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APPROXIMATION OF JENSEN TYPE RECIPROCAL MAPPINGS VIA FIXED POINT TECHNIQUE

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Abstract. In this paper, we solve a new Jensen type m-dimensional multiplicative inverse functional equation and then its various stability problems in the setting of non-negative real numbers and non-Archimedean spaces via fixed point method. The functional equation dealt in this study is linked with the famous relationship between arithmetic and harmonic mean of m values. The role of harmonic mean is very significant in many other fields such as traffic flow theory, industrial engineering, communication system, etc. By inversing the arithmetic mean of reciprocal values, we attain the harmonic mean. This property could be analyzed as an inverse problem via the functional equation dealt in this investigation.

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1. INTRODUCTION & PRELIMINARIES

The query of Ulam [25] regarding homomorphisms and subsequent brilliant reply provided by Hyers [12] are the foundations for the emerging research area of the stability of various forms of functional, difference, differential and integral equations. The result presented by Hyers is known as Ulam-Hyers Stability (or UHS) and the method of proof adopted by Hyers is called as 'Direct method'. Later, Hyers' result is generalized in different versions by many mathematicians, viz., Aoki [1], T. Rassias (Ulam-Hyers-T. Rassias Stability or UHTRS) [20] , J. Rassias (Ulam-Hyers-J. Rassias Stability or UHJRS) [19], Găvruţa (Ulam-Rassias-Găvruţa Stability or URGS) [11] and these results are the fundamental stabilities of a functional equation. These results inspired many mathematicians and there are a lot of novel, interesting and instigating results in the literature pertinent to different category of functional equations and their fundamental stabilities, refer [2, 15–18].

Fixed point method is another method of proving stability of functional equations. The stability problems of Jensen's and Cauchy functional equations were dealt in [9, 10] via fixed point technique.

For the first time, the fundamental stabilities of the ensuing multiplicative inverse functional equation

$$m_i(u+v) = \frac{m_i(u)m_i(v)}{m_i(u)+m_i(v)} \tag{1.1}$$

were dealt in [22], where $m_i : \mathbb{R}^\star \longrightarrow \mathbb{R}$ is a multiplicative inverse function with $\mathbb{R}^\star = \mathbb{R} - \{0\}$.

The stability problem of equation (1.1) triggered many authors to solve URGS of various types of multiplicative inverse functional equations such as generalized version in several variables, adjoint, difference, quadratic, cubic, quartic, quintic, sextic functional equations and their stabilities and non-stabilities (one can see [3–8, 13, 21]). Multiplicative inverse functional equations and their applications in real time are dealt in [23, 24].

In this paper, we deal with a new rational type functional equation of the form

$$r_j\left(\frac{1}{m}\sum_{k=1}^{m}\alpha_k\right) = \frac{m\prod_{k=1}^{m} r_j(\alpha_k)}{\sum_{k=1}^{m}\left[\prod_{i=1, i\neq k}^{m} r_j(\alpha_i)\right]} \tag{1.2}$$

where $m \geq 2$ is a positive integer and $r_j : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}$ is a function with the conditions $\sum_{k=2}^{m-1}\alpha_k \neq -\alpha_1$, $\sum_{k=1}^{m}\left[\prod_{i=1, i\neq k}^{m} r_j(\alpha_i)\right] \neq 0$ and $r_j(\alpha_k) \neq 0$ for $k = 1, \ldots, m$.

We solve equation (1.2) and obtain its various fundamental stabilities in the domain of non-negative real numbers and non-Archimedean fields through fixed point technique. We also demonstrate the relevance of equation (1.2) with arithmetic and harmonic means.

In the sequel, we evoke a theorem which serves as the backbone in obtaining our desired results.

Theorem 1. *Let (T, ρ) be a complete generalized metric space and let $\mu : T \longrightarrow T$ be a strictly contractive mapping (that is $\rho(\mu(\alpha), \mu(\beta)) \leq L\rho(\alpha, \beta)$, for all $\alpha, \beta \in T$ and a Lipschitz constant $L < 1$). Then for each given element $\alpha \in T$, either*

$$\rho\left(\mu^k\alpha, \mu^{k+1}\alpha\right) = \infty$$

for all integers $k \geq 0$ or there exists an integer $k_0 > 0$ such that

1. *$\rho\left(\mu^k\alpha, \mu^{k+1}\alpha\right) < \infty$ for all $k \geq k_0$;*
2. *the sequence $\{\mu^k\alpha\}$ converges to a fixed point $\beta^\star$ of μ;*
3. *$\beta^\star$ is the unique fixed point of μ in the set $\Gamma = \{\beta \in T : \rho(\mu^{k_0}\alpha, \beta) < \infty\}$;*
4. *$d(\beta, \beta^\star) \leq \frac{1}{1-L} d(\beta, \mu\beta)$ for all $\beta \in \Gamma$.*

Now, we recall a few fundamental notions of non-Archimedean field and fixed point alternative theorem in non-Archimedean settings. Throughout this paper, let $\mathbb{R}$ denote the set of real numbers.

Definition 1. Let $\mathbb{P}$ be a field with a mapping (valuation) $|\cdot|$ from $\mathbb{P}$ into $[0, \infty)$. Then $\mathbb{P}$ is said to be a non-Archimedean field if the ensuing conditions hold.

(i) $|\alpha| = 0$ if and only if $\alpha = 0$;
(ii) $|\alpha_1\alpha_2| = |\alpha_1||\alpha_2|$;
(iii) $|\alpha_1 + \alpha_2| \leq max\{|\alpha_1|, |\alpha_2|\}$

for all $\alpha_1, \alpha_2 \in \mathbb{P}$.

It is obvious that $|1| = |-1| = 1$ and $|\alpha| \leq 1$ for all $\alpha \in \mathbb{N}$. Furthermore, we assume that $|\cdot|$ is non-trivial, that is, there exists an $\ell_0 \in \mathbb{P}$ such that $|\ell_0| \neq 0, 1$.

Suppose W is a vector space over a scalar field $\mathbb{P}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : W \longrightarrow \mathbb{R}$ is a *non-Archimedean norm* (*valuation*) if it satisfies the following conditions:

(i) $\|p\| = 0$ if and only if $p = 0$;
(ii) $\|\lambda p\| = |\lambda| \, \|p\| \quad (\lambda \in \mathbb{P}, p \in W)$;
(iii) the strong triangle inequality (ultrametric); namely,

$$\|p+q\| \leq \max\{\|p\|, \|q\|\} \quad (p, q \in W).$$

Then $(W, \|\cdot\|)$ is known as a non-Archimedean space. In lieu of the following inequality

$$\|u_\ell - u_k\| \leq \max\left\{\left\|u_{j+1} - u_j\right\| : k \leq j \leq \ell - 1\right\} \quad (\ell > k),$$

a sequence $\{u_k\}$ is Cauchy if and only if $\{u_{k+1} - u_k\}$ converges to 0 in a non-Archimedean space. If every Cauchy sequence is convergent in the space, then it is called as complete non-Archimedean space.

Definition 2. Let K be a nonempty set. Suppose $\rho : K \times K \longrightarrow [0, \infty]$ satisfies the following axioms:

(i) $\rho(r, s) = 0$ if and only if $r = s$;
(ii) $\rho(r, s) = \rho(s, r)$ (symmetry);
(iii) $\rho(r, t) \leq \max\{\rho(r, s), \rho(s, t)\}$ (strong triangle inequality)

for all $r, s, t \in K$. Then (K, ρ) is called a generalized non-Archimedean metric space. Suppose every Cauchy sequence in K is convergent, then (K, ρ) is called complete.

Example 1. *Let $\mathbb{P}$ be a non-Archimedean field and U_1 and U_2 be two non-Archimedean spaces over $\mathbb{P}$. If U_2 is complete and $\chi : U_1 \longrightarrow [0, \infty)$, for every $r, s : U_1 \longrightarrow U_2$, define $\rho(r, s) = \inf\{\mu > 0 : |r(\alpha) - s(\alpha)| \leq \mu\chi(\alpha),\ \forall \alpha \in U_1\}$. Then (U_1, ρ) is a generalized non-Archimedean metric space.*

Using (Theorem 2.5, [10]), Mirmostafaee [14] introduced a different version of the alternative fixed point principle in the setting of non-Archimedean space which is presented in the following theorem.

Theorem 2. [14] (Alternative fixed point principle in non-Archimedean scheme) *Suppose (H, ρ) is a non-Archimedean generalized metric space. Let a mapping $\Lambda : H \longrightarrow H$ be a strictly contractive, (that is $\rho(\Lambda(u), \Lambda(v)) \leq L\rho(v, u)$, for all $u, v \in H$ and a Lipschitz constant $\rho < 1$), then either*

(i) $\rho\left(\Lambda^p(u), \Lambda^{p+1}u\right) = \infty$ *for all* $p \geq 0$, *or;*

(ii) *there exists some* $p_0 \geq 0$ *such that* $\rho\left(\Lambda^p(u), \Lambda^{p+1}(u)\right) < \infty$ *for all* $p \geq p_0$*;*

the sequence $\{\Lambda^p(u)\}$ *is convergent to a fixed point* $u^\star$ *of* Λ*;* $u^\star$ *is the distinct invariant point of* Λ *in the set* $Y = \{y \in X : \rho\left(\Lambda^{p_0}(u), v\right) < \infty\}$ *and* $\rho\left(v, u^\star\right) \leq \rho(v, \Lambda(v))$ *for all* v *in this set.*

2. Solution of equation (1.2)

In this section, we achieve the solution of equation (1.2). Then using the solution of equation (1.2), we define Jensen type m-dimensional multiplicative inverse mapping and functional equation.

Theorem 3. *Let a rational function* $r_j : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}$ *be defined by* $r_j(\alpha) = \frac{1}{c_1\alpha + c_2}$*, for* $\alpha \in \mathbb{R}^+ \cup \{0\}$ *and* c_1, c_2 *be any arbitrary constants with* $c_2 \neq 0$ *and* $c_1\alpha + c_2 \neq 0$*. Then* r_j *is a solution of equation* (1.2)*.*

Proof. In order to prove the result, first let us rewrite equation (1.2) as

$$\frac{1}{r_j\left(\frac{1}{m}\sum_{k=1}^m \alpha_m\right)} = \frac{1}{m}\left[\sum_{k=1}^m \frac{1}{r_j(\alpha_k)}\right]. \tag{2.1}$$

Assuming $r_j(0) = \frac{1}{c_2}$, letting $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$ and then plugging α_1 into α in (2.1), we obtain

$$\frac{1}{r_j\left(\frac{\alpha}{m}\right)} = \frac{1}{m}\left[\frac{1}{r_j(\alpha)} + (m-1)c_2\right]. \tag{2.2}$$

Again, equation (2.2) can be modified as

$$r_j\left(\frac{\alpha}{m}\right) = \frac{m}{\frac{1}{r_j(\alpha)} + (m-1)c_2}. \tag{2.3}$$

By virtue of equation (2.3), the equation (2.1) can be written as

$$\frac{\prod_{k=1}^m r_j(\alpha_k)}{\sum_{k=1}^m \prod_{i=1, i\neq k}^m r_j(\alpha_i)} = \frac{1}{\frac{1}{r_j(\sum_{k=1}^m \alpha_k)} + (m-1)c_2}. \tag{2.4}$$

Equation (2.4) can be transformed into

$$\frac{1}{r_j(\sum_{k=1}^m \alpha_m)} = \left(\sum_{k=1}^m \frac{1}{r_j(\alpha_k)}\right) - (m-1)c_2. \tag{2.5}$$

Let $\mathcal{A}(\alpha) = \frac{1}{r_j(\alpha)} - c_2$. Then, we have

$$\mathcal{A}\left(\sum_{k=1}^m \alpha_k\right) = \frac{1}{r_j\left(\sum_{k=1}^m \alpha_k\right)} - c_2$$

$$= \left[\left(\sum_{k=1}^{m} \frac{1}{r_j(\alpha_k)} \right) - (m-1)c_2 \right] - c$$
$$= \sum_{k=1}^{m} \left(\frac{1}{r_j(\alpha)} - c_2 \right)$$
$$= \sum_{k=1}^{m} \mathcal{A}(\alpha_k) \tag{2.6}$$

which indicates that $\mathcal{A}$ is a Cauchy functional equation in m variables and its solution is an additive function $\mathcal{A}(\alpha) = c_1\alpha$, where c_1 is a constant. Hence, it implies that the rational function $r_j(\alpha) = \frac{1}{c_1\alpha + c_2}$ is a solution of equation (1.2). □

The upcoming definition follows from the above theorem.

Definition 3. A function r_j is called as a Jensen type m-dimensional multiplicative inverse mapping if it satisfies equation (1.2). Hence equation (1.2) is said to be Jensen type m-dimensional multiplicative inverse functional equation.

3. VARIOUS CLASSICAL STABILITIES OF EQUATION (1.2)

In order to prove our main results, let us assume that $\sum_{k=2}^{m-1} \alpha_k \neq -\alpha_1$ for all $\alpha_k \in \mathbb{R}^+ \cup \{0\}$, $k = 1, \ldots, m$. We solve URGS problem of equation (1.2) in the domain of non-zero real numbers. Then, we apply URGS result of equation (1.2), to investigate the stability problems involving a positive fixed constant and sum of powers of norms.

Theorem 4. *Suppose that a function $r_j : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}$ satisfies the condition that $r_j(\alpha)$ approaches $\frac{1}{c_2}$ as $\alpha \to 0$, where c_2 is a constant and an inequality*

$$\left| \frac{1}{m} r_j \left(\frac{1}{m} \sum_{k=1}^{m} \alpha_k \right) - \frac{1}{\sum_{k=1}^{m} \frac{1}{r_j(\alpha_k)}} \right| \leq \upsilon(\alpha_1, \alpha_2, \ldots, \alpha_m) \tag{3.1}$$

for all $\alpha_k \in \mathbb{R}^+ \cup \{0\}$, where $\upsilon : \underbrace{\mathbb{R}^+ \cup \{0\} \times \cdots \times \mathbb{R}^+ \cup \{0\}}_{m\text{-times}} \longrightarrow [0, \infty)$ is a given function. Suppose that there exists $L < 1$ such that the mapping $\alpha \mapsto \Upsilon(\alpha) = \upsilon(\alpha, \underbrace{0, \ldots, 0}_{(m-1)\text{-times}})$ has the property $\Upsilon\left(\frac{\alpha}{m}\right) \leq mL\Upsilon(\alpha)$ for all $\alpha \in \mathbb{R}^+ \cup \{0\}$. If the function υ has the property

$$\lim_{n\to\infty} m^{-n} \upsilon\left(m^{-n}\alpha_1, \ldots, m^{-n}\alpha_m\right) = 0, \tag{3.2}$$

for all $\alpha_k \in \mathbb{R}^+ \cup \{0\}$, then there exists a unique Jensen type m-dimensional multiplicative inverse mapping $R_j : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}$ such that

$$\left| r_j(\alpha) - R_j(\alpha) \right| \leq \frac{1}{1-L} \Upsilon(\alpha) \tag{3.3}$$

for all $\alpha \in \mathbb{R}^+ \cup \{0\}$.

Proof. Let T be a set defined as $T = \{\phi : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}\}$. Assume ρ is a generalized metric on T which is described as:

$$\rho(\psi,\phi) = \rho_\Upsilon(\psi,\phi) = \inf\{\lambda > 0 : |\psi(\alpha) - \phi(\alpha)| \leq \lambda\Upsilon(\alpha),\ \text{for all } \alpha \in \mathbb{R}^+ \cup \{0\}\}. \tag{3.4}$$

The definition of ρ implies that the set T is complete. Now, let us define a mapping $\mu : T \longrightarrow T$ by

$$\mu\phi(\alpha) = \frac{1}{m}\phi\left(\frac{\alpha}{m}\right) \qquad (\alpha \in \mathbb{R}^+ \cup \{0\}) \tag{3.5}$$

for all $\phi \in T$. Next, let us show that μ is a strictly contractive mapping on the set T. For given $\psi,\phi \in T$, suppose $0 \leq \lambda_{\psi\phi} \leq \infty$ is an arbitrary constant with $\rho(\psi,\phi) \leq \lambda_{\psi\phi}$. Therefore, we have

$$\begin{aligned}
\rho(\psi,\phi) < \lambda_{\psi\phi} &\Longrightarrow |\psi(\alpha) - \phi(\alpha)| \leq \lambda_{\psi\phi}\Upsilon(\alpha), \quad (\forall \alpha \in \mathbb{R}^+ \cup \{0\}) \\
&\Longrightarrow \left|\frac{1}{m}\psi\left(\frac{\alpha}{m}\right) - \frac{1}{m}\phi\left(\frac{\alpha}{m}\right)\right| \leq \frac{1}{m}\lambda_{\psi\phi}\Upsilon\left(\frac{\alpha}{m}\right), \quad (\forall \alpha \in \mathbb{R}^+ \cup \{0\}) \\
&\Longrightarrow \left|\frac{1}{m}\psi\left(\frac{\alpha}{m}\right) - \frac{1}{m}\phi\left(\frac{\alpha}{m}\right)\right| \leq L\lambda_{\psi\phi}\Upsilon(\alpha), \quad (\forall \alpha \in \mathbb{R}^+ \cup \{0\}) \\
&\Longrightarrow \rho(\mu\psi,\mu\phi) \leq L\lambda_{\psi\phi}.
\end{aligned}$$

The above inequality implies that $\rho(\mu\psi,\mu\phi) \leq L\rho(\psi,\phi)$ for all $\psi,\phi \in T$, which inturn indicates that μ is a strictly contractive mapping of T, with the Lipschitz constant L. Now, plugging $(\alpha_1,\ldots,\alpha_m)$ by $(\alpha, \underbrace{0,\ldots,0}_{(m-1)\text{-times}})$ in (3.1), we get

$$\left|\frac{1}{m}r_j\left(\frac{\alpha}{m}\right) - \frac{1}{\frac{1}{r_j(\alpha)} + (m-1)c_2}\right| \leq \upsilon(\alpha, \underbrace{0,\ldots,0}_{(m-1)\text{-times}}) = \Upsilon(\alpha)$$

which produces

$$\left|\frac{1}{2}r_j\left(\frac{\alpha}{2}\right) - r_j(\alpha)\right| \leq \left|\frac{1}{m}r_j\left(\frac{\alpha}{m}\right) - \frac{1}{\frac{1}{r_j(\alpha)} + (m-1)c_2}\right| \leq \Upsilon(\alpha)$$

for all $\alpha \in \mathbb{R}^+ \cup \{0\}$. Hence (3.4) produces that $\rho(\mu r_j, r_j) \leq 1$. So, by employing the fixed point alternative Theorem 1, there exists a function $R_j : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}$ satisfying the following:

(1) R_j is a fixed point of ρ, that is,

$$R_j\left(\frac{\alpha}{m}\right) = mR_j(\alpha) \tag{3.6}$$

for all $\alpha \in \mathbb{R}^+ \cup \{0\}$. The mapping R_j is the unique fixed point of μ in the set

$$\Gamma = \{\phi \in T : \rho(R_j, r_j) < \infty\}.$$

This implies that R_j is the unique mapping satisfying (3.6) such that there exists $0<\lambda<\infty$ satisfying

$$|R_j(\alpha)-r_j(\alpha)|\leq \lambda\Upsilon(\alpha),\ \forall\alpha\in\mathbb{R}^+\cup\{0\}.$$

(2) $\rho\left(\mu^n r_j, R_j\right)\to 0$ as $n\to\infty$. Thus, we have

$$\lim_{n\to\infty} m^{-n} r_j\left(m^{-n}\alpha\right)=R_j(\alpha) \tag{3.7}$$

for all $\alpha\in\mathbb{R}^+\cup\{0\}$.

(3) $\rho(R_j,r_j)\leq \frac{1}{1-L}\rho(R_j,\mu r_j)$, which implies $\rho(R_j,r_j)\leq\frac{1}{1-L}$.

So, the inequality (3.3) holds. On the other hand, from (3.1), (3.2) and (3.7), we have

$$\begin{aligned}
&\left|\frac{1}{m}R_j\left(\frac{1}{m}\sum_{k=1}^{m}\alpha_k\right)-\frac{1}{\sum_{k=1}^{m}\frac{1}{R_j(\alpha_k)}}\right| \\
&\qquad=\lim_{n\to\infty} m^{-n}\left|\frac{1}{m}r_j\left(\frac{1}{m}\sum_{k=1}^{m}m^{-n}\alpha_k\right)-\frac{1}{\sum_{k=1}^{m}\frac{1}{r_j(m^{-n}\alpha_k)}}\right| \\
&\qquad\leq \lim_{n\to\infty} m^{-n}\upsilon\left(m^{-n}\alpha_1,\ldots,m^{-n}\alpha_m\right)=0
\end{aligned}$$

for all $\alpha_k\in\mathbb{R}^+\cup\{0\}$, which shows that R_j is a solution of the equation (1.2) and hence $R_j:\mathbb{R}^+\cup\{0\}\longrightarrow\mathbb{R}$ is a Jensen type m-dimensional multiplicative inverse function. □

The upcoming theorem is the dual of Theorem 4. The proof is obtained by similar arguments as in Theorem 4 and hence for completeness, we present only the statement.

Theorem 5. *Suppose that the mapping $r_j:\mathbb{R}^+\cup\{0\}\longrightarrow\mathbb{R}$ satisfies the condition $r_j(0)=\frac{1}{c_2}$ and the inequality (3.1), with a given function $\upsilon:\underbrace{\mathbb{R}^+\cup\{0\}\times\cdots\times\mathbb{R}^+\cup\{0\}}_{m\text{-times}}\longrightarrow[0,\infty)$. If there exists $L<1$ such that the mapping $\alpha\mapsto\Upsilon(\alpha)=m\upsilon(m\alpha,0)$ has the property $\Upsilon(m\alpha)\leq\frac{1}{m}L\Upsilon(\alpha)$, for all $\alpha\in\mathbb{R}^+\cup\{0\}$. If the function υ has the property*

$$\lim_{n\to\infty} m^n\upsilon\left(m^n\alpha_1,\ldots,m^n\alpha_m\right)=0, \tag{3.8}$$

for all $\alpha_k\in\mathbb{R}^+\cup\{0\}$, then there exists a unique Jensen type m-dimensional multiplicative inverse function mapping $R_j:\mathbb{R}^+\cup\{0\}\longrightarrow\mathbb{R}$ such that

$$\left|r_j(\alpha)-R_j(\alpha)\right|\leq\frac{1}{1-L}\Upsilon(\alpha) \tag{3.9}$$

for all $\alpha\in\mathbb{R}^+\cup\{0\}$.

The following corollaries are the investigation of various stabilities of equation (1.2) pertinent to UHS and UHR stability. The proofs of corollaries directly follow from the above theorems.

Corollary 1. *Let $r_j : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}$ be a mapping for which there exists a constant η (independent of $\alpha_1, \ldots, \alpha_m$)$\geq 0$ such that the functional inequality*

$$\left| \frac{1}{m} r_j \left(\frac{1}{m} \sum_{k=1}^{m} \alpha_k \right) - \frac{1}{\sum_{k=1}^{m} \frac{1}{r_j(\alpha_k)}} \right| \leq \frac{\eta}{2}$$

holds for all $\alpha_k \in \mathbb{R}^+ \cup \{0\}$. Then there exists a unique Jensen type m-dimensional multiplicative inverse function $R_j : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}$ satisfying (1.2) and

$$|r_j(\alpha) - R_j(\alpha)| \leq \eta$$

for all $\alpha \in \mathbb{R}^+ \cup \{0\}$.

Proof. The proof is obtained by considering $\upsilon(\alpha_1, \ldots, \alpha_m) = \frac{\eta}{2}$ and $L = \frac{1}{m}$ in Theorem 4, we attain the desired result. □

Corollary 2. *Let $r_j : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}$ be a mapping and let there exist real numbers $\ell \neq -1$ and $\delta \geq 0$ such that*

$$\left| \frac{1}{m} r_j \left(\frac{1}{m} \sum_{k=1}^{m} \alpha_k \right) - \frac{1}{\sum_{k=1}^{m} \frac{1}{r_j(\alpha_k)}} \right| \leq \delta \sum_{k=1}^{m} \alpha_k^{\ell}$$

for all $\alpha_k \in \mathbb{R}^+ \cup \{0\}$. Then, there exists a unique Jensen type m-dimensional multiplicative inverse function $R_j : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}$ which satisfies (1.2) and

$$|r_j(\alpha) - R_j(\alpha)| \leq \begin{cases} \frac{m^{\ell+1}\delta}{m^{\ell+1}-1} \alpha^{\ell}, & \ell > -1 \\ \frac{m^{\ell}\delta}{1-m^{\ell+1}} \alpha^{\ell}, & \ell < -1 \end{cases}$$

for all $\alpha \in \mathbb{R}^+ \cup \{0\}$.

Proof. The required result is obtained by the application of Theorems 4 and 5 with $\upsilon(\alpha_1, \ldots, \alpha_m) = \delta \sum_{k=1}^{m} \alpha_k^{\ell}$, for all $\alpha_k \in \mathbb{R}^+ \cup \{0\}$ and then $L = m^{-\ell-1}$ and $L = m^{\ell+1}$, respectively. □

Remark 1. If upper bound is taken as $\upsilon(\alpha_1, \ldots, \alpha_m) = \delta_2 \prod_{k=1}^{m} \alpha_k^{\ell}$ and then letting $\alpha_k = 0$, $k = 2, 3, \ldots, m$ in Theorems 4 and 5, we observe that the upper bound becomes 0 and it is obvious that the function r_j satisfies the Jensen type m-dimensional multiplicative inverse functional equation (1.2). Similarly, if upper bound is taken as $\upsilon(\alpha_1, \ldots, \alpha_m) = \delta_3 \left(\sum_{k=1}^{m} \alpha_k^{\ell} + \left(\prod_{k=1}^{m} \alpha_k^{\frac{\ell}{m}} \right) \right)$ in Theorems 4 and 5, then the result is the same as obtained Corollary 2. Hence in this study, we exclude the investigation of stabilities involving product of powers of norms and mixed product-sum of powers of norms.

4. Non-Archimedean stabilities of equation (1.2)

In this section, we solve various stability problems of equation (1.2) via fixed point scheme in non-Archimedean fields. In the upcoming results, let us presume that $\mathbb{K}$ and $\mathbb{L}$ are a non-Archimedean field and a complete non-Archimedean field, respectively. We denote $\mathbb{K}^\star = \mathbb{K} - \{0\}$. For the sake of simplicity, we describe the difference operator $\Delta_{r_j} : \underbrace{\mathbb{K}^\star \times \cdots \times \mathbb{K}^\star}_{m\text{-times}} \longrightarrow \mathbb{L}$ by

$$\Delta_{r_j}(\alpha_1, \ldots, \alpha_m) = \frac{1}{m} r_j \left(\frac{1}{m} \sum_{k=1}^{m} \alpha_k \right) - \frac{1}{\sum_{k=1}^{m} \frac{1}{r_j(\alpha_k)}}$$

for all $\alpha_k \in \mathbb{K}^\star$.

Theorem 6. *Suppose a function $r_j : \mathbb{K}^\star \longrightarrow \mathbb{L}$ satisfies the following inequality*

$$\left|\Delta_{r_j}(\alpha_1, \ldots, \alpha_m)\right| \leq \varphi(\alpha_1, \ldots, \alpha_m) \tag{4.1}$$

for all $\alpha_k \in \mathbb{K}^\star$ with the condition that $r_j(0) = \frac{1}{c_2}$, where $\varphi : \underbrace{\mathbb{K}^\star \times \cdots \times \mathbb{K}^\star}_{m\text{-times}} \longrightarrow \mathbb{L}$ is a given function. If $L \in (0,1)$,

$$|m|^{-1} \varphi\left(m^{-1}\alpha_1, \ldots, m^{-1}\alpha_m\right) \leq L\varphi(\alpha_1, \ldots, \alpha_m) \tag{4.2}$$

for all $\alpha_k \in \mathbb{K}^\star$, then there exists a unique Jensen type m-dimensional multiplicative inverse function $R_j : \mathbb{K}^\star \longrightarrow \mathbb{L}$ satisfying (1.2) and

$$|r_j(\alpha) - R_j(\alpha)| \leq L\varphi(\alpha_1, \underbrace{0, \ldots, 0}_{(m-1)\text{-times}}) \tag{4.3}$$

for all $\alpha \in \mathbb{K}^\star$.

Proof. Let the set $\mathcal{A}$ be defined as: $\mathcal{A} = \{p | p : \mathbb{K}^\star \longrightarrow \mathbb{L}\}$, and define

$$\rho(r_j, s_j) = \inf\{\mu > 0 : |r_j(\alpha) - s_j(\alpha)| \leq \mu\varphi(\alpha, \underbrace{0, \ldots, 0}_{(m-1)\text{-times}}), \text{ for all } \alpha \in \mathbb{K}^\star\}.$$

Example 1 produces ρ to be a complete generalized non-Archimedean complete metric on $\mathcal{A}$. Let $\Gamma : \mathcal{A} \longrightarrow \mathcal{A}$ be a mapping defined by $\Gamma(r_j)(\alpha) = m^{-1} r_j\left(m^{-1}\alpha\right)$ for all $\alpha \in \mathbb{K}^\star$ and $r_j \in \mathcal{A}$. We claim that Γ is strictly contractive on $\mathcal{A}$. For given $r_j, s_j \in \mathcal{A}$, let $\mu > 0$ be a constant with $\left|r_j(\alpha) - s_j(\alpha)\right| \leq \mu\varphi(\alpha, \underbrace{0, \ldots, 0}_{(m-1)\text{-times}})$, for all $\alpha \in \mathbb{K}^\star$. Then using (4.2), we obtain

$$\begin{aligned}
|\Gamma(r_j)(\alpha) - \Gamma(s_j)(\alpha)| &= |m|^{-1} \left| r_j\left(m^{-1}\alpha\right) - s_j\left(m^{-1}\alpha\right)\right| \\
&\leq \mu|m|^{-1}\varphi\left(m^{-1}\alpha, \underbrace{0, \ldots, 0}_{(m-1)\text{-times}}\right)
\end{aligned}$$

$$\leq \mu L\varphi(\alpha, \underbrace{0,\ldots,0}_{(m-1)\text{-times}}) \qquad (\alpha \in \mathbb{K}^\star).$$

The above inequality implies that $\rho(\Gamma(r_j),\Gamma(s_j)) \leq L\rho(r_j,s_j)$, for $r_j,s_j \in \mathcal{A}$. Consequently, the mapping Γ is strictly contractive with Lipschitz constant L. Now, plugging $(\alpha_1,\ldots,\alpha_m)$ into $(\alpha, \underbrace{0,\ldots,0}_{(m-1)\text{-times}})$ in (4.1), we have

$$\left|\frac{1}{m}r_j\left(\frac{\alpha}{m}\right) - \frac{1}{\frac{1}{r_j(\alpha)}+(m-1)c_2}\right| \leq \varphi(\alpha, \underbrace{0,\ldots,0}_{(m-1)\text{-times}}) \leq L\varphi(\alpha, \underbrace{0,\ldots,0}_{(m-1)\text{-times}}) \quad \alpha \in \mathbb{K}^\star)$$

which inturn implies

$$\begin{aligned}|\Gamma(r_j)(\alpha) - r_j(\alpha)| &\leq \left|\frac{1}{m}r_j\left(\frac{\alpha}{m}\right) - \frac{1}{\frac{1}{r_j(\alpha)}+(m-1)c_2}\right| \\ &\leq L\varphi(\alpha, \underbrace{0,\ldots,0}_{(m-1)\text{-times}}) \quad (\alpha \in \mathbb{K}^\star).\end{aligned}$$

This indicates that $d(\Gamma(r_j),r_j) \leq L$. Due to Theorem 2 (ii), Γ has a unique fixed point $R_j : \mathbb{K}^\star \longrightarrow \mathbb{L}$ in the set $G = \{g \in \mathcal{A} : \rho(r_j,g) < \infty\}$ and for each $\alpha \in \mathbb{K}^\star$, $R_j(\alpha) = \lim\limits_{s\to\infty} \Gamma^s r_j(\alpha) = \lim\limits_{s\to\infty} m^{-s} r_j(m^{-s}\alpha)$ $(\alpha \in \mathbb{K}^\star)$. Therefore, for all $\alpha_1,\alpha_2,\ldots,\alpha_m \in \mathbb{K}^\star$,

$$\begin{aligned}\left|\Delta_{R_j}(\alpha_1,\ldots,\alpha_m)\right| &= \lim_{s\to\infty} |m|^{-s}\left|\Delta_{r_j}\left(m^{-s}\alpha_1,\ldots,m^{-s}\alpha_m\right)\right| \\ &\leq \lim_{s\to\infty} |m|^{-s}\varphi\left(m^{-s}\alpha_1,\ldots,m^{-s}\alpha_m\right) \\ &\leq \lim_{s\to\infty} L^s\varphi(\alpha_1,\ldots,\alpha_m) = 0\end{aligned}$$

which implies that the mapping R_j is a Jensen type m-dimensional multiplicative inverse function. Theorem 2 (ii) implies $\rho(r_j,R_j(\alpha)) \leq \rho(\Gamma(r_j),r_j)$, that is,

$$|r_j(\alpha) - R_j(\alpha)| \leq L\varphi(\alpha, \underbrace{0,\ldots,0}_{(m-1)\text{-times}}) \quad (\alpha \in \mathbb{K}^\star).$$

Let $R'_j : \mathbb{K}^\star \longrightarrow \mathbb{L}$ be another Jensen type m-dimensional multiplicative inverse function which satisfies (4.3), then R'_j is a fixed point of Γ in $\mathcal{A}$. However, by Theorem 2, Γ has only one fixed point in G. This completes the uniqueness assertion of the theorem. □

The succeeding theorem is dual of Theorem 6. We omit the proof as it is similar to Theorem 3.

Theorem 7. *Suppose the mapping $r_j : \mathbb{K}^\star \longrightarrow \mathbb{L}$ satisfies the inequality (4.2) with the condition $r_j(0) = \frac{1}{c_2}$. If $L \in (0,1)$,*

$$|m|\varphi(m\alpha_1,\ldots,m\alpha_m) \leq L\varphi(\alpha_1,\ldots,\alpha_m)$$

for all $\alpha_k \in \mathbb{K}^\star$, *then there exists a unique Jensen type m-dimensional multiplicative inverse function* $R_j : \mathbb{K}^\star \longrightarrow \mathbb{L}$ *satisfying* (1.2) *and*

$$|r_j(\alpha) - R_j(\alpha)| \leq L\varphi\left(\frac{\alpha}{m}, \underbrace{0,\ldots,0}_{(m-1)\text{-times}}\right),$$

for all $\alpha \in \mathbb{K}^\star$.

The ensuing corollaries are immediate consequences of Theorems 6 and 7. In the following corollaries, we presume that $|m| < 1$ for a non-Archimedean field $\mathbb{K}^\star$.

Corollary 3. *Let* ε *(independent of* $\alpha_1,\ldots,\alpha_m$*) be a non-negative constant. Suppose that there exists a mapping* $r_j : \mathbb{K}^\star \longrightarrow \mathbb{L}$ *such that the inequality*

$$\left|\Delta_{r_j}(\alpha_1,\ldots,\alpha_m)\right| \leq \frac{\varepsilon}{2},$$

for all $\alpha_k \in \mathbb{K}^\star$. *Then there exists a unique Jensen type m-dimensional multiplicative inverse function* $R_j : \mathbb{K}^\star \longrightarrow \mathbb{L}$ *satisfying* (1.2) *and* $|r_j(\alpha) - R_j(\alpha)| \leq \varepsilon$, *for all* $\alpha \in \mathbb{K}^\star$.

Proof. Considering $\varphi(\alpha_1,\ldots,\alpha_m) = \varepsilon$ and then choosing $L = |m|^{-1}$ in Theorem 6, we get the desired result. □

Corollary 4. *Let* $\lambda \neq -1$ *and* $\theta_1 \geq 0$ *be real numbers. Suppose that there exists a mapping* $r_j : \mathbb{K}^\star \longrightarrow \mathbb{L}$ *such that* $\left|\Delta_{r_j}(\alpha_1,\ldots,\alpha_m)\right| \leq \theta_1\left(\sum_{k=1}^m |\alpha_k|^\lambda\right)$, *for all* $\alpha_k \in \mathbb{K}^\star$. *Then there exists a unique Jensen type m-dimensional multiplicative inverse function* $R_j : \mathbb{K}^\star \longrightarrow \mathbb{L}$ *satisfying* (1.2) *and*

$$|r_j(\alpha) - R_j(\alpha)| \leq \begin{cases} \frac{\theta_1}{|m|^{\lambda+1}}|\alpha|^\lambda, & \lambda > -1 \\ |m|\theta_1|\alpha|^\lambda, & \lambda < -1 \end{cases}$$

for all $\alpha \in \mathbb{K}^\star$.

Proof. The proof follows by assigning $\varphi(\alpha_1,\ldots,\alpha_m) = \theta_1\left(\sum_{k=1}^m |\alpha_k|^\lambda\right)$ in Theorems 6 and 7 and then assuming $L = |m|^{-\lambda-1}$, $\lambda > -1$ and $L = |m|^{\lambda+1}$, $\lambda < -1$, respectively. □

5. Pertinence of equation (1.2)

We close this article, with an interpretation of equation (1.2) with arithmetic and harmonic mean of m values. Equation (1.2) can be considered as

$$r_j\left(\frac{1}{m}\sum_{k=1}^m \alpha_k\right) = \frac{m}{\sum_{k=1}^m \frac{1}{r_j(\alpha_k)}}.$$

We construe that the above equation maps the arithmetic mean of m values to harmonic mean of the images of m values. The vital applications of harmonic mean like calculating average speed in traffic flow theory, manipulating the performance measures of objective factors in industrial engineering, analyzing the performance of transmission communication system with relays in electrical engineering render that the equation (1.2) dealt in our study is constructive in these fields.

6. Conclusion

In this study, we have obtained the solution of equation (1.2) as a rational function. Also, from the results obtained in this investigation, we infer that the URG stability problem and other associated stability problems of equation (1.2) have solution in the setting of non-zero real numbers and non-Archimedean fields. The equation (1.2) is interpreted with the relationship between arithmetic and harmonic mean of m values.

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