Miskolc Mathematical Notes
Vol. 23 (2022), No. 1, pp. 381–387
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2022.3655

COFINITELY ESSENTIAL G-SUPPLEMENTED MODULES

CELIL NEBIYEV AND HASAN HÜSEYIN ÖKTEN

Received 17 January, 2021

Abstract. Let M be an $R-$module. If every cofinite essential submodule of M has a g-supplement in M, then M is called a cofinitely essential g-supplemented (or briefly cofinitely eg-supplemented) module. In this work, some properties of these modules are investigated. It is proved that every factor module and every homomorphic image of a cofinitely eg-supplemented module are cofinitely eg-supplemented. Let M be a cofinitely eg-supplemented module. Then every $M-$generated $R-$module is cofinitely eg-supplemented.

2010 *Mathematics Subject Classification:* 16D10; 16D70

Keywords: cofinite submodules, essential submodules, g-small submodules, g-supplemented modules

1. Introduction

Throughout this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an $R-$module. We denote a submodule N of M by $N \leq M$. A submodule U of an $R-$module M is called a *cofinite submodule* of M if M/U is finitely generated. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*) submodule of M and denoted by $N \ll M$. A submodule N of an R -module M is called an *essential* submodule, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. Let M be an $R-$module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) submodule of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, we denote this by $K \ll_g M$ (in [15], it is called an *e-small submodule* of M and denoted by $K \ll_e M$). Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. M is said to be *cofinitely supplemented* if every cofinite submodule of M has a supplement in M. M is said to be *essential supplemented* (briefly, *e-supplemented*) if every essential submodule of M has a supplement in M. M is said to be *cofinitely* *essential supplemented* (briefly, *cofinitely e-supplemented*) if every cofinite essential submodule of M has a supplement in M. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. M is said to be *essential g-supplemented* if every essential submodule of M has a g-supplement in M. M is said to be *cofinitely g-supplemented* if every cofinite submodule of M has a g-supplement in M. The intersection of all maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. The intersection of all essential maximal submodules of an R-module M is called the *generalized radical* (briefly, *g-radical*) of M and denoted by $Rad_g M$ (in [15], it is denoted by $Rad_e M$). If M have no essential maximal submodules, then we denote $Rad_g M = M$. Let M be an $R-$module and $K \leq V \leq M$. We say V *lies above* K in M if $V/K \ll M/K$.

More details about supplemented modules are in [3, 14]. More informaitons about cofinitely supplemented modules are in [1]. More details about essential supplemented modules are in [11, 12]. More details about cofinitely essential supplemented modules are in [7, 8]. More informations about g-small submodules and g-supplemented modules are in [5, 6]. The definition of cofinitely g-supplemented modules and more informations about these modules are in [4]. The definition of essential g-supplemented modules and some properties of them are in [9].

Lemma 1. *Let M be an R -module and $K, N \leq M$. Consider the following conditions.*

(1) *If $K \leq N$ and N is a generalized small submodule of M, then K is a generalized small submodule of M.*
(2) *If K is contained in N and a generalized small submodule of N, then K is a generalized small submodule in submodules of M which contain N.*
(3) *If $K \ll_g L$ and $N \ll_g T$ with $L, T \leq M$, then $K + N \ll_g L + T$.*
(4) $Rad_g M = \sum_{L \ll_g M} L$.
(5) *Let T be an $R-$module and $f : M \to T$ be an R -module homomorphism. If $K \ll_g M$, then $f(K) \ll_g T$. Here $f(Rad_g M) \leq Rad_g T$.*

Proof. See [6, Lemma 1 and Lemma 3]. □

2. Cofinitely essential g-supplemented modules

Definition 1. Let M be an $R-$module. If every cofinite essential submodule of M has a g-supplement in M, then M is called a cofinitely essential g-supplemented (or briefly cofinitely eg-supplemented) module. (See also [10]).

Clearly we can see that every essential g-supplemented module is cofinitely eg-supplemented. But the converse is not true in general (see Example 1 and Example 2). Every cofinitely essential supplemented module is cofinitely eg-supplemented.

Proposition 1. *Let M be a cofinitely eg-supplemented $R-$module. If every nonzero submodule of M is essential in M, then M is cofinitely supplemented.*

Proof. Clear from definitions. □

Lemma 2. *Let M be a finitely generated $R-$module. Then M is essential g-supplemented if and only if M is cofinitely eg-supplemented.*

Proof. Clear, since every submodule of M is cofinite. □

Lemma 3. *Let M be a cofinitely eg-supplemented module. Then M/Rad_gM have no proper cofinite essential submodules.*

Proof. Let U/Rad_gM be a cofinite essential submodule M/Rad_gM. Then $U \trianglelefteq M$ and since $\frac{M}{U} \cong \frac{M/Rad_gM}{U/Rad_gM}$, U is a cofinite essential submodule of M. Since M is cofinitely eg-supplemented, U has a g-supplement V in M. Here $M = U + V$ and $U \cap V \ll_g V$. Since $U \cap V \ll_g V$, by Lemma 1, $U \cap V \leq Rad_gM$. Then $M/Rad_gM = (U+V)/Rad_gM = U/Rad_gM + (V + Rad_gM)/Rad_gM$ and $U/Rad_gM \cap (V + Rad_gM)/Rad_gM = (U \cap V + Rad_gM)/Rad_gM = Rad_gM/Rad_gM = 0$. Hence $M/Rad_gM = U/Rad_gM \oplus (V + Rad_gM)/Rad_gM$ and since $U/Rad_gM \trianglelefteq M/Rad_gM$, $U/Rad_gM = M/Rad_gM$. Thus M/Rad_gM have no proper cofinite essential submodules. □

Lemma 4. *Let M be an $R-$module, U be a cofinite essential submodule of M and $N \leq M$. If $U + N$ has a g-supplement in M and N is cofinitely eg-supplemented, then U has a g-supplement in M.*

Proof. Let X be a g-supplement of $U+N$ in M. Then $M = U+N+X$ and $(U+N) \cap X \ll_g X$. Since $U \trianglelefteq M$, $U + X \trianglelefteq M$ and $N \cap (U+X) \trianglelefteq N$. Since U is a cofinite submodule of M, $U+X$ is a cofinite submodule of M. Hence by $\frac{M}{U+X} = \frac{U+N+X}{U+X} \cong \frac{N}{N\cap(U+X)}$, $N \cap (U+X)$ is a cofinite essential submodule of M. Since N is cofinitely eg-supplemented, $N \cap (U+X)$ has a g-supplement Y in N. Since Y is a g-supplement of $N \cap (U+X)$ in N, $N = N \cap (U+X) + Y$ and $(U+X) \cap Y = N \cap (U+X) \cap Y \ll_g Y$. Then $M = U+N+X = U+N \cap (U+X)+Y+X = U+X+Y$ and, by Lemma 1, $U \cap (X+Y) \leq (U+Y) \cap X + (U+X) \cap Y \leq (U+N) \cap X + (U+X) \cap Y \ll_g X+Y$. Hence $X+Y$ is a g-supplement of U in M. □

Corollary 1. *Let M be an $R-$module, U be a cofinite essential submodule of M and $N_1, N_2, ..., N_k \leq M$. If $U + N_1 + N_2 + ... + N_k$ has a g-supplement in M and N_i is cofinitely eg-supplemented for $i = 1, 2, ..., k$, then U has a g-supplement in M.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let* $M = \sum_{\lambda\in\Lambda} M_\lambda$. *If* M_λ *is cofinitely eg-supplemented for every* $\lambda \in \Lambda$, *then* M *is also cofinitely eg-supplemented.*

Proof. Let U be a cofinite essential submodule of M. Since U is a cofinite submodule of M, it is easy to see that there exist $\lambda_1, \lambda_2, ..., \lambda_n \in \Lambda$ such that $M = U + M_{\lambda_1} + M_{\lambda_2} + ... + M_{\lambda_n}$. Then $U + M_{\lambda_1} + M_{\lambda_2} + ... + M_{\lambda_n}$ has a trivial g-supplement 0 in M and since M_{λ_i} is cofinitely eg-supplemented for every $i = 1, 2, ..., n$, by Corollary 1, U has a g-supplement in M. Hence M is cofinitely eg-supplemented. □

Lemma 6. *Let* $f : M \longrightarrow N$ *be an* $R-$*module epimorphism,* $U, V \leq M$ *and* $Kef \leq U$. *If* V *is a g-supplement of* U *in* M*, then* $f(V)$ *is a g-supplement of* $f(U)$ *in* N.

Proof. Since V is a g-supplement of U in M, $M = U + V$ and $U \cap V \ll_g V$. Then $N = f(M) = f(U+V) = f(U) + f(V)$. Let $x \in f(U) \cap f(V)$. Then there exist $u \in U$ and $v \in V$ with $x = f(u) = f(v)$. Here $f(v-u) = f(v) - f(u) = 0$ and $v - u \in Kef \leq U$. Then $v = v - u + u \in U$ and since $v \in V$, $v \in U \cap V$. Hence $x = f(v) \in f(U \cap V)$ and $f(U) \cap f(V) \leq f(U \cap V)$. Here clearly we can see that $f(U \cap V) \leq f(U) \cap f(V)$ and $f(U) \cap f(V) = f(U \cap V)$. Since $U \cap V \ll_g V$, by Lemma 1, $f(U) \cap f(V) = f(U \cap V) \ll_g f(V)$. Hence $f(V)$ is a g-supplement of $f(U)$ in N, as desired. □

Lemma 7. *Every homomorphic image of a cofinitely eg-supplemented module is cofinitely eg-supplemented.*

Proof. Let M be an cofinitely eg-supplemented $R-$module and $f : M \longrightarrow N$ be an $R-$module epimorphism. Let U be a cofinite essential submodule of N. Since $U \trianglelefteq N$, by [14, 17.3 (3)], $f^{-1}(U) \trianglelefteq M$. Let $p : N \longrightarrow N/U$ be a canonical epimorphism. Since $(pf)(x) = p(f(x)) = f(x) + U = U$ for every $x \in f^{-1}(U)$, $x \in Ke(pf)$ and $f^{-1}(U) \leq Ke(pf)$. Let $y \in Ke(pf)$. Then $U = (pf)(y) = p(f(y)) = f(y) + U$ and $f(y) \in U$. Hence $y \in f^{-1}(U)$ and $Ke(pf) \leq f^{-1}(U)$. Since $f^{-1}(U) \leq Ke(pf)$, $Ke(pf) = f^{-1}(U)$. Hence $M/f^{-1}(U) \cong N/U$ and $f^{-1}(U)$ is a cofinite submodule of M. Moreover, $f^{-1}(U) \trianglelefteq M$. Since M is cofinitely eg-supplemented, $f^{-1}(U)$ has a g-supplement V in M. Since $Kef \leq f^{-1}(U)$, by Lemma 6, $f(V)$ is a g-supplement of $f\left(f^{-1}(U)\right) = U$ in N. Hence N is cofinitely eg-supplemented, as desired. □

Corollary 2. *Every factor module of a cofinitely eg-supplemented module is cofinitely eg-supplemented.*

Proof. Clear from Lemma 7. □

Lemma 8. *Let* M *be a cofinitely eg-supplemented* $R-$*module. Then every* $M-$*generated* $R-$*module is cofinitely eg-supplemented.*

Proof. Let N be a $M-$generated $R-$module. Then there exist an index set Λ and an $R-$module epimorphism $f : M^{(\Lambda)} \longrightarrow N$. Since M is cofinitely eg-supplemented, by Lemma 5, $M^{(\Lambda)}$ is cofinitely eg-supplemented. Then by Lemma 7, N is cofinitely eg-supplemented, as desired. □

Proposition 2. *Let R be a ring. Then the $R-$module $_RR$ is essential g-supplemented if and only if every $R-$module is cofinitely eg-supplemented.*

Proof. ($\Longrightarrow$) Clear from Lemma 8.
($\Longleftarrow$) Clear from Lemma 2, since $_RR$ is finitely generated. □

Definition 2. Let M be an $R-$module and $X \leq M$. If X is a g-supplement of a cofinite essential submodule of M, then X is called a ceg-supplement submodule in M.

Let M be an $R-$module. It is defined the relation $'\beta^{*\prime}$ on the set of submodules of an $R-$module M by $X\beta^{*}Y$ if and only if $Y+K=M$ for every $K \leq M$ such that $X+K=M$ and $X+T=M$ for every $T \leq M$ such that $Y+T=M$ (See [2]). It is defined the relation $'\beta_g^{*}\,'$ on the set of submodules of an $R-$module M by $X\beta_g^{*}Y$ if and only if $Y+K=M$ for every $K \trianglelefteq M$ such that $X+K=M$ and $X+T=M$ for every $T \trianglelefteq M$ such that $Y+T=M$ (See [13]).

Lemma 9. *Let M be an $R-$module. If every cofinite essential submodule of M is β_g^{*} equivalent to a ceg-supplement submodule in M, then M is cofinitely eg-supplemented.*

Proof. Let X be a cofinite essential submodule of M. By hypothesis, there exists a ceg-supplement submodule V in M with $X\beta_g^{*}V$. Let V be a g-supplement of a cofinite essential submodule U in M. Then $M=U+V$ and $U\cap V \ll_g V$. Since U is a cofinite essential submodule of M, by hypothesis, there exists a ceg-supplement submodule Y in M with $U\beta_g^{*}Y$. Let S be a cofinite essential submodule of M and Y be a g-supplement of S in M. Then $M=S+Y$ and $S\cap Y \ll_g Y$. Since $X\beta_g^{*}V$ and $M=U+V$, $M=X+U$ and since $U\beta_g^{*}Y$ and $X \trianglelefteq M$, $M=X+Y$. Assume $X+T=M$ with $T \trianglelefteq Y$. Then $Y=Y\cap M=Y\cap(X+T)=X\cap Y+T$. By using [3, Lemma 1.24], we can see that $M=S+Y=X\cap Y+S+T=Y+X\cap(S+T)=U+X\cap(S+T)=X+U\cap(S+T)=V+U\cap(S+T)=U\cap V+S+T$. Since $U\cap V \ll_g M$ and $S+T \trianglelefteq M$, $M=S+T$ and since Y is a g-supplement of S in M and $T \trianglelefteq Y$, $T=Y$. Hence Y is a g-supplement of X in M. Thus M is cofinitely eg-supplemented. □

Corollary 3. *Let M be an $R-$module. If every cofinite essential submodule of M is β^{*} equivalent to an ceg-supplement submodule in M, then M is cofinitely eg-supplemented.*

Proof. Clear from Lemma 9. □

Corollary 4. *Let M be an $R-$module. If every cofinite essential submodule of M lies above an ceg-supplement submodule in M, then M is cofinitely eg-supplemented.*

Proof. Clear from Corollary 3. □

Corollary 5. *Let M be an $R-$module. If every cofinite essential submodule of M is a ceg-supplement submodule in M, then M is cofinitely eg-supplemented.*

Proof. Clear from Corollary 4. □

Lemma 10. *Let M be an $R-$module. If every cofinite submodule of M is β^* equivalent to a ceg-supplement submodule in M, then M is cofinitely g-supplemented.*

Proof. Let X be a cofinite submodule of M. By hypothesis, there exists a ceg-supplement submodule V in M with $X\beta^* V$. Let V be a g-supplement of a cofinite essential submodule U in M. Then $M = U + V$ and $U \cap V \ll_g V$. Since U is a cofinite submodule of M, by hypothesis, there exists a ceg-supplement submodule Y in M with $U\beta^*Y$. Let S be a cofinite essential submodule of M and Y be a g-supplement of S in M. Then $M = S+Y$ and $S\cap Y \ll_g Y$. Since $X\beta^* V$ and $M = U+V$, $M = X + U$ and since $U\beta^*Y$, $M = X + Y$. Assume $X + T = M$ with $T \trianglelefteq Y$. Then $Y = Y \cap M = Y \cap (X + T) = X \cap Y + T$. By using [3, Lemma 1.24], we can see that $M = S+Y = X\cap Y+S+T = Y+X\cap(S+T) = U+X\cap(S+T) = X+U\cap(S+T) = V+U\cap(S+T) = U\cap V+S+T$. Since $U\cap V \ll_g M$ and $S+T \trianglelefteq M$, $M = S+T$ and since Y is a g-supplement of S in M and $T \trianglelefteq Y$, $T = Y$. Hence Y is a g-supplement of X in M. Thus M is cofinitely g-supplemented. □

Corollary 6. *Let M be an $R-$module. If every cofinite submodule of M lies above a ceg-supplement submodule in M, then M is cofinitely g-supplemented.*

Proof. Clear from Lemma 10. □

Corollary 7. *Let M be an $R-$module. If every cofinite submodule of M is a ceg-supplement submodule in M, then M is cofinitely g-supplemented.*

Proof. Clear from Lemma 10. □

Example 1. Consider the $\mathbb{Z}-$module $\mathbb{Q}$. Since ${}_{\mathbb{Z}}\mathbb{Q}$ have no proper cofinite essential submodules, ${}_{\mathbb{Z}}\mathbb{Q}$ is cofinitely eg-supplemented. But, since ${}_{\mathbb{Z}}\mathbb{Q}$ is not supplemented and every nonzero submodule of ${}_{\mathbb{Z}}\mathbb{Q}$ is essential in ${}_{\mathbb{Z}}\mathbb{Q}$, ${}_{\mathbb{Z}}\mathbb{Q}$ is not essential g-supplemented.

Example 2. Consider the $\mathbb{Z}-$module $\mathbb{Q}\oplus\mathbb{Z}_{p^2}$ for a prime p. It is easy to check that $Rad_g\mathbb{Z}_{p^2} \neq \mathbb{Z}_{p^2}$. By [6, Lemma 4], $Rad_g\left(\mathbb{Q}\oplus\mathbb{Z}_{p^2}\right) = Rad_g\mathbb{Q}\oplus Rad_g\mathbb{Z}_{p^2} \neq \mathbb{Q}\oplus\mathbb{Z}_{p^2}$. Since $\mathbb{Q}$ and $\mathbb{Z}_{p^2}$ are cofinitely eg-supplemented, by Lemma 5, $\mathbb{Q}\oplus\mathbb{Z}_{p^2}$ is cofinitely eg-supplemented. But $\mathbb{Q}\oplus\mathbb{Z}_{p^2}$ is not essential g-supplemented.

REFERENCES

[1] R. Alizade, G. Bilhan, and P. F. Smith, "Modules whose maximal submodules have supplements," *Comm. Algebra*, vol. 29, no. 6, pp. 2389–2405, 2001, doi: 10.1081/AGB-100002396.

[2] G. F. Birkenmeier, F. T. Mutlu, C. Nebiyev, N. Sokmez, and A. Tercan, "Goldie*-supplemented modules," *Glasgow Mathematical Journal*, vol. 52A, pp. 41–52, 2010, doi: 10.1017/S0017089510000212.

[3] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting Modules: Supplements and Projectivity in Module Theory (Frontiers in Mathematics)*, 2006th ed. Basel: Birkhäuser, 8 2006.
[4] B. Koşar, "Cofinitely g-supplemented modules," *British Journal of Mathematics & Computer Science*, vol. 17, no. 4, pp. 1–6, 2016, doi: 10.9734/BJMCS/2016/26697.
[5] B. Koşar, C. Nebiyev, and N. Sökmez, "G-supplemented modules," *Ukrainian Mathematical Journal*, vol. 67, no. 6, pp. 861–864, 2015, doi: 10.1007/s11253-015-1127-8.
[6] B. Koşar, C. Nebiyev, and A. Pekin, "A generalization of g-supplemented modules," *Miskolc Math. Notes*, vol. 20, no. 1, pp. 345–352, 2019, doi: 10.18514/MMN.2019.2586.
[7] B. Koşar and C. Nebiyev, "Cofinitely essential supplemented modules," *Turkish Studies Information Technologies and Applied Sciences*, vol. 13, no. 29, pp. 83–88, 2018.
[8] B. Koşar and C. Nebiyev, "Amply cofinitely essential supplemented modules," *Archives of Current Research International*, vol. 19, no. 1, pp. 1–4, 2019, doi: 10.9734/acri/2019/v19i130146.
[9] C. Nebiyev and H. H. Ökten, "Essential g-supplemented modules," *Turkish Studies Information Technologies and Applied Sciences*, vol. 14, no. 1, pp. 83–89, 2019.
[10] C. Nebiyev and H. H. Ökten, "Cofinitely eg-supplemented modules," in *3rd International E-Conference on Mathematical Advances and Applications (ICOMAA-2020)*, 2020.
[11] C. Nebiyev, H. H. Ökten, and A. Pekin, "Amply essential supplemented modules," *Journal of Scientific Research and Reports*, vol. 24, no. 4, pp. 1–4, 2018, doi: 10.9734/JSRR/2018/45651.
[12] C. Nebiyev, H. H. Ökten, and A. Pekin, "Essential supplemented modules," *International Journal of Pure and Applied Mathematics*, vol. 120, no. 2, pp. 253–257, 2018.
[13] C. Nebiyev and N. Sökmez, "Beta *g*-star relation on modules," *Eur. J. Pure Appl. Math.*, vol. 11, no. 1, pp. 238–243, 2018, doi: 10.29020/nybg.ejpam.v11i1.2741.
[14] R. Wisbauer, *Foundations of module and ring theory*, german ed., ser. Algebra, Logic and Applications. Gordon and Breach Science Publishers, Philadelphia, PA, 1991, vol. 3, a handbook for study and research.
[15] D. X. Zhou and X. R. Zhang, "Small-Essential Submodules and Morita Duality," *Southeast Asian Bulletin of Mathematics*, vol. 35, pp. 1051–1062, 2011.

Authors' addresses

Celil Nebiyev
Ondokuz Mayıs University, Department of Mathematics, Atakum, Samsun, Turkey
E-mail address: cnebiyev@omu.edu.tr

Hasan Hüseyin Ökten
(**Corresponding author**) Amasya University, Technical Sciences Vocational School, Amasya, Turkey
E-mail address: hokten@gmail.com