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A PARAMETRIC TYPE OF CAUCHY POLYNOMIALS

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Abstract. In this paper, we introduce a parametric type of Cauchy polynomials and study their characteristic and combinatorial properties. In particular, we show some determinant expressions.

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1. INTRODUCTION

By using the Taylor expansion of the two functions $e^{pt}\cos qt$ and $e^{pt}\sin qt$ in [15], a parametric type of Bernoulli polynomials is introduced and their basic properties are presented ([14]). Precisely, two kinds of bivariate Bernoulli polynomials are introduced as

$$\frac{te^{pt}}{e^t-1}\cos qt=\sum_{n=0}^\infty B_n^{(c)}(p,q)\frac{t^n}{n!} \tag{1.1}$$

and

$$\frac{te^{pt}}{e^t-1}\sin qt=\sum_{n=0}^\infty B_n^{(s)}(p,q)\frac{t^n}{n!}\,. \tag{1.2}$$

In [13], by defining two specific exponential generating functions, a kind of Euler polynomials is introduced and its basic properties are studied in detail. In [17], a kind of parametric Fubini-type polynomials is defined and some fundamental properties of these parametric-kind Fubini-type polynomials are studied. In [18], a type of generalized parametric Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials is introduced and systematically their basic properties are studied. Then, in [19], it is shown that the real and imaginary parts of a general set of complex Appell polynomials can be represented in terms of the Chebyshev polynomials of the first and second kind. In [7], some special polynomials related to Euler and Bernoulli polynomials are studied and some identities are given. Recently, in [6], identities, relations and combinatorial sums of several polynomials including trigonometric type polynomials are given. In [5], generating functions of parametric Hermite-based Milne–Thomson type polynomials are constructed.

In 1935, D. H. Lehmer [12] introduced and investigated generalized Euler numbers W_n, defined by the generating function

$$\frac{3}{e^t+e^{\omega t}+e^{\omega^2 t}}=\sum_{n=0}^\infty W_n\frac{t^n}{n!},$$

where $\omega=\frac{-1+\sqrt{-3}}{2}$ and $\omega^2=\bar\omega=\frac{-1-\sqrt{-3}}{2}$ are the primitive cube roots of unity. Notice that $W_n=0$ unless $n\equiv 0 \pmod 3$. More general Lehmer's type of Euler numbers were considered in [1,9]. Since

$$\frac{e^t+e^{\omega t}+e^{\omega^2 t}}{3}=\sum_{n=0}^\infty\frac{t^{3n}}{(3n)!},$$

the function on the left-hand side can be considered as 3 level generalization of the level 2 function

$$\cosh t=\sum_{n=0}^\infty\frac{t^{2n}}{(2n)!}.$$

Therefore, it may be natural to consider analogous bivariate Cauchy polynomials with level 2 of (1.1) and (1.2) with level 2. Nevertheless, it is even possible to introduce a slightly more general case. In fact, in [8], a parametric type of Bernoulli polynomials with level 3 is introduced and their characteristic and combinatorial properties are studied.

For real numbers p,q, define *bivariate Cauchy polynomials* $c_n^{(3,i)}(p,q)$ *with level* 3 $(i=0,1,2)$ as

$$\frac{t f_q^{(3,i)}(t)}{(1+t)^p\log(1+t)}=\sum_{n=0}^\infty c_n^{(3,i)}(p,q)\frac{t^n}{n!},\tag{1.3}$$

where

$$f_q^{(3,i)}(t)=\sum_{n=0}^\infty(-1)^n\frac{(qt)^{3n+i}}{(3n+i)!}.\tag{1.4}$$

Their complementary polynomials $\widehat{c}_n^{(3,i)}(p,q)$ $(i=0,1,2)$ are defined as

$$\frac{t \widehat{f}_q^{(3,i)}(t)}{(1+t)^p\log(1+t)}=\sum_{n=0}^\infty \widehat{c}_n^{(3,i)}(p,q)\frac{t^n}{n!},\tag{1.5}$$

where

$$\widehat{f}_q^{(3,i)}(t)=\sum_{n=0}^\infty\frac{(qt)^{3n+i}}{(3n+i)!}.\tag{1.6}$$

When $q=i=0$, $c_n(x)=c_n^{(3,0)}(x,0)=\widehat{c}_n^{(3,0)}(x,0)$ is the Cauchy polynomial, defined by

$$\frac{t}{(1+t)^x\log(1+t)}=\sum_{n=0}^\infty c_n(x)\frac{t^n}{n!}\tag{1.7}$$

(see [3]). When $x=0$, $c_n=c_n(0)$ is the classical Cauchy number. In [11], convolution identities for Cauchy numbers are studied.

Since for $i=0,1,2$

$$\widehat{f}_1^{(3,i)}(t)=\frac{e^t+\omega^{2i}e^{\omega t}+\omega^i e^{\omega^2 t}}{3}$$

and

$$f_1^{(3,i)}(t)=(-1)^i\frac{e^{-t}+\omega^{2i}e^{-\omega t}+\omega^i e^{-\omega^2 t}}{3},$$

we see that

$$f_q^{(3,i)}(-t)=(-1)^i\widehat{f}_q^{(3,i)}(t), \tag{1.8}$$

$$\widehat{f}_q^{(3,i)}(-t)=(-1)^i f_q^{(3,i)}(t). \tag{1.9}$$

It is trivial to see the following.

Proposition 1. *For $n\ge 0$ and $i,j=0,1,2$,*

$$c_n^{(3,i)}(p,\omega^j q)=\omega^{ij}c_n^{(3,i)}(p,q),$$

$$\widehat{c}_n^{(3,i)}(p,\omega^j q)=\omega^{ij}\widehat{c}_n^{(3,i)}(p,q),$$

where ω is a complex root of $\omega^3=1$.

In the next section, we show several properties of bivariate Cauchy polynomials with level 3. In particular, Theorem 1 entails fundamental recurrence formulas. By using these formulas, we give determinant expressions of bivariate Cauchy polynomials with level 3. In special cases, we can get determinant expressions of the classical Cauchy polynomials and numbers. In a similar method, we also have determinant expressions of bivariate Cauchy polynomials with level 2.

2. Basic properties

In this section, we show several properties of bivariate Cauchy polynomials with level 3. We introduce the auxiliary polynomials $G_n^{(3,i)}(p,q)$ and $\widehat{G}_n^{(3,i)}(p,q)$ as

$$\frac{f_q^{(3,i)}(t)}{(1+t)^p}=\sum_{n=0}^\infty G_n^{(3,i)}(p,q)\frac{t^n}{n!}, \tag{2.1}$$

$$\frac{\widehat{f}_q^{(3,i)}(t)}{(1+t)^p}=\sum_{n=0}^\infty \widehat{G}_n^{(3,i)}(p,q)\frac{t^n}{n!}. \tag{2.2}$$

respectively. From the definition in (2.1), we have the following.

Proposition 2. *For $n\ge 0$ and $i=0,1,2$,*

$$G_n^{(3,i)}(p,q)=\sum_{k=0}^{\lfloor\frac{n-i}{3}\rfloor}(-1)^k\binom{n}{3k+i}(-p)_{n-3k-i}q^{3k+i}, \tag{2.3}$$

$$\widehat{G}_n^{(3,i)}(p,q)=\sum_{k=0}^{\lfloor\frac{n-i}{3}\rfloor}\binom{n}{3k+i}(-p)_{n-3k-i}q^{3k+i}\,, \tag{2.4}$$

where $(x)_\ell=x(x-1)\cdots(x-\ell+1)$ *(*$\ell\ge 1$*) denotes the falling factorial with* $(x)_0=1$.

$c_n^{(3,i)}(p,q)$ (respectively, $\widehat{c}_n^{(3,i)}(p,q)$) can be written in terms of $G_n^{(3,i)}(p,q)$ (respectively, $\widehat{G}_n^{(3,i)}(p,q)$).

Proposition 3. *For* $n\ge 0$ *and* $i=0,1,2$*,*

$$c_n^{(3,i)}(p,q)=\sum_{k=0}^{n}\binom{n}{k}c_{n-k}G_k^{(3,i)}(p,q)\,, \tag{2.5}$$

$$\widehat{c}_n^{(3,i)}(p,q)=\sum_{k=0}^{n}\binom{n}{k}c_{n-k}\widehat{G}_k^{(3,i)}(p,q)\,. \tag{2.6}$$

On the contrary to Proposition 3, $G_n^{(3,i)}(p,q)$ (respectively, $\widehat{G}_n^{(3,i)}(p,q)$) can be written in terms of $c_n^{(3,i)}(p,q)$ (respectively, $\widehat{c}_n^{(3,i)}(p,q)$).

Proposition 4. *For* $n\ge 0$ *and* $i=0,1,2$*,*

$$G_n^{(3,i)}(p,q)=\sum_{k=0}^{n}\frac{(-1)^{n-k}n!}{(n-k+1)k!}c_k^{(3,i)}(p,q)\,, \tag{2.7}$$

$$\widehat{G}_n^{(3,i)}(p,q)=\sum_{k=0}^{n}\frac{(-1)^{n-k}n!}{(n-k+1)k!}\widehat{c}_k^{(3,i)}(p,q)\,. \tag{2.8}$$

We have a summation formula for $c_n^{(3,i)}(p,q)$ (respectively, $\widehat{c}_n^{(3,i)}(p,q)$).

Theorem 1. *For* $n\ge 0$ *and* $i=0,1,2$*,*

$$\sum_{k=0}^{n}\binom{n+1}{k}c_k^{(3,i)}(p,q)\frac{d}{dp}(p)_{n-k+1} =\begin{cases}-(-1)^{(n-i)/3}(n+1)q^n & \text{if } n\equiv i \pmod 3;\\ 0 & \text{otherwise}\,,\end{cases} \tag{2.9}$$

$$\sum_{k=0}^{n}\binom{n+1}{k}\widehat{c}_k^{(3,i)}(p,q)\frac{d}{dp}(p)_{n-k+1} =\begin{cases}-(n+1)q^n & \text{if } n\equiv i \pmod 3;\\ 0 & \text{otherwise}\,.\end{cases} \tag{2.10}$$

When $p=0$ in Theorem 1, by

$$\frac{d}{dp}(p)_{n-k+1}\Big|_{p=0}=(-1)^{n-k}(n-k)!$$

we have simpler recurrence relations.

Corollary 1. *For $n\ge 0$ and $i=0,1,2$,*

$$\sum_{k=0}^{n}\frac{(-1)^{n-k}c_k^{(3,i)}(p,q)}{(n-k+1)k!}=\begin{cases}-(-1)^{(n-i)/3}\dfrac{q^n}{n!} & \text{if } n\equiv i \pmod 3;\\ 0 & \text{otherwise},\end{cases}$$

$$\sum_{k=0}^{n}\frac{(-1)^{n-k}\widehat{c}_k^{(3,i)}(p,q)}{(n-k+1)k!}=\begin{cases}-\dfrac{q^n}{n!} & \text{if } n\equiv i \pmod 3;\\ 0 & \text{otherwise}.\end{cases}$$

Proof of Theorem 1. From the definition in (1.3) and the proof of Proposition 4, we have

$$\begin{aligned}
f_q^{(3,i)}(t)&=(1+t)^p\cdot\frac{\log(1+t)}{t}\sum_{n=0}^{\infty}c_n^{(3,i)}(p,q)\frac{t^n}{n!}\\
&=\left(\sum_{m=0}^{\infty}\frac{(p)_m}{m!}t^m\right)\left(\sum_{l=0}^{\infty}\sum_{k=0}^{l}\frac{(-1)^{l-k+1}}{(l-k+1)k!}c_k^{(3,i)}(p,q)t^l\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{(p)_{n-l}}{(n-l)!}\sum_{k=0}^{l}\frac{(-1)^{l-k+1}n!}{(l-k+1)k!}c_k^{(3,i)}(p,q)\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}c_k^{(3,i)}(p,q)\sum_{l=k}^{n}\binom{n}{l}\binom{l}{k}\frac{(-p)^{n-l}}{(l-k+1)}\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^{k-1}c_k^{(3,i)}(p,q)}{k!}\sum_{l=k}^{n}\frac{(-1)^l n!(p)_{n-l}}{(l-k+1)(n-l)!}\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^{k-1}c_k^{(3,i)}(p,q)}{k!}(-1)^n n!\sum_{j=0}^{n-k}\frac{(-1)^j(p)_j}{(n-k-j+1)j!}\right)\frac{t^n}{n!}.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{j=0}^{n-k}\frac{(-1)^j(p)_j}{(n-k-j+1)j!}&=\frac{1}{(n-k+1)!}\sum_{l=0}^{n-k}(-1)^l(l+1)\begin{bmatrix}n-k+1\\ l+1\end{bmatrix}p^l\\
&=\frac{1}{(n-k+1)!}\frac{d}{dp}(-1)^{n-k}(p)_{n-k+1},
\end{aligned}$$

comparing the coefficients with

$$f_q^{(3,i)}(t)=\sum_{n=0}^\infty(-1)^n\frac{(qt)^{3n+i}}{(3n+i)!}$$

in (1.4), we get the identity (2.9). The identity (2.10) is similarly proved. □

From Theorem 1, we have the recurrence relations

$$\begin{aligned}c_n^{(3,i)}(p,q)&=-n!\sum_{k=0}^{n-1}\frac{c_k^{(3,i)}(p,q)}{k!}\frac{d}{dp}\frac{(p)_{n-k+1}}{(n-k+1)!}\\&\quad-\begin{cases}(-1)^{(n-i)/3}q^n & \text{if } n\equiv i \pmod 3;\\ 0 & \text{otherwise},\end{cases}\end{aligned}\tag{2.11}$$

$$\begin{aligned}\widehat{c}_n^{(3,i)}(p,q)&=-n!\sum_{k=0}^{n-1}\frac{\widehat{c}_k^{(3,i)}(p,q)}{k!}\frac{d}{dp}\frac{(p)_{n-k+1}}{(n-k+1)!}\\&\quad-\begin{cases}q^n & \text{if } n\equiv i \pmod 3;\\ 0 & \text{otherwise}\end{cases}\end{aligned}\tag{2.12}$$

with

$$c_0^{(3,i)}(p,q)=\widehat{c}_0^{(3,i)}(p,q)=\begin{cases}1 & \text{if } i=0;\\ 0 & \text{if } i=1,2.\end{cases}$$

Using recurrence relations, we can obtain the exact values of $c_n^{(3,i)}(p,q)$ and $\widehat{c}_n^{(3,i)}(p,q)$ for small n. We list some initial values in Appendix.

Theorem 2. *For $n\ge 0$ and $i=0,1,2$,*

$$c_n^{(3,i)}(p+1,q)-c_n^{(3,i)}(p,q)=-nc_{n-1}^{(3,i)}(p+1,q),\tag{2.13}$$

$$\widehat{c}_n^{(3,i)}(p+1,q)-\widehat{c}_n^{(3,i)}(p,q)=-n\widehat{F}_{n-1}^{(3,i)}(p+1,q).\tag{2.14}$$

Proof.

$$\begin{aligned}\sum_{n=0}^\infty c_n^{(3,i)}(p+1,q)\frac{t^n}{n!}-\sum_{n=0}^\infty c_n^{(3,i)}(p,q)\frac{t^n}{n!}&=\frac{tf_q^{(3,i)}(t)}{(1+t)^{p+1}\log(1+t)}-\frac{tf_q^{(3,i)}(t)}{(1+t)^{p}\log(1+t)}\\&=-\frac{t^2f_q^{(3,i)}(t)}{(1+t)^{p+1}\log(1+t)}\\&=-t\sum_{n=0}^\infty c_n^{(3,i)}(p+1,q)\frac{t^n}{n!}\\&=-\sum_{n=0}^\infty nc_{n-1}^{(3,i)}(p+1,q)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients on both sides, we get the identity (2.13). The identity (2.14) is similarly proved. □

Theorem 3. *For* $n \geq 0$ *and* $i = 0, 1, 2$,

$$c_n^{(3,i)}(p+r,q) = \sum_{k=0}^{n} \binom{n}{k} c_k^{(3,i)}(p,q)(-r)_{n-k}\,, \tag{2.15}$$

$$\widehat{c}_n^{(3,i)}(p+r,q) = \sum_{k=0}^{n} \binom{n}{k} \widehat{c}_k^{(3,i)}(p,q)(-r)_{n-k}\,. \tag{2.16}$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} c_n^{(3,i)}(p+r,q)\frac{t^n}{n!} &= \frac{t f_q^{(3,i)}(t)}{(1+t)^{p+r}\log(1+t)} \\ &= \left(\sum_{n=0}^{\infty} c_n^{(3,i)}(p,q)\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty} \frac{(-r)_l t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} c_k^{(3,i)}(p,q)(-r)_{n-k}\frac{t^n}{n!}\,.\end{aligned}$$

Comparing the coefficients on both sides, we get the identity (2.15). The identity (2.16) is similarly proved. □

Theorem 4. *For* $n \geq 0$ *and* $i = 0, 1, 2$,

$$\sum_{k=0}^{n} \frac{c_k^{(3,i)}(p,q)}{k!} = -\frac{c_n^{(3,i)}(p+1,q)}{n!}\,, \tag{2.17}$$

$$\sum_{k=0}^{n} \frac{\widehat{c}_k^{(3,i)}(p,q)}{k!} = -\frac{\widehat{c}_n^{(3,i)}(p+1,q)}{n!}\,. \tag{2.18}$$

Proof. By using Theorem 3 (2.15) with $r = 1$ and $r = 0$,

$$\begin{aligned}c_{n+1}^{(3,i)}(p+1,q) - c_{n+1}^{(3,i)}(p,q) &= \sum_{k=0}^{n} \binom{n+1}{k} c_k^{(3,i)}(p,q)\cdot(-1)^{n-k+1}(n-k+1)! \\ &= (n+1)!\sum_{k=0}^{n}(-1)^{n-k+1}\frac{c_k^{(3,i)}(p,q)}{k!}\,.\end{aligned}$$

Together with Theorem 2 (2.13), we get the identity (2.17). The identity (2.18) is similarly proved. □

Theorem 5. *For* $n \geq 1$,

$$\frac{\partial}{\partial p} c_n^{(3,i)}(p,q) = \sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{c_k^{(3,i)}(p,q)}{(n-k)k!} \quad (i = 0, 1, 2)\,, \tag{2.19}$$

$$\frac{\partial}{\partial q}c_n^{(3,i)}(p,q)=\begin{cases}-nc_{n-1}^{(3,2)}(p,q) & \text{if } i=0;\\ nc_{n-1}^{(3,i-1)}(p,q) & \text{if } i=1,2,\end{cases} \tag{2.20}$$

$$\frac{\partial}{\partial p}\widehat{c}_n^{(3,i)}(p,q)=\sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{\widehat{c}_k^{(3,i)}(p,q)}{(n-k)k!}\quad (i=0,1,2), \tag{2.21}$$

$$\frac{\partial}{\partial q}\widehat{c}_n^{(3,i)}(p,q)=\begin{cases}n\widehat{c}_{n-1}^{(3,2)}(p,q) & \text{if } i=0;\\ n\widehat{c}_{n-1}^{(3,i-1)}(p,q) & \text{if } i=1,2.\end{cases} \tag{2.22}$$

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\partial}{\partial p}c_n^{(3,i)}(p,q)\frac{t^n}{n!}&=-\frac{tf_q^{(3,i)}(t)}{(1+t)^p}\\ &=\left(\sum_{m=1}^{\infty}(-1)^{m-1}\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}c_n^{(3,i)}(p,q)\frac{t^n}{n!}\right)\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{c_k^{(3,i)}(p,q)}{(n-k)k!}\frac{t^n}{n!},\end{aligned}$$

yielding (2.19). Since

$$\begin{aligned}\frac{\partial}{\partial q}f_q^{(3,0)}(t)&=t\sum_{n=1}^{\infty}\frac{(-1)^n(qt)^{3n-1}}{(3n-1)!}\\ &=-tf_q^{(3,2)}(t)\end{aligned}$$

and

$$\frac{\partial}{\partial q}f_q^{(3,i)}(t)=tf_q^{(3,i-1)}(t)\quad (i=1,2),$$

we get (2.20). The identities (2.21) and (2.22) are similarly proved. □

3. Determinants

Determinant is one of the most useful tools for analyzing numbers and polynomials. Our result is motivated by Glaisher's results in 1875 [4], where he gave determinantal expressions of several numbers, including Bernoulli, Euler and Cauchy numbers by solving the systems of equations. See [16] for a good collection of many types of determinants with their backgrounds and techniques. Our method is mainly from the recurrence relations and the inverse formula. Namely, we need the following equivalent relations (see, e.g., [10]).

Lemma 1.

$$\sum_{k=0}^{n}(-1)^{n-k}x_{n-k}z_k=0\quad \textit{with}\quad x_0=z_0=1$$

$$\Longleftrightarrow \quad x_n = \begin{vmatrix} z_1 & 1 & & \\ z_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ z_n & \cdots & z_2 & z_1 \end{vmatrix} \quad \Longleftrightarrow \quad z_n = \begin{vmatrix} x_1 & 1 & & \\ x_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ x_n & \cdots & x_2 & x_1 \end{vmatrix}.$$

When $i = 0$, bivariate Cauchy polynomials $c_n^{(3,0)}(p,q)$ with level 3 and their complementary numbers $\widehat{c}_n(p) := \widehat{c}_n^{(3,0)}(p,q)$ have determinant expressions.

Theorem 6. *For $n \ge 1$,*

$$c_n^{(3,0)}(p,q) = n! \begin{vmatrix} r_p(2) & 1 & 0 & \cdots & 0 \\ r_p(3) & r_p(2) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ r_p(n) & r_p(n-1) & \cdots & r_p(2) & 1 \\ r_p^*(n+1) & r_p^*(n) & \cdots & r_p^*(3) & r_p^*(2) \end{vmatrix}$$

and

$$\widehat{c}_n^{(3,0)}(p,q) = n! \begin{vmatrix} r_p(2) & 1 & 0 & \cdots & 0 \\ r_p(3) & r_p(2) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ r_p(n) & r_p(n-1) & \cdots & r_p(2) & 1 \\ \widehat{r}_p(n+1) & \widehat{r}_p(n) & \cdots & \widehat{r}_p(3) & \widehat{r}_p(2) \end{vmatrix},$$

where

$$r_p(n) = \frac{d}{dp}\frac{(-1)^{n-1}(p)_n}{n!} = \frac{1}{n!}\sum_{\ell=0}^{n-1}(-1)^\ell(\ell+1)\begin{bmatrix} n \\ \ell+1 \end{bmatrix} p^\ell$$

with

$$r_p^*(n) = r_p(n) - \begin{cases} \frac{q^{n-1}}{(n-1)!} & \text{if } n \equiv 1 \pmod 3; \\ 0 & \text{otherwise} \end{cases} \quad (n \ge 2)$$

and

$$\widehat{r}_p(n) = r_p(n) + \begin{cases} \frac{(-1)^n q^{n-1}}{(n-1)!} & \text{if } n \equiv 1 \pmod 3; \\ 0 & \text{otherwise} \end{cases} \quad (n \ge 2).$$

When $q = i = 0$, Cauchy polynomials $c_n(p) := c_n^{(3,0)}(p,0) = \widehat{c}_n^{(3,0)}(p,0) = \widehat{c}_n(p)$ in (1.7) have simpler determinant expressions.

Corollary 2. *For $n \ge 1$,*

$$c_n(p) = \widehat{c}_n(p) = n! \begin{vmatrix} r_p(2) & 1 & 0 & \cdots & 0 \\ r_p(3) & r_p(2) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ r_p(n) & r_p(n-1) & \cdots & r_p(2) & 1 \\ r_p(n+1) & r_p(n) & \cdots & r_p(3) & r_p(2) \end{vmatrix} .$$

Remark 1. When $p = 0$ in Corollary 2, by

$$r_0(n) = \frac{d}{dp} \frac{(-1)^{n-1}(p)_n}{n!}\bigg|_{p=0} = \frac{1}{n},$$

we have a determinant expression of Cauchy numbers c_n as follows [4, p. 50]:

$$c_n = n! \begin{vmatrix} \frac{1}{2} & 1 & 0 & \cdots & 0 \\ \frac{1}{3} & \frac{1}{2} & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{2} & 1 \\ \frac{1}{n+1} & \frac{1}{n} & \cdots & \frac{1}{3} & \frac{1}{2} \end{vmatrix} .$$

Proof of Theorem 6. From Theorem 1 with the recurrence relation in (2.11), we have

$$\frac{c_n^{(3,0)}(p,q)}{n!} = \sum_{k=0}^{n-1} (-1)^{n-k+1} r_p(n-k+1) \frac{c_k^{(3,0)}(p,q)}{k!} - \begin{cases} \frac{(-1)^{n/3} q^n}{n!} & \text{if } n \equiv 0 \pmod 3; \\ 0 & \text{otherwise}. \end{cases}$$

By Lemma 1, this relation is equivalent to the determinant if it is expanded along the first row continually. More precisely, we see that $c_0^{(3,0)}(p,q) = 1$ and $c_1^{(p,q)} = r_p(2)$. And by induction, the first determinant in Theorem 6 is equal to

$$r_p(2) \frac{c_{n-1}^{(3,0)}(p,q)}{(n-1)!} - \begin{vmatrix} r_p(3) & 1 & 0 & \cdots & 0 \\ r_p(4) & r_p(2) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ r_p(n) & r_p(n-2) & \cdots & r_p(2) & 1 \\ r_p^*(n+1) & r_p^*(n-1) & \cdots & r_p^*(3) & r_p^*(2) \end{vmatrix}$$

$$= r_p(2) \frac{c_{n-1}^{(3,0)}(p,q)}{(n-1)!} - r_p(3) \frac{c_{n-2}^{(3,0)}(p,q)}{(n-2)!} + \cdots + (-1)^{n-1} r_p(n-1) \frac{c_2^{(3,0)}(p,q)}{2!}$$

$$
\begin{aligned}
&+(-1)^n\begin{vmatrix} r_p(n) & 1 \\ r_p^*(n+1) & r_p^*(2) \end{vmatrix}\\
&= r_p(2)\frac{c_{n-1}^{(3,0)}(p,q)}{(n-1)!} - r_p(3)\frac{c_{n-2}^{(3,0)}(p,q)}{(n-2)!} + \cdots + (-1)^{n-1} r_p(n-1)\frac{c_2^{(3,0)}(p,q)}{2!}\\
&\quad + (-1)^n r_p(n) c_1^{(3,0)}(p,q) + (-1)^{n+1} r_p^*(n+1) c_0^{(3,0)}(p,q)\\
&= \frac{c_n^{(3,0)}(p,q)}{n!}.
\end{aligned}
$$

Another identity can be proved similarly. □

By using Lemma 1 again, we have the inversion relation of Corollary 2.

Corollary 3. *For $n \ge 1$,*

$$
r_p(n+1) = \begin{vmatrix}
c_1(p) & 1 & 0 & \cdots & 0 \\
\frac{c_2(p)}{2!} & c_1(p) & 1 & & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
\frac{c_{n-1}(p)}{(n-1)!} & \frac{c_{n-2}(p)}{(n-2)!} & \cdots & c_1(p) & 1 \\
\frac{c_n(p)}{n!} & \frac{c_{n-1}(p)}{(n-1)!} & \cdots & \frac{c_2(p)}{2!} & c_1(p)
\end{vmatrix}.
$$

We shall use Trudi's formula to obtain different explicit expressions and inversion relations for the numbers $c_n^{(3,i)}(p,0)$.

Lemma 2. *For $n \ge 1$, we have*

$$
\begin{vmatrix}
a_1 & a_0 & 0 & \cdots & \\
a_2 & a_1 & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
a_{n-1} & & \cdots & a_1 & a_0 \\
a_n & a_{n-1} & \cdots & a_2 & a_1
\end{vmatrix}
= \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-a_0)^{n-t_1-\cdots-t_n} a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n},
$$

where $\binom{t_1+\cdots+t_n}{t_1,\dots,t_n} = \frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$ are the multinomial coefficients.

This relation is known as Trudi's formula [16, Vol.3, p.214], [20] and the case $a_0 = 1$ of this formula is known as Brioschi's formula [2], [16, Vol.3, pp.208–209].

By Corollary 2 and Corollary 3 with Lemma 2, we get different expressions of $c_n(p)$ and $r_p(n)$.

Corollary 4. *For $n \ge 1$,*

$$
\begin{aligned}
c_n(p) = &\sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n}\\
&\times (-1)^{n-t_1-\cdots-t_n} \big(r_p(2)\big)^{t_1} \big(r_p(3)\big)^{t_2} \cdots \big(r_p(n+1)\big)^{t_n},
\end{aligned}
$$

$$r_p(n+1)=\sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}\\ \times(-1)^{n-t_1-\cdots-t_n}\bigl(c_1(p)\bigr)^{t_1}\left(\frac{c_2(p)}{2!}\right)^{t_2}\cdots\left(\frac{c_n(p)}{n!}\right)^{t_n}.$$

3.1. *Determinant expressions for $c_n^{(3,1)}(p,q)$ and $c_n^{(3,2)}(p,q)$*

When $i=1$, bivariate Cauchy polynomials $c_n^{(3,1)}(p,q)$ with level 3 and their complementary polynomials $\widehat{c}_n^{(3,1)}(p,q)$ have also determinant expressions.

Theorem 7. *For $n\ge 2$,*

$$c_n^{(3,1)}(p,q)=n!q\begin{vmatrix} r_p(2) & 1 & 0 & \cdots & 0\\ r_p(3) & r_p(2) & 1 & & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ r_p(n-1) & r_p(n-2) & \cdots & r_p(2) & 1\\ r_p^\dagger(n) & r_p^\dagger(n-1) & \cdots & r_p^\dagger(3) & r_p^\dagger(2)\end{vmatrix}$$

and

$$\widehat{c}_n^{(3,1)}(p,q)=n!q\begin{vmatrix} r_p(2) & 1 & 0 & \cdots & 0\\ r_p(3) & r_p(2) & 1 & & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ r_p(n-1) & r_p(n-2) & \cdots & r_p(2) & 1\\ \widehat{r}_p^\dagger(n) & \widehat{r}_p^\dagger(n-1) & \cdots & \widehat{r}_p^\dagger(3) & \widehat{r}_p^\dagger(2)\end{vmatrix},$$

where

$$r_p^\dagger(n)=r_p(n)-\begin{cases}\frac{q^{n-1}}{n!} & \text{if } n\equiv 1 \pmod 3;\\ 0 & \text{otherwise}\end{cases}\qquad (n\ge 2)$$

and

$$\widehat{r}_p^\dagger(n)=r_p(n)+\begin{cases}\frac{(-1)^n q^{n-1}}{n!} & \text{if } n\equiv 1 \pmod 3;\\ 0 & \text{otherwise}\end{cases}\qquad (n\ge 2).$$

Proof. When $i=1$, by $c_0^{(3,1)}(p,q)=0$ and $c_1^{(3,1)}(p,q)=q$, from Theorem 1 with the recurrence relation in (2.11), we have for $n\ge 2$

$$\frac{c_n^{(3,1)}(p,q)}{n!}=\sum_{k=1}^{n-1}(-1)^{n-k+1}r_p(n-k+1)\frac{c_k^{(3,1)}(p,q)}{k!}\\ +\begin{cases}\frac{(-1)^{(n-1)/3}q^n}{n!} & \text{if } n\equiv 1 \pmod 3;\\ 0 & \text{otherwise}.\end{cases}$$

Putting $c_n^\dagger = c_n^{(3,1)}(p,q)/q$, we get for $n \ge 2$

$$\frac{c_n^\dagger}{n!} = \sum_{k=1}^{n-1}(-1)^{n-k+1}r_p(n-k+1)\frac{c_k^\dagger}{k!} - \begin{cases}\frac{(-1)^{(n-1)/3}q^{n-1}}{n!} & \text{if } n \equiv 1 \pmod 3;\\ 0 & \text{otherwise}\end{cases}$$

with $c_1^\dagger = 1$. Then, the first identity is proved similarly to Theorem 6 because the first determinant is equal to $c_n^\dagger/n!$. The proof of the second identity is similar and omitted. □

When $i=2$, bivariate Cauchy polynomials $c_n^{(3,2)}(p,q)$ with level 3 and their complementary polynomials $\widehat{c}_n^{(3,2)}(p,q)$ have also determinant expressions.

Theorem 8. *For $n \ge 3$,*

$$c_n^{(3,2)}(p,q) = \frac{n!q^2}{2}\begin{vmatrix} r_p(2) & 1 & 0 & \cdots & 0\\ r_p(3) & r_p(2) & 1 & & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ r_p(n-2) & r_p(n-3) & \cdots & r_p(2) & 1\\ r_p^\star(n-1) & r_p^\star(n-2) & \cdots & r_p^\star(3) & r_p^\star(2)\end{vmatrix}$$

and

$$\widehat{c}_n^{(3,2)}(p,q) = \frac{n!q^2}{2}\begin{vmatrix} r_p(2) & 1 & 0 & \cdots & 0\\ r_p(3) & r_p(2) & 1 & & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ r_p(n-2) & r_p(n-3) & \cdots & r_p(2) & 1\\ \widehat{r}_p^\star(n-1) & \widehat{r}_p^\star(n-2) & \cdots & \widehat{r}_p^\star(3) & \widehat{r}_p^\star(2)\end{vmatrix},$$

where

$$r_p^\star(n) = r_p(n) - \begin{cases}\frac{2q^{n-1}}{(n+1)!} & \textit{if } n \equiv 1 \pmod 3;\\ 0 & \textit{otherwise}\end{cases} \qquad (n \ge 2)$$

and

$$\widehat{r}_p^\star(n) = r_p(n) + \begin{cases}\frac{(-1)^n 2q^{n-1}}{(n+1)!} & \textit{if } n \equiv 1 \pmod 3;\\ 0 & \textit{otherwise}\end{cases} \qquad (n \ge 2)\,.$$

Proof. When $i=2$, by $c_0^{(3,1)}(p,q) = c_1^{(3,1)}(p,q) = 0$ and $c_2^{(3,1)}(p,q) = q^2$, from Theorem 1 with the recurrence relation in (2.11), we have for $n \ge 3$

$$\frac{c_n^{(3,2)}(p,q)}{n!} = \sum_{k=2}^{n-1}(-1)^{n-k+1}r_p(n-k+1)\frac{c_k^{(3,2)}(p,q)}{k!} - \begin{cases}\frac{(-1)^{(n-2)/3}q^n}{n!} & \text{if } n \equiv 2 \pmod 3;\\ 0 & \text{otherwise}\,.\end{cases}$$

Putting $c_n^\star=2c_n^{(3,2)}(p,q)/q^2$, we get for $n\ge 3$

$$\frac{c_n^\star}{n!}=\sum_{k=1}^{n-1}(-1)^{n-k+1}r_p(n-k+1)\frac{c_k^\star}{k!}-\begin{cases}\frac{(-1)^{(n-2)/3}2q^{n-2}}{n!} & \text{if } n\equiv 2 \pmod 3;\\ 0 & \text{otherwise}\end{cases}$$

with $c_2^\star/2!=1$. Then, the first identity is proved similarly to Theorem 6 because the first determinant is equal to $c_n^\star/n!$. The proof of the second identity is similar and omitted. □

4. Bivariate Cauchy polynomials with level 2

Using the functions

$$\cos qt=f_q^{(2,0)}(t),\qquad \sin qt=f_q^{(2,1)},$$
$$\cosh qt=\widehat{f}_q^{(2,0)}(t),\qquad \sinh qt=\widehat{f}_q^{(2,1)},$$

we can define *bivariate Cauchy polynomials* $c_n^{(2,i)}(p,q)$ *with level* 2 $(i=0,1)$ as

$$\frac{t f_q^{(2,i)}(t)}{(1+t)^p\log(1+t)}=\sum_{n=0}^\infty c_n^{(2,i)}(p,q)\frac{t^n}{n!}\,, \tag{4.1}$$

where

$$f_q^{(2,i)}(t)=\sum_{n=0}^\infty(-1)^n\frac{(qt)^{2n+i}}{(2n+i)!}\,. \tag{4.2}$$

Their complementary polynomials $\widehat{c}_n^{(2,i)}(p,q)$ $(i=0,1)$ are defined as

$$\frac{t \widehat{f}_q^{(2,i)}(t)}{(1+t)^p\log(1+t)}=\sum_{n=0}^\infty \widehat{c}_n^{(2,i)}(p,q)\frac{t^n}{n!}\,, \tag{4.3}$$

where

$$\widehat{f}_q^{(2,i)}(t)=\sum_{n=0}^\infty\frac{(qt)^{2n+i}}{(2n+i)!}\,. \tag{4.4}$$

When $q=0$, $c_n(x)=c_n^{(2,i)}(x,0)=\widehat{c}_n^{(2,i)}(x,0)$.

Similarly to those with level 3, there are characteristic properties for bivariate Cauchy polynomials with level 2. We have the main recurrence relations for $c_n^{(2,i)}(p,q)$ (respectively, $\widehat{c}_n^{(2,i)}(p,q)$).

Theorem 9. *For $n\ge 0$ and $i=0,1,2$,*

$$\sum_{k=0}^n\binom{n+1}{k}c_k^{(2,i)}(p,q)\frac{d}{dp}(p)_{n-k+1}=\begin{cases}-(-1)^{(n-i)/2}(n+1)q^n & \textit{if } n\equiv i \pmod 2;\\ 0 & \textit{otherwise}\,,\end{cases} \tag{4.5}$$

$$\sum_{k=0}^{n}\binom{n+1}{k}\widehat{c}_k^{(2,i)}(p,q)\frac{d}{dp}(p)_{n-k+1}=\begin{cases}-(n+1)q^n & \text{if } n\equiv i \pmod 2;\\ 0 & \text{otherwise}.\end{cases} \tag{4.6}$$

When $p=0$ in Theorem 9, we have simpler recurrence relations.

Corollary 5. *For $n\ge 0$ and $i=0,1,2$,*

$$\sum_{k=0}^{n}\frac{(-1)^{n-k}c_k^{(2,i)}(p,q)}{(n-k+1)k!}=\begin{cases}-(-1)^{(n-i)/3}\dfrac{q^n}{n!} & \text{if } n\equiv i \pmod 2;\\ 0 & \text{otherwise},\end{cases}$$

$$\sum_{k=0}^{n}\frac{(-1)^{n-k}\widehat{c}_k^{(2,i)}(p,q)}{(n-k+1)k!}=\begin{cases}-\dfrac{q^n}{n!} & \text{if } n\equiv i \pmod 2;\\ 0 & \text{otherwise}.\end{cases}$$

It is analogous to obtain determinant expressions of $c_n^{(2,i)}(p,q)$ and $\widehat{c}_n^{(2,i)}(p,q)$, defined in (4.1) and (4.3), respectively. The details are similar to Theorem 6 and Theorem 7, respectively, and are omitted.

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APPENDIX

$$c_0^{(3,0)}(p,q)=1,$$
$$c_1^{(3,0)}(p,q)=-p+\frac{1}{2},$$
$$c_2^{(3,0)}(p,q)=p^2-\frac{1}{6},$$
$$c_3^{(3,0)}(p,q)=-p^3-\frac{3p^2}{2}-q^3+\frac{1}{4},$$
$$c_4^{(3,0)}(p,q)=p^4+4p^3+4p^2+4q^3p-2q^3-\frac{19}{30},$$
$$c_5^{(3,0)}(p,q)=-p^5-\frac{15p^4}{2}-\frac{55p^3}{3}-5(2q^3+3)p^2+\frac{5q^3}{3}+\frac{9}{4},$$
$$c_6^{(3,0)}(p,q)=p^6+12p^5+\frac{105p^4}{2}+20(q^3+5)p^3+6(5q^3+12)p^2+q^6-5q^3-\frac{863}{84}.$$

$$c_0^{(3,1)}(p,q)=0,$$

$$c_1^{(3,1)}(p,q)=q\,,$$
$$c_2^{(3,1)}(p,q)=-2qp+q\,,$$
$$c_3^{(3,1)}(p,q)=3qp^2-\frac{q}{2}\,,$$
$$c_4^{(3,1)}(p,q)=-4qp^3+6qp^2-q^4+q\,,$$
$$c_5^{(3,1)}(p,q)=5qp^4+20qp^3+20qp^2+5q^4p-\frac{5q^4}{2}-\frac{19q}{6}\,,$$
$$c_6^{(3,1)}(p,q)=-6qp^5-45qp^4-110qp^3-15q(q^3+6)p^2+\frac{5q^4}{2}+\frac{27q}{2}\,.$$

$$c_0^{(3,2)}(p,q)=0\,,$$
$$c_1^{(3,2)}(p,q)=0\,,$$
$$c_2^{(3,2)}(p,q)=q^2\,,$$
$$c_3^{(3,2)}(p,q)=-3q^2p+\frac{3q^2}{2}\,,$$
$$c_4^{(3,2)}(p,q)=6q^2p^2-q^2\,,$$
$$c_5^{(3,2)}(p,q)=-10q^2p^3-15q^2p^2-q^5+\frac{5q^2}{2}\,,$$
$$c_6^{(3,2)}(p,q)=15q^2p^4+60q^2p^3+60q^2p^2+6q^5p-3q^5-\frac{19q^2}{2}\,.$$

$$\widehat{c}_0^{(3,0)}(p,q)=1\,,$$
$$\widehat{c}_1^{(3,0)}(p,q)=-p+\frac{1}{2}\,,$$
$$\widehat{c}_2^{(3,0)}(p,q)=p^2-\frac{1}{6}\,,$$
$$\widehat{c}_3^{(3,0)}(p,q)=-p^3-\frac{3p^2}{2}+q^3+\frac{1}{4}\,,$$
$$\widehat{c}_4^{(3,0)}(p,q)=p^4+4p^3+4p^2-4q^3p+2q^3-\frac{19}{30}\,,$$
$$\widehat{c}_5^{(3,0)}(p,q)=-p^5-\frac{15p^4}{2}-\frac{55p^3}{3}+5(2q^3-3)p^2-\frac{5q^3}{3}+\frac{9}{4}\,,$$
$$\widehat{c}_6^{(3,0)}(p,q)=p^6+12p^5+\frac{105p^4}{2}-20(q^3-5)p^3-6(5q^3-12)p^2+q^6+5q^3-\frac{863}{84}\,.$$

$$\widehat{c}_0^{(3,1)}(p,q) = 0\,,$$
$$\widehat{c}_1^{(3,1)}(p,q) = q\,,$$
$$\widehat{c}_2^{(3,1)}(p,q) = -2qp + q\,,$$
$$\widehat{c}_3^{(3,1)}(p,q) = 3qp^2 - \frac{q}{2}\,,$$
$$\widehat{c}_4^{(3,1)}(p,q) = -4qp^3 + 6qp^2 + q^4 + q\,,$$
$$\widehat{c}_5^{(3,1)}(p,q) = 5qp^4 + 20qp^3 + 20qp^2 - 5q^4p + \frac{5q^4}{2} - \frac{19q}{6}\,,$$
$$\widehat{c}_6^{(3,1)}(p,q) = -6qp^5 - 45qp^4 - 110qp^3 + 15q(q^3-6)p^2 - \frac{5q^4}{2} + \frac{27q}{2}\,.$$

$$\widehat{c}_0^{(3,2)}(p,q) = 0\,,$$
$$\widehat{c}_1^{(3,2)}(p,q) = 0\,,$$
$$\widehat{c}_3^{(3,2)}(p,q) = -3q^2p + \frac{3q^2}{2}\,,$$
$$\widehat{c}_4^{(3,2)}(p,q) = 6q^2p^2 - q^2\,,$$
$$\widehat{c}_5^{(3,2)}(p,q) = -10q^2p^3 - 15q^2p^2 + q^5 + \frac{5q^2}{2}\,,$$
$$\widehat{c}_6^{(3,2)}(p,q) = 15q^2p^4 + 60q^2p^3 + 60q^2p^2 - 6q^5p + 3q^5 - \frac{19q^2}{2}\,.$$

References

[1] R. Barman and T. Komatsu, "Lehmer's generalized Euler numbers in hypergeometric functions," *J. Korean Math. Soc.*, vol. 56, no. 2, pp. 485–505, 2019, doi: 10.4134/JKMS.j180227.

[2] F. Brioschi, "Sulle funzioni Bernoulliane ed Euleriane," *Annali de Mat., i.* , vol. 36, no. 2, pp. 260–263, 1858.

[3] G.-S. Cheon, S.-G. Hwang, and S.-G. Lee, "Several polynomials associated with the harmonic numbers," *Discrete Appl. Math.*, vol. 155, no. 18, pp. 2573–2584, 2007, doi: 10.1016/j.dam.2007.07.008.

[4] J. W. L. Glaisher, "Expressions for Laplace's coefficients, Bernoullian and Eulerian numbers etc. as determinants," Messenger (2) VI. 49-63, 1875.

[5] N. Kilar and Y. Simsek, "Computational formulas and identities for new classes of Hermite-based Milne-Thomson type polynomials: Analysis of generating functions with Euler's formula," *Math. Methods Appl. Sci.*, vol. 44, no. 8, pp. 6731–6762, 2021, id/No 28, doi: 10.1002/mma.7220.

[6] N. Kilar and Y. Simsek, "Identities and relations for Hermite-based Milne-Thomson polynomials associated with Fibonacci and Chebyshev polynomials," *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, vol. 115, no. 1, p. 20 pages, 2021, id/No 28, doi: 10.1007/s13398-020-00968-3.

[7] T. Kim and C. S. Ryoo, “Some identities for Euler and Bernoulli polynomials and their zeros,” *Axioms*, vol. 7, no. 3, p. 19 pages, 2018, id/No 56, doi: 10.3390/axioms7030056.
[8] T. Komatsu, “A parametric type of Bernoulli polynomials with level 3,” *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, vol. 114, no. 3, p. 19 pages, 2020, id/No 151, doi: 10.1007/s13398-020-00886-4.
[9] T. Komatsu and Y. Ohno, “Congruence properties of Lehmer’s generalized Euler numbers,” *preprint*.
[10] T. Komatsu and J. L. Ramírez, “Some determinants involving incomplete Fubini numbers,” *An. Ştiinţ. Univ. “Ovidius” Constanţa, Ser. Mat.*, vol. 26, no. 3, pp. 143–170, 2018, doi: 10.2478/auom-2018-0038.
[11] T. Komatsu and Y. Simsek, “Third and higher order convolution identities for Cauchy numbers,” *Filomat*, vol. 30, no. 4, pp. 1053–1060, 2016, doi: 10.2298/FIL1604053K.
[12] D. H. Lehmer, “Lacunary recurrence formulas for the numbers of Bernoulli and Euler,” *Ann. Math. (2)*, vol. 36, pp. 637–649, 1935, doi: 10.2307/1968647.
[13] M. Masjed-Jamei, M. R. Beyki, and W. Koepf, “A new type of Euler polynomials and numbers,” *Mediterr. J. Math.*, vol. 15, no. 3, p. 17 pages, 2018, id/No 138, doi: 10.1007/s00009-018-1181-1.
[14] M. Masjed-Jamei, M. R. Beyki, and W. Koepf, “An extension of the Euler-MacLaurin quadrature formula using a parametric type of Bernoulli polynomials,” *Bull. Sci. Math.*, vol. 156, p. 26 pages, 2019, id/No 102798, doi: 10.1016/j.bulsci.2019.102798.
[15] M. Masjed-Jamei and W. Koepf, “Symbolic computation of some power-trigonometric series,” *J. Symb. Comput.*, vol. 80, pp. 273–284, 2017, doi: 10.1016/j.jsc.2016.03.004.
[16] T. Muir, *The theory of determinants in the historical order of development. Four volumes*, 4th ed. Dover Publications, New York, 1960.
[17] H. M. Srivastava and C. Kızılateş, “A parametric kind of the Fubini-type polynomials,” *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, vol. 113, no. 4, pp. 3253–3267, 2019, doi: 10.1007/s13398-019-00687-4.
[18] H. M. Srivastava, M. Masjed-Jamei, and M. R. Beyki, “Some new generalizations and applications of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials,” *Rocky Mt. J. Math.*, vol. 49, no. 2, pp. 681–697, 2019, doi: 10.1216/RMJ-2019-49-2-681.
[19] H. M. Srivastava, P. E. Ricci, and P. Natalini, “A family of complex Appell polynomial sets,” *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, vol. 113, no. 3, pp. 2359–2371, 2019, doi: 10.1007/s13398-018-00622-z.
[20] N. Trudi, “Intorno ad alcune formole di sviluppo,” *Rendic. dell’ Accad. Napoli*, vol. 36, no. 2, pp. 135–143, 1862.

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