}$. A ring equipped with an involution is called ring with involution or $*$-ring. An ideal I of $\mathcal{R}$ is called $*$-ideal if $I^* = I$. An element x in a ring with involution is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of $\mathcal{R}$ will be denoted by $\mathcal{H}(\mathcal{R})$ and $\mathcal{S}(\mathcal{R})$, respectively. The involution is called the first kind if $\mathcal{Z}(\mathcal{R}) \subseteq \mathcal{H}(\mathcal{R})$, otherwise it is said to be of the second kind. In the later case $\mathcal{S}(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R}) \neq (0)$.

An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be a derivation of $\mathcal{R}$ if $d(st) = d(s)t + sd(t)$ for all $s, t \in \mathcal{R}$. An additive mapping $\mathcal{F} : \mathcal{R} \to \mathcal{R}$ is called a generalized derivation of $\mathcal{R}$ if there exists a derivation d of $\mathcal{R}$ such that $\mathcal{F}(st) = \mathcal{F}(s)t + sd(t)$ for all $s, t \in \mathcal{R}$. Obviously, any derivation is a generalized derivation, but the converse is not true in general. A significative example is a map of the form $\mathcal{F}(s) = as + sb$ for some $a, b \in \mathcal{R}$; such generalized derivations are called inner. Over the last few decades, several authors have studied on rings with involution involving additive mappings like derivations, cenralizers, generalized derivations etc. (viz.; [1–4] and references therein).

In [8], Herstein proved the following result: If $\mathcal{R}$ is a prime ring of characteristic not two admitting a nonzero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative. In [10], Lanski prove that if L is a noncommutative Lie ideal of a 2-torsion free prime ring $\mathcal{R}$ and d, h are nonzero derivations of $\mathcal{R}$ such that $[d(x), h(x)] \in C$ for all $x \in L$, then $h = \lambda d$, where $\lambda \in C$. Very recently, Ali et al. [2] prove the following result: Let $\mathcal{R}$ be a noncommutative prime ring with involution of the second kind such that char$(\mathcal{R}) \neq 2$. If $\mathcal{R}$ admits a nonzero generalized derivation $\mathcal{F}$ such that $[\mathcal{F}(x), \mathcal{F}(x^*)] = 0$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is an order in a central simple algebra of dimension at most 4 over its center and $\mathcal{F}(x) = ax + xb$ for all $x \in \mathcal{R}$ and fixed $a, b \in Q$ such that $a - b \in C$. In the same paper, they also proved that: Let $\mathcal{R}$ be a prime ring with involution of the second kind such that char$(\mathcal{R}) \neq 2$. If $\mathcal{R}$ admits a generalized derivation $\mathcal{F}$ such that $\mathcal{F}(x) \circ \mathcal{F}(x^*) = 0$ for all $x \in \mathcal{R}$, then $\mathcal{F} = 0$.

The main objective of this paper is to study the above mentioned results for a pair of generalized derivations in prime rings with involution. Throughout the paper we assume that $\mathcal{S}(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R}) \neq (0)$.

2. THE RESULTS

We shall do a great deal of calculations with commutators and anti-commutators, routinely using the following basic identities. For all $x, y, z \in \mathcal{R}$;

$$
\begin{aligned}
[xy, z] &= x[y, z] + [x, z]y, \\
[x, yz] &= [x, y]z + y[x, z], \\
x \circ (yz) &= (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z, \\
(xy) \circ z &= x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z].
\end{aligned}
$$

We start our investigation with some well known facts in rings which will be used frequently throughout the text.

Fact 1. *If $\mathcal{R}$ is a prime ring and $0 \neq b \in \mathcal{Z}(\mathcal{R})$ and $ab \in \mathcal{Z}(\mathcal{R})$, then $a \in \mathcal{Z}(\mathcal{R})$.*

Fact 2. *Let $\mathcal{R}$ be a prime ring with involution $'*'$ of second kind such that* char$(\mathcal{R})$ *$\neq 2$. Let d be a nonzero derivation of $\mathcal{R}$ such that $d(h) = 0$ for all $h \in \mathcal{H}(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R})$. Then $d(x) = 0$ for all $x \in \mathcal{Z}$.*

Theorem 1. *Let $\mathcal{R}$ be a 2-torsion free noncommutative prime $*$-ring and I be a nonzero $*$-ideal of $\mathcal{R}$. If $\mathcal{R}$ admit nonzero generalized derivations $\mathcal{F}$ and $\mathcal{G}$ associated with derivations d and g respectively such that $[\mathcal{F}(x), \mathcal{G}(x^*)] = 0$ for all $x \in I$, then $\mathcal{F} = \lambda \mathcal{G}$ for some $\lambda \in C$.*

Proof. By the assumption, we have

$$[\mathcal{F}(x), \mathcal{G}(x^*)] = 0 \text{ for all } x \in I.$$

Taking $x = x+y$, we get

$$[\mathcal{F}(x), \mathcal{G}(y^*)] + [\mathcal{F}(y), \mathcal{G}(x^*)] = 0 \text{ for all } x, y \in I. \tag{2.1}$$

Replacing y by yh (where $h \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{H}(\mathcal{R})$) in (2.1), we obtain

$$[\mathcal{F}(x), y^*]g(h) + [y, \mathcal{G}(x^*)]d(h) = 0 \text{ for all } x, y \in I \text{ and } h \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{H}(\mathcal{R}). \tag{2.2}$$

Replacing y by yk (where $k \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{S}(\mathcal{R})$) in (2.2), we get

$$(-[\mathcal{F}(x), y^*]g(h) + [y, \mathcal{G}(x^*)]d(h))k = 0 \text{ for all } x, y \in I.$$

Since $\mathcal{Z}(\mathcal{R}) \cap \mathcal{S}(\mathcal{R}) \neq \{0\}$, so application of Fact 1 yields that

$$-[\mathcal{F}(x), y^*]g(h) + [y, \mathcal{G}(x^*)]d(h) = 0 \text{ for all } x, y \in I \text{ and } h \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{H}(\mathcal{R}). \tag{2.3}$$

Combining (2.2) and (2.3), we obtain that $2[\mathcal{F}(x), y^*]g(h) = 0$. Since $\mathcal{R}$ is a 2-torsion free prime ring, the last relation gives that either $[\mathcal{F}(x), y^*] = 0$ or $g(h) = 0$.

Consider the first case $[\mathcal{F}(x), y^*] = 0$ for all $x, y \in I$. This implies that $[\mathcal{F}(x), t] = 0$ for all $x, t \in I$. Since I and $\mathcal{R}$ satisfy the same differential identities (see [11, Theorem 2]), hence $[\mathcal{F}(x), t] = 0$ for all $x, t \in \mathcal{R}$. Therefore, in view of Lemma 3 in [9], $\mathcal{F} = 0$. This leads a contradiction.

Now, consider the second case $g(h) = 0$, then (2.2) gives that $[y, \mathcal{G}(x^*)]d(h) = 0$. Using the same arguments as we have used above, we get $d(h) = 0$. Thus, the conclusion of both cases are $d(h) = g(h) = 0$ for all $h \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{H}(\mathcal{R})$. By Fact 2, $d(z) = g(z) = 0$ for all $z \in \mathcal{Z}(\mathcal{R})$. Substitute $y = yk$ in (2.1), we get

$$(-[\mathcal{F}(x), \mathcal{G}(y^*)] + [\mathcal{F}(y), \mathcal{G}(x^*)])k = 0 \text{ for all } x, y \in I.$$

This gives

$$-[\mathcal{F}(x), \mathcal{G}(y^*)] + [\mathcal{F}(y), \mathcal{G}(x^*)] = 0 \text{ for all } x, y \in I. \tag{2.4}$$

Combining (2.1) and (2.4), we obtain $2[\mathcal{F}(y), \mathcal{G}(x^*)] = 0$ for all $x, y \in I$. Taking $x^* = t$ and using the fact that $\mathcal{R}$ is 2-torsion free, we conclude that $[\mathcal{F}(y), \mathcal{G}(t)] = 0$ for all $y, t \in I$. Since I and $\mathcal{R}$ satisfy the same differential identities (see [11, Theorem 2]), so we have $[\mathcal{F}(y), \mathcal{G}(t)] = 0$ for all $y, t \in \mathcal{R}$. By [9, Theorem 2] there exists $\lambda \in C$ such that $\mathcal{F} = \lambda\mathcal{G}$. This completes the proof of the theorem. □

The immediate consequences of the above theorem are the following results:

Corollary 1. *Let $\mathcal{R}$ be a 2-torsion free noncommutative prime $*$-ring. If $\mathcal{R}$ admit nonzero generalized derivations $\mathcal{F}$ and $\mathcal{G}$ such that $[\mathcal{F}(x), \mathcal{G}(x^*)] = 0$ for all $x \in \mathcal{R}$, then $\mathcal{F} = \lambda\mathcal{G}$ for some $\lambda \in C$.*

Corollary 2. *Let $\mathcal{R}$ be a 2-torsion free noncommutative prime $*$-ring. If $\mathcal{R}$ admit nonzero generalized derivations $\mathcal{F}$ and $\mathcal{G}$ such that $[\mathcal{F}(x), \mathcal{G}(y^*)] = 0$ for all $x, y \in \mathcal{R}$, then $\mathcal{F} = \lambda\mathcal{G}$ for some $\lambda \in C$.*

Corollary 3. [3, Main Theorem] *Let $\mathcal{R}$ be a 2-torsion free noncommutative prime $*$-ring. If $\mathcal{R}$ admit nonzero derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $[\mathcal{D}_1(x), \mathcal{D}_2(x^*)] = 0$ for all $x \in \mathcal{R}$, then $\mathcal{D}_1 = \lambda \mathcal{D}_2$ for some $\lambda \in C$.*

Theorem 2. *Let $\mathcal{R}$ be a 2-torsion free noncommutative prime $*$-ring and I be a nonzero $*$-ideal of $\mathcal{R}$. If $\mathcal{R}$ admit nonzero generalized derivations $\mathcal{F}$ and $\mathcal{G}$ associated with derivations d and g respectively such that $[\mathcal{F}(x), \mathcal{G}(x^*)] = \mathcal{G}([x,x^*])$ for all $x \in I$, then there exists $\lambda \in C$ such that $\mathcal{F} = \lambda \mathcal{G}$.*

Proof. Linearize the given condition, we have

$$[\mathcal{F}(x), \mathcal{G}(y^*)] + [\mathcal{F}(y), \mathcal{G}(x^*)] = \mathcal{G}([x,y^*]) + \mathcal{G}([y,x^*]) \text{ for all } x,y \in I. \tag{2.5}$$

Substitute yh in place of y, we reach at

$$[\mathcal{F}(x), y^*]g(h) + [y, \mathcal{G}(x^*)]d(h) = 0 \text{ for all } x,y \in I.$$

This is same as (2.2) and thus following the same technique, we get $d(z) = g(z) = 0$ for all $z \in \mathcal{Z}(\mathcal{R})$. Now replacing y by yk in (2.5), we get

$$(-[\mathcal{F}(x), \mathcal{G}(y^*)] + [\mathcal{F}(y), \mathcal{G}(x^*)])k = (-\mathcal{G}([x,y^*]) + \mathcal{G}([y,x^*]))k \text{ for all } x,y \in I.$$

Since $\mathcal{Z}(\mathcal{R}) \cap \mathcal{S}(\mathcal{R}) \neq \{0\}$ and $\mathcal{R}$ is prime, we have

$$-[\mathcal{F}(x), \mathcal{G}(y^*)] + [\mathcal{F}(y), \mathcal{G}(x^*)] = -\mathcal{G}([x,y^*]) + \mathcal{G}([y,x^*]) \text{ for all } x,y \in I. \tag{2.6}$$

Combining (2.5) and (2.6), we obtain that $[\mathcal{F}(x), \mathcal{G}(y^*)] = \mathcal{G}([x,y^*])$ for all $x,y \in I$. In particular, for $y^* = x$, we get $[\mathcal{F}(x), \mathcal{G}(x)] = 0$ for all $x,y \in I$. Thus, in view of [9, Theorem 2] there exists $\lambda \in C$ such that $\mathcal{F} = \lambda \mathcal{G}$. This completes the proof of the theorem. □

Corollary 4. *Let $\mathcal{R}$ be a 2-torsion free noncommutative prime $*$-ring. If $\mathcal{R}$ admit nonzero generalized derivations $\mathcal{F}$ and $\mathcal{G}$ such that $[\mathcal{F}(x), \mathcal{G}(x^*)] = \mathcal{G}([x,x^*])$ for all $x \in \mathcal{R}$, then there exists $\lambda \in C$ such that $\mathcal{F} = \lambda \mathcal{G}$.*

Theorem 3. *Let $\mathcal{R}$ be a 2-torsion free prime $*$-ring and I be a nonzero $*$-ideal of $\mathcal{R}$. If $\mathcal{R}$ admit generalized derivations $\mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{F}(x) \circ \mathcal{G}(x^*) = 0$ or $\mathcal{F}(x^*) \circ \mathcal{G}(x) = 0$ for all $x \in I$, then in both cases either $\mathcal{F} = 0$ or $\mathcal{G} = 0$.*

Proof. Suppose d and g are associated derivations of $\mathcal{F}$ and $\mathcal{G}$ respectively. Assume the relation

$$\mathcal{F}(x) \circ \mathcal{G}(x^*) = 0 \text{ for all } x \in I.$$

Linearize the above equation, we get

$$\mathcal{F}(x) \circ \mathcal{G}(y^*) + \mathcal{F}(y) \circ \mathcal{G}(x^*) = 0 \text{ for all } x,y \in I. \tag{2.7}$$

Putting yh (where $h \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{H}(\mathcal{R})$) in place of y and we obtain

$$(\mathcal{F}(x) \circ y^*)g(h) + (y \circ \mathcal{G}(x^*))d(h) = 0 \text{ for all } x,y \in I. \tag{2.8}$$

Taking $y = yk$, we get

$$(-(\mathcal{F}(x) \circ y^*)g(h) + (y \circ \mathcal{G}(x^*))d(h))k = 0 \text{ for all } x,y \in I.$$

This implies that

$$-(\mathcal{F}(x)\circ y^*)g(h)+(y\circ\mathcal{G}(x^*))d(h)=0 \text{ for all } x,y\in I. \tag{2.9}$$

Combining (2.8) and (2.9), we find that $(\mathcal{F}(x)\circ y^*)g(h)=0$ for all $x,y\in I$. By the primeness of $\mathcal{R}$, we have $\mathcal{F}(x)\circ y^*=0$ or $g(h)=0$. Firstly, if $\mathcal{F}(x)\circ y^*=0$, then $\mathcal{F}(x)\circ x=0$ for all $x\in I$. Therefore, in view of [6, Theorem 1], $\mathcal{F}(x)=0$ for all $x\in I$. For any $r\in\mathcal{R}$, $0=\mathcal{F}(xr)=\mathcal{F}(x)r+xd(r)=xd(r)$. This implies $d=0$ on $\mathcal{R}$. In the other way, for any $r\in\mathcal{R}$, $0=\mathcal{F}(rx)=\mathcal{F}(r)x+rd(x)=\mathcal{F}(r)x$ and hence $\mathcal{F}=0$ on $\mathcal{R}$. Now, consider the second case $g(h)=0$. From (2.8), we have $(y\circ\mathcal{G}(x^*))d(h)=0$ for all $x,y\in I$. Primeness of $\mathcal{R}$ yields that $y\circ\mathcal{G}(x^*)=0$ or $d(h)=0$. If $y\circ\mathcal{G}(x^*)=0$, then $\mathcal{G}=0$ on $\mathcal{R}$. If $d(h)=0$, then by Fact 2, we have $d(z)=g(z)=0$ for all $z\in\mathcal{Z}(\mathcal{R})$. Substituting $y=yk$ in (2.7), we obtain

$$-\mathcal{F}(x)\circ\mathcal{G}(y^*)+\mathcal{F}(y)\circ\mathcal{G}(x^*)=0 \text{ for all } x,y\in I. \tag{2.10}$$

Subtracting (2.10) from (2.7), we get

$$2(\mathcal{F}(x)\circ\mathcal{G}(y^*))=0 \text{ for all } x,y\in I.$$

Since $\mathcal{R}$ is 2-torsion free and taking $y=y^*$, we obtain that

$$\mathcal{F}(x)\circ\mathcal{G}(y)=0 \text{ for all } x,y\in I.$$

Since I and $\mathcal{R}$ satisfy the same differential identities (see [11, Theorem 2]), so $\mathcal{F}(x)\circ\mathcal{G}(y)=0$ for all $x,y\in\mathcal{R}$. Thus in view of [7, Theorem 2.7], we conclude that either $\mathcal{F}=0$ or $\mathcal{G}=0$ on $\mathcal{R}$. The proof in case we have the relation $\mathcal{F}(x^*)\circ\mathcal{G}(x)=0$ for all $x\in I$ goes through similarly. □

Following corollaries are the immediate consequences of the above theorem.

Corollary 5. *Let $\mathcal{R}$ be a 2-torsion free prime $*$-ring. If $\mathcal{R}$ admit generalized derivations $\mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{F}(x)\circ\mathcal{G}(x^*)=0$ or $\mathcal{F}(x^*)\circ\mathcal{G}(x)=0$ for all $x\in\mathcal{R}$, then in both cases either $\mathcal{F}=0$ or $\mathcal{G}=0$.*

Corollary 6. *Let $\mathcal{R}$ be a 2-torsion free prime $*$-ring and I be a nonzero $*$-ideal of $\mathcal{R}$. If $\mathcal{R}$ admit a derivation $\mathcal{D}$ and a generalized derivation $\mathcal{F}$ such that $\mathcal{F}(x)\circ\mathcal{D}(x^*)=0$ or $\mathcal{F}(x^*)\circ\mathcal{D}(x)=0$ for all $x\in I$, then in both cases either $\mathcal{F}=0$ or $\mathcal{D}=0$.*

Corollary 7. *Let $\mathcal{R}$ be a 2-torsion free prime $*$-ring. If $\mathcal{R}$ admit derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $\mathcal{D}_1(x)\circ\mathcal{D}_2(x^*)=0$ or $\mathcal{D}_1(x^*)\circ\mathcal{D}_2(x)=0$ for all $x\in\mathcal{R}$, then in both cases either $\mathcal{D}_1=0$ or $\mathcal{D}_2=0$.*

Corollary 8. *Let $\mathcal{R}$ be a 2-torsion free prime $*$-ring and I be a nonzero $*$-ideal of $\mathcal{R}$. If $\mathcal{R}$ admits generalized derivation $\mathcal{F}$ such that $\mathcal{F}(x)\circ\mathcal{F}(x^*)=0$ for all $x\in I$, then $\mathcal{F}=0$.*

Corollary 9. [2, Theorem 1.2.] *Let $\mathcal{R}$ be a 2-torsion free prime $*$-ring. If $\mathcal{R}$ admits generalized derivation $\mathcal{F}$ such that $\mathcal{F}(x)\circ\mathcal{F}(x^*)=0$ for all $x\in\mathcal{R}$, then $\mathcal{F}=0$.*

Theorem 4. *Let $\mathcal{R}$ be a 2-torsion free prime $*$-ring and I be a nonzero $*$-ideal of $\mathcal{R}$. If $\mathcal{R}$ admit nonzero generalized derivations $\mathcal{F}$ and $\mathcal{G}$ associated with derivations d and g respectively such that $\mathcal{F}(x)\mathcal{G}(x^*) = 0$ or $\mathcal{F}(x^*)\mathcal{G}(x) = 0$ for all $x \in I$, then in both cases $\mathcal{F}$ and $\mathcal{G}$ are of the form $\mathcal{F}(x) = xp$ and $\mathcal{G}(x) = qx$ for all $x \in \mathcal{R}$, where $p \in Q_l$, $q \in Q_r$ are fixed elements such that $pq = 0$.*

Proof. Firstly, we suppose that

$$\mathcal{F}(x)\mathcal{G}(x^*) = 0 \text{ for all } x \in I.$$

Linearization of the above condition gives that

$$\mathcal{F}(x)\mathcal{G}(y^*) + \mathcal{F}(y)\mathcal{G}(x^*) = 0 \text{ for all } x,y \in I. \tag{2.11}$$

Replacing y by yh, we obtain

$$\mathcal{F}(x)y^*g(h) + y\mathcal{G}(x^*)d(h) = 0 \text{ for all } x,y \in I. \tag{2.12}$$

Write yk in place of y, we get

$$(-\mathcal{F}(x)y^*g(h) + y\mathcal{G}(x^*)d(h))k = 0 \text{ for all } x,y \in I.$$

Primeness of $\mathcal{R}$ yields that

$$-\mathcal{F}(x)y^*g(h) + y\mathcal{G}(x^*)d(h) = 0 \text{ for all } x,y \in I.$$

Combining the last relation with (2.12), we can find that $\mathcal{F}(x)y^*g(h) = 0$ and hence primeness of $\mathcal{R}$ yields that $\mathcal{F}(x)y^* = 0$ or $g(h) = 0$. Firstly, consider the case $\mathcal{F}(x)y^* = 0$ for all $x,y \in I$. This implies that $\mathcal{F}(x) = 0$ for all $x \in I$, which contradicts the fact that $\mathcal{F} \neq 0$. Now, consider the second case $g(h) = 0$ for all $h \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{H}(\mathcal{R})$. From (2.12), we have $y\mathcal{G}(x^*)d(h) = 0$. This further implies that $y\mathcal{G}(x^*) = 0$ or $d(h) = 0$. If $y\mathcal{G}(x^*) = 0$ for all $x,y \in I$, then again we get a contradiction. Thus, there is the only possible case $d(h) = 0$ and $g(h) = 0$ for all $h \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{H}(\mathcal{R})$ and hence $d(k) = g(k) = 0$ for all $k \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{S}(\mathcal{R})$. Now, replacing y by yk in (2.11), we arrive at

$$(-\mathcal{F}(x)\mathcal{G}(y^*) + \mathcal{F}(y)\mathcal{G}(x^*))k = 0 \text{ for all } x,y \in I.$$

This implies that

$$-\mathcal{F}(x)\mathcal{G}(y^*) + \mathcal{F}(y)\mathcal{G}(x^*) = 0 \text{ for all } x,y \in I.$$

Subtracting the last relation from (2.11) and using the fact that $\mathcal{R}$ is 2-torsion free, we get that $\mathcal{F}(x)\mathcal{G}(y^*) = 0$ for all $x,y \in I$. In particular, for $y^* = x$ this gives that $\mathcal{F}(x)\mathcal{G}(x) = 0$ for all $x \in I$. Since I and $\mathcal{R}$ satisfy the same differential identities (see [11, Theorem 2]), so $\mathcal{F}(x)\mathcal{G}(x) = 0$ for all $x \in \mathcal{R}$. Therefore, by the Theorem 2.2 in [7], we get the required result. Similarly, we can prove the result for the case $\mathcal{F}(x^*)\mathcal{G}(x) = 0$ for all $x \in I$. Therefore the proof of the theorem is completed. □

Corollary 10. *Let $\mathcal{R}$ be a 2-torsion free prime $*$-ring. If $\mathcal{R}$ admit a derivation $\mathcal{D}$ and a generalized derivation $\mathcal{F}$ such that $\mathcal{D}(x)\mathcal{F}(x^*) = 0$ or $\mathcal{F}(x^*)\mathcal{D}(x) = 0$ for all $x \in \mathcal{R}$, then $\mathcal{F} = 0$ or $\mathcal{D} = 0$.*

Proof. For the first case $\mathcal{D}(x)\mathcal{F}(x^*) = 0$ for all $x \in \mathcal{R}$. On the contrary, suppose that $\mathcal{D} \neq 0$ and $\mathcal{F} \neq 0$. Then by the Theorem 4, there exists $p \in Q_l$ such that $\mathcal{D}(x) = xp$ for all $x \in \mathcal{R}$. Therefore $\mathcal{D}(xy) = xyp = \mathcal{D}(x)y + x\mathcal{D}(y)$. This gives that $\mathcal{D}(x)y = 0$ for all $x, y \in \mathcal{R}$. This implies that $\mathcal{D} = 0$, which leads a contradiction to our supposition. Similar proof for the case $\mathcal{F}(x^*)\mathcal{D}(x) = 0$ for all $x \in \mathcal{R}$. □

Corollary 11. *Let $\mathcal{R}$ be a 2-torsion free prime $*$-ring. If $\mathcal{R}$ admit derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $\mathcal{D}_1(x)\mathcal{D}_2(x^*) = 0$ or $\mathcal{D}_1(x^*)\mathcal{D}_2(x) = 0$ for all $x \in \mathcal{R}$, then $\mathcal{D}_1 = 0$ or $\mathcal{D}_2 = 0$.*

The following example justifies the fact that Theorems 1, 3 & 4 are not true for semiprime rings.

Example 1. Let $\mathcal{R} = \left\{ \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \middle| \, c_1, c_2, c_3, c_4 \in \mathbb{C} \right\}$, where $\mathbb{C}$ is a ring of complex numbers. Of course, $\mathcal{R}$ with matrix addition and matrix multiplication is a noncommutative prime ring. Define mappings $*_1, d_1, \mathcal{F}_1 : \mathcal{R} \longrightarrow \mathcal{R}$ such that

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}^{*_1} = \begin{pmatrix} \bar{c_4} & \bar{c_2} \\ \bar{c_3} & \bar{c_1} \end{pmatrix}, \quad d_1 \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & -c_2 \\ c_3 & 0 \end{pmatrix}$$

$$\text{and} \quad \mathcal{F}_1 \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} c_1 & 2c_2 \\ -c_3 & 0 \end{pmatrix}.$$

It can be easily checked that $*_1$ is an involution of the second kind and $\mathcal{F}_1$ is a generalized derivation of $\mathcal{R}$ associated with the derivation d_1. Let $\mathbb{H}$ be a ring of real quaternions. Clearly, a mapping $*_2$ such that $q^{*_2} = \alpha - i\beta - j\gamma - k\delta$ is an involution on $\mathbb{H}$. Next, define a mapping $\mathcal{G}_1$ on $\mathbb{H}$ such that $\mathcal{G}_1(q) = 2iq - qi$. Then $\mathcal{G}_1$ is a generalized derivation of $\mathbb{H}$ associated with the derivation $g_1 = g_i$ (where g_i is an inner derivation on $\mathbb{H}$ determined by $i \in \mathbb{H}$).

Let $\mathcal{L} = \mathcal{R} \times \mathbb{H}$. Then $\mathcal{L}$ is a 2-torsion free noncommutative semiprime ring. Now define an involution $*$ on $\mathcal{L}$, as $(x, y)^* = (x^{*_1}, y*_2)$. Clearly, $*$ is an involution of the second kind. Further, we define the mappings $\mathcal{F}$ and $\mathcal{G}$ from $\mathcal{L}$ to $\mathcal{L}$ such that $\mathcal{F}(x, y) = (\mathcal{F}_1(x), 0)$ and $\mathcal{G}(x, y) = (0, \mathcal{G}_2(x))$ for all $(x, y) \in \mathcal{L}$. It can be easily checked that $\mathcal{F}$ and $\mathcal{G}$ are nonzero generalized derivations on $\mathcal{L}$ and satisfying $[\mathcal{F}(X), \mathcal{G}(X^*)] = 0, \mathcal{F}(X) \circ \mathcal{G}(X^*) = 0$ and $\mathcal{F}(X)\mathcal{G}(X^*) = 0$ for all $X \in \mathcal{L}$, but the conclusions of Theorems 1, 3 & 4 are not held. Hence, in these results the hypothesis of primeness is essential.

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Author's address

Mohammad Salahuddin Khan
Department of Applied Mathematics, Z. H. College of Engineering & Technology, Aligarh Muslim University, Aligarh-202002, India
E-mail address: salahuddinkhan50@gmail.com