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DEFERRED CESÀRO STATISTICAL CONVERGENCE OF MARTINGALE SEQUENCE AND KOROVKIN-TYPE APPROXIMATION THEOREMS

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Abstract. In the present paper, we introduce and study the concepts of statistical convergence and statistical summability for martingale sequences of random variables via deferred Cesàro mean. We then establish an inclusion theorem concerning the relation between these two beautiful and potentially useful concepts. Also, based upon our proposed notions, we state and prove new Korovkin-type approximation theorems with algebraic test functions for a martingale sequence over a Banach space. Moreover, we demonstrate that our theorems effectively extend and improve most (if not all) of the previously existing results (in statistical and classical versions). Finally, by using the generalized Bernstein polynomials, we present an illustrative example of a martingale sequence in order to demonstrate that our established theorems are stronger than their traditional and statistical versions.

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1. Introduction and motivation

Let $(\Omega, \mathcal{F}, P)$ be a probability measurable space and suppose that (X_n) is a random variable defined over this space. Also let $\mathcal{F}_n \subseteq \mathcal{F}$ $(n \in \mathbb{N})$ be a monotonically increasing sequence of σ-fields of measurable sets. Now, considering the random variable (X_n) and the measurable functions $(\mathcal{F}_n)$, we adopt a stochastic sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$.

A given stochastic sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is said to be a martingale sequence if

(i) $\mathbb{E}|X_n| < \infty$,

(ii) $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ almost surely (a.s.) and

(iii) $(\mathcal{F}_n)$ is a measurable sequence of functions,

where $\mathbb{E}$ is the mathematical expectation.

We now recall the definition of convergence of martingale sequences of random variables.

Definition 1. A martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ with $\mathbb{E}|X_n|$ is bounded and $\text{Prob}(X_n) = 1$ (that is, with probability 1) is said to be convergent to a martingale $(X_0, \mathcal{F}_0)$, if

$$\lim_{n\to\infty}(X_n, \mathcal{F}_n) \longrightarrow (X_0, \mathcal{F}_0) \qquad (\mathbb{E}|X_0| < \infty).$$

Recently, the study of statistical convergence has been one of the beautiful aspects of the theory of sequence spaces. The investigation and study of statistical convergence are potentially useful in sequence space because it is more general than the usual convergence. Such a concept was first introduced independently by two eminent mathematicians, Fast [6] and Steinhaus [24]. Subsequently, by using this nice concept with different settings, various researchers developed many interesting and useful results in several fields of mathematics such as summability theory, Fourier series, approximation theory, probability theory, measure theory, and so on. Moreover, the introduction of statistical probability convergence has enhanced the glory of this develoment. For some recent research works in this direction, see [2–4, 7–9, 11, 17, 21] and [22].

Let $\mathfrak{U} \subseteq \mathbb{N}$. Also let

$$\mathfrak{U}_n = \{i : i \leqq n \quad \text{and} \quad i \in \mathfrak{U}\}.$$

Then the natural density $d(\mathfrak{U})$ of $\mathfrak{U}$ is defined by

$$d(\mathfrak{U}) = \lim_{n\to\infty} \frac{|\mathfrak{U}_n|}{n} = b,$$

where b is a real and finite number and $|\mathfrak{U}_n|$ is the cardinality of $\mathfrak{U}_n$.
A given sequence (a_n) is statistically convergent to λ if, for each $\varepsilon > 0$,

$$\mathfrak{U}_\varepsilon = \{i : i \in \mathbb{N} \quad \text{and} \quad |a_i - \lambda| \geqq \varepsilon\}$$

has zero natural density (see [6] and [24]). Thus, for each $\varepsilon > 0$, we have

$$d(\mathfrak{U}_\varepsilon) = \lim_{n\to\infty} \frac{|\mathfrak{U}_\varepsilon|}{n} = 0.$$

We write

$$\text{stat} \lim_{n\to\infty} a_n = \lambda.$$

We now introduce the definition of statistical convergence of martingale sequence.

Definition 2. A bounded martingale sequence $(X_n, \mathcal{F}_n, n \in \mathbb{N})$ having its probability 1 is said to be statistically convergent to a martingale $(X_0, \mathcal{F}_0)$ with $\mathbb{E}|X_0| < \infty$ if, for all $\varepsilon > 0$,

$$\mathfrak{U}_\varepsilon = \{i : i \leqq n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| \geqq \varepsilon\}$$

has zero natural density. This means that, for every $\varepsilon > 0$, we have

$$d(\mathfrak{U}_\varepsilon) = \lim_{n\to\infty} \frac{|\mathfrak{U}_\varepsilon|}{n} = 0.$$

We write

$$\text{stat}_{\text{mart}} \lim_{n\to\infty} (X_n, \mathcal{F}_n) = (X_0, \mathcal{F}_0).$$

Example 1. Let $(\mathcal{F}_n, n \in \mathbb{N})$ be a monotonically increasing sequence of 0-mean independent random variables over σ-fields. Also let $(X_n) \in \mathcal{F}_n$ be such that

$$X_n = \begin{cases} 1 & (n = m^2;\ m \in \mathbb{N}) \\ 0 & (\text{otherwise}). \end{cases}$$

It is easy to see that the martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is statistically convergent to zero, but not simply martingale convergent.

Based on our proposed definition, we now establish a theorem concerning a relation between the ordinary and statistical versions of convergence of martingale sequences.

Theorem 1. *If a martingale sequence $(X_n, \mathcal{F}_n, n \in \mathbb{N})$ is convergent to a martingale $(X_0, \mathcal{F}_0)$ with $\mathbb{E}|X_0| < \infty$, then it is statistically convergent to the same martingale.*

Proof. Let the martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ be bounded and convergent with probability 1, then there exists a martingale $(X_0, \mathcal{F}_0)$ with $\mathbb{E}|X_0| < \infty$, that is,

$$\lim_{n\to\infty} (X_n, \mathcal{F}_n) \longrightarrow (X_0, \mathcal{F}_0).$$

As the given martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is bounded with probability 1, then, for every $\varepsilon > 0$, we have

$$\frac{1}{n}\{i : i \leqq n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| \geqq \varepsilon\} \subseteq \lim_{n\to\infty} |(X_n, \mathcal{F}_n) - (X_0, \mathcal{F}_0)| < \varepsilon.$$

Consequently, by Definition 2, we obtain

$$d(\mathfrak{U}_\varepsilon) = \lim_{n\to\infty} \frac{|\mathfrak{U}_\varepsilon|}{n} = 0,$$

where, just as in Definition 2, we have

$$\mathfrak{U}_\varepsilon := \{i : i \leqq n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| \geqq \varepsilon\}.$$

This evidently completes the demonstration of Theorem 1. □

Motivated essentially by the above-mentioned investigations, here we introduce and study the concepts of statistical convergence and statistical summability for martingale sequences of random variables via deferred Cesàro mean. We then establish an inclusion theorem with associated example concerning a relation between these new concepts. Also, based upon our proposed methods, we state and prove the new Korovkin-type approximation theorems, with algebraic test functions, involving a martingale sequence over a Banach space. Moreover, we demonstrate that our theorems effectively extend and improve most (if not all) of the previously existing results (in both the statistical and classical versions). Finally, by considering the generalized Bernstein polynomials, we present an illustrative example of a martingale sequence in order to demonstrate that our established theorems are stronger than their traditional and statistical versions.

2. DEFERRED CESÀRO MARTINGALE SEQUENCE

Let (u_n) and (v_n) be sequences of non-negative integers such that $u_n < v_n$ and

$$\lim_{n\to\infty} v_n = +\infty.$$

Then the deferred Cesàro mean for the martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is defined by

$$\begin{aligned}\mathcal{D}(X_n, \mathcal{F}_n) &= \frac{(X_{u_n+1}, \mathcal{F}_{u_n+1}) + (X_{u_n+2}, \mathcal{F}_{u_n+2}) + \cdots + (X_{v_n}, \mathcal{F}_{v_n})}{v_n - u_n} \\ &= \frac{1}{v_n - u_n} \sum_{k=u_n+1}^{v_n} (X_k, \mathcal{F}_k).\end{aligned}$$

We now present the definitions of deferred Cesàro statistical convergence and statistically deferred Cesàro summability of martingale sequences.

Definition 3. Let (u_n) and (v_n) be sequences of non-negative integers. A bounded martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ having probability 1 is deferred Cesàro statistically convergent to a martingale $(X_0, \mathcal{F}_0)$ with $\mathbb{E}|X_0| < \infty$ if, for all $\varepsilon > 0$,

$$\mathfrak{U}_\varepsilon = \{i : u_n < i \leqq v_n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| \geqq \varepsilon\}$$

has zero natural density. This means that, for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{|\{i : u_n < i \leqq v_n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| \geqq \varepsilon\}|}{u_n - v_n} = 0.$$

We write

$$\mathrm{DM}_{\text{stat}} \lim_{n\to\infty} (X_n, \mathcal{F}_n) = (X_0, \mathcal{F}_0).$$

Definition 4. Let (u_n) and (v_n) be sequences of non-negative integers. A bounded martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ having probability 1 is statistically deferred Cesàro summable to a martingale $(X_0, \mathcal{F}_0)$ with $\mathbb{E}|X_0| < \infty$ if, for all $\varepsilon > 0$,

$$\mathfrak{U}_\varepsilon = \{i : u_n < i \leqq v_n \quad \text{and} \quad |\mathcal{D}(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| \geqq \varepsilon\}$$

has zero natural density. This means that, for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{|\{i : u_n < i \leqq v_n \quad \text{and} \quad |\mathcal{D}(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| \geqq \varepsilon\}|}{u_n - v_n} = 0.$$

We write

$$\text{stat}_{\text{DM}} \lim_{n\to\infty} \mathcal{D}(X_i,\mathcal{F}_i) = (X_0,\mathcal{F}_0).$$

We now establish an inclusion theorem concerning the two new and interesting notions that every deferred Cesàro statistical convergent martingale sequence is statistically deferred Cesàro summable, but the converse is not generally true.

Theorem 2. *If a given martingale sequence* $(X_n,\mathcal{F}_n;n\in\mathbb{N})$ *is deferred Cesàro statistically convergent to a martingale* $(X_0,\mathcal{F}_0)$ *with* $\mathbb{E}|X_0|<\infty$, *then it is statistically deferred Cesàro summable to the same martingale, but not conversely.*

Proof. Suppose the given martingale sequence $(X_n,\mathcal{F}_n;n\in\mathbb{N})$ is deferred Cesàro statistically convergent to a martingale $(X_0,\mathcal{F}_0)$ with $\mathbb{E}|X_0|<\infty$. Then, by Definition 3, we have

$$\lim_{n\to\infty} \frac{|\{i : u_n < i \leqq v_n \quad \text{and} \quad |(X_i,\mathcal{F}_i)-(X_0,\mathcal{F}_0)| \geqq \varepsilon\}|}{u_n - v_n} = 0.$$

Now, for the following two sets:

$$\mathcal{W}_\varepsilon = \{i : u_n < i \leqq v_n \quad \text{and} \quad |(X_i,\mathcal{F}_i)-(X_0,\mathcal{F}_0)| \geqq \varepsilon\}$$

and

$$\mathcal{W}_\varepsilon^c = \{i : u_n < i \leqq v_n \quad \text{and} \quad |(X_i,\mathcal{F}_i)-(X_0,\mathcal{F}_0)| < \varepsilon\},$$

we find that

$$\begin{aligned}
|\mathcal{D}(X_n,\mathcal{F}_n)-(X_0,\mathcal{F}_0)| &= \left| \frac{1}{v_n-u_n} \sum_{k=u_n+1}^{v_n} (X_k,\mathcal{F}_k)-(X_0,\mathcal{F}_0) \right| \\
&\leqq \left| \frac{1}{v_n-u_n} \sum_{k=u_n+1}^{v_n} [(X_k,\mathcal{F}_k)-(X_0,\mathcal{F}_0)] \right| \\
&\quad + \left| \frac{1}{v_n-u_n} \sum_{k=u_n+1}^{v_n} (X_0,\mathcal{F}_0)-(X_0,\mathcal{F}_0) \right| \\
&\leqq \frac{1}{v_n-u_n} \sum_{\substack{k=u_n+1 \\ (i\in\mathcal{W}_\varepsilon)}}^{v_n} |\mathcal{D}(X_n,\mathcal{F}_n)-(X_0,\mathcal{F}_0)| \\
&\quad + \frac{1}{v_n-u_n} \sum_{\substack{k=u_n+1 \\ (i\in\mathcal{W}_\varepsilon^c)}}^{v_n} |\mathcal{D}(X_n,\mathcal{F}_n)-(X_0,\mathcal{F}_0)| \\
&\quad + |(X_0,\mathcal{F}_0)| \left| \frac{1}{v_n-u_n} \sum_{\lambda=u_n+1}^{v_n} -1 \right| \\
&\leqq \frac{1}{v_n-u_n} |\mathcal{W}_\varepsilon| + \frac{1}{v_n-u_n} |\mathcal{W}_\varepsilon^c| = 0.
\end{aligned}$$

Thus, clearly, we obtain

$$|\mathcal{D}(X_n, \mathcal{F}_n) - (X_0, \mathcal{F}_0)| < \varepsilon.$$

Therefore, the martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is statistically deferred Cesàro summable to the martingale $(X_0, \mathcal{F}_0)$ with $\mathbb{E}|X_0| < \infty$.

Next, in support of the non-validity of the converse statement, we present here an example demonstrating that a statistically deferred Cesàro summable martingale sequence is not necessarily deferred Cesàro statistically convergent.

Example 2. Let us set

$$u_n = 2n \qquad \text{and} \qquad v_n = 4n \qquad (n \in \mathbb{N}).$$

Also let $(\mathcal{F}_n, n \in \mathbb{N})$ be a monotonically increasing sequence of 0-mean independent random variables of σ-fields with $(X_n) \in \mathcal{F}_n$ such that

$$X_n = \begin{cases} 1 & (n = 2m;\ m \in \mathbb{N}) \\ -1 & (n = 2m+1;\ m \in \mathbb{N}). \end{cases}$$

It is easy to see that, the martingale sequence $(X_n, \mathcal{F}_n, n \in \mathbb{N})$ is neither convergent nor deferred Cesàro statistically convergent; however, it is deferred Cesàro summable to $\frac{1}{2}$. Therefore, it is statistically deferred Cesàro summable to $\frac{1}{2}$.

□

3. KOROVKIN-TYPE THEOREMS FOR MARTINGALE SEQUENCE

Quite recently, a number of researchers worked toward extending (or generalizing) the approximation aspect of the Korovkin-type theorems in different fields of mathematics such as (for example) sequence spaces, Banach space, probability space, measurable space, and so on. This concept is extremely valuable in real analysis, functional analysis, harmonic analysis, and other related areas. Here, in this connection, we choose to refer the interested readers to the recent works [5, 12–15, 17–19] and [20].

We establish here the statistical versions of new Korovkin-type approximation theorems for martingale sequences of positive linear operators via deferred Cesàro mean.

Let $\mathcal{C}([0,1])$ be the space of all real-valued continuous functions defined on $[0,1]$ under the norm $\|\cdot\|_\infty$. Also let $\mathcal{C}[0,1]$ be a Banach space. Then, for $f \in \mathcal{C}[0,1]$, the norm of f denoted by $\|f\|$ is given by

$$\|f\|_\infty = \sup_{x\in[0,1]} \{|f(x)|\}.$$

We say that the operator $\mathfrak{L}$ is a martingale sequence of positive linear operators, provided that

$$\mathfrak{L}(f;x) \geqq 0 \quad \text{whenever} \quad f \geqq 0 \text{ with } \mathfrak{L}(f;x) < \infty \text{ and } \operatorname{Prob}\big(\mathfrak{L}(f;x)\big) = 1.$$

Theorem 3. *Let*

$$\mathfrak{L}_m : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$$

be a martingale sequence of positive linear operators. Then, for all $f \in \mathcal{C}[0,1]$,

$$\mathrm{DM}_{\mathrm{stat}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_\infty = 0 \tag{3.1}$$

if and only if

$$\mathrm{DM}_{\mathrm{stat}} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x) - 1\|_\infty = 0, \tag{3.2}$$

$$\mathrm{DM}_{\mathrm{stat}} \lim_{m\to\infty} \|\mathfrak{L}_m(2x;x) - 2x\|_\infty = 0 \tag{3.3}$$

and

$$\mathrm{DM}_{\mathrm{stat}} \lim_{m\to\infty} \|\mathfrak{L}_m(3x^2;x) - 3x^2\|_\infty = 0. \tag{3.4}$$

Proof. Since each of the following functions:

$$f_0(x) = 1, \quad f_1(x) = 2x \quad \text{and} \quad f_2(x) = 3x^2$$

belong to $\mathcal{C}[0,1]$ and are continuous, the implication given by (3.1) implies that conditions (3.2) to (3.4) are obvious.

In order to complete the proof of Theorem 3, we first assume that conditions (3.2) to (3.4) hold true. If $f \in \mathcal{C}[0,1]$, then there exists a constant $\mathcal{A} > 0$ such that

$$|f(x)| \leqq \mathcal{A} \quad (\forall\, x \in [0,1]).$$

We thus find that

$$|f(t) - f(x)| \leqq 2\mathcal{A} \qquad (t,x \in [0,1]). \tag{3.5}$$

Clearly, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(t) - f(x)| < \varepsilon \tag{3.6}$$

whenever

$$|t-x| < \delta \quad \text{for all} \quad t,x \in [0,1].$$

Let us choose

$$\varphi_1 = \varphi_1(t,x) = 4(t-x)^2.$$

If $|t-x| \geqq \delta$, then we find that

$$|f(t) - f(x)| < \frac{2\mathcal{A}}{\delta^2}\varphi_1(t,x). \tag{3.7}$$

Thus, from the equations (3.6) and (3.7), we get

$$|f(t) - f(x)| < \varepsilon + \frac{2\mathcal{A}}{\delta^2}\varphi_1(t,x),$$

which implies that

$$-\varepsilon - \frac{2\mathcal{A}}{\delta^2}\varphi_1(t,x) \leqq f(t) - f(x) \leqq \varepsilon + \frac{2\mathcal{A}}{\delta^2}\varphi_1(t,x). \tag{3.8}$$

Now, since $\mathfrak{L}_m(1;x)$ is monotone and linear, by applying the operator $\mathfrak{L}_m(1;x)$ to this inequality, we have

$$\mathfrak{L}_m(1;x)\left(-\varepsilon-\frac{2\mathcal{A}}{\delta^2}\varphi_1(t,x)\right) \leqq \mathfrak{L}_m(1;x)(f(t)-f(x)) \leqq \mathfrak{L}_m(1;x)\left(\varepsilon+\frac{2\mathcal{A}}{\delta^2}\varphi_1(t,x)\right).$$

We note that x is fixed and so $f(x)$ is a constant number. Therefore, we have

$$-\varepsilon\mathfrak{L}_m(1;x)-\frac{2\mathcal{A}}{\delta^2}\mathfrak{L}_m(\varphi_1;x) \leqq \mathfrak{L}_m(f;x)-f(x)\mathfrak{L}_m(1;x) \leqq \varepsilon\mathfrak{L}_m(1;x)+\frac{2\mathcal{A}}{\delta^2}\mathfrak{L}_m(\varphi_1;x). \tag{3.9}$$

We know also that that

$$\mathfrak{L}_m(f;x)-f(x) = [\mathfrak{L}_m(f;x)-f(x)\mathfrak{L}_m(1;x)]+f(x)[\mathfrak{L}_m(1;x)-1]. \tag{3.10}$$

So, by using (3.9) and (3.10), we have

$$\mathfrak{L}_m(f;x)-f(x) < \varepsilon\mathfrak{L}_m(1;x)+\frac{2\mathcal{A}}{\delta^2}\mathfrak{L}_m(\varphi_1;x)+f(x)[\mathfrak{L}_m(1;x)-1]. \tag{3.11}$$

We now estimate $\mathfrak{L}_m(\varphi_1;x)$ as follows:

$$\begin{aligned}\mathfrak{L}_m(\varphi_1;x) &= \mathfrak{L}_m((2t-2x)^2;x) = \mathfrak{L}_m(2t^2-8xt+4x^2;x)\\ &= \mathfrak{L}_m(4t^2;x)-8x\mathfrak{L}_m(t;x)+4x^2\mathfrak{L}_m(1;x)\\ &= 4[\mathfrak{L}_m(t^2;x)-x^2]-8x[\mathfrak{L}_m(t;x)-x]+4x^2[\mathfrak{L}_m(1;x)-1].\end{aligned}$$

Thus, by using (3.11), we obtain

$$\begin{aligned}\mathfrak{L}_m(f;x)-f(x) &< \varepsilon\mathfrak{L}_m(1;x)+\frac{2\mathcal{A}}{\delta^2}\{4[\mathfrak{L}_m(t^2;x)-x^2]\\ &\quad -8x[\mathfrak{L}_m(t;x)-x]+4x^2[\mathfrak{L}_m(1;x)-1]\}+f(x)[\mathfrak{L}_m(1;x)-1].\\ &= \varepsilon[\mathfrak{L}_m(1;x)-1]+\varepsilon+\frac{2\mathcal{A}}{\delta^2}\{4[\mathfrak{L}_m(t^2;x)-x^2]\\ &\quad -8x[\mathfrak{L}_m(t;x)-x]+4x^2[\mathfrak{L}_m(1;x)-1]\}+f(x)[\mathfrak{L}_m(1;x)-1].\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we can write

$$\begin{aligned}|\mathfrak{L}_m(f;x)-f(x)| &\leqq \varepsilon+\left(\varepsilon+\frac{8\mathcal{A}}{\delta^2}+\mathcal{A}\right)|\mathfrak{L}_m(1;x)-1|\\ &\quad +\frac{16\mathcal{A}}{\delta^2}|\mathfrak{L}_m(t;x)-x|+\frac{8\mathcal{A}}{\delta^2}|\mathfrak{L}_m(t^2;x)-x^2|\\ &\leqq \mathcal{E}(|\mathfrak{L}_m(1;x)-1|+|\mathfrak{L}_m(t;x)-x|+|\mathfrak{L}_m(t^2;x)-x^2|),\end{aligned} \tag{3.12}$$

where

$$\mathcal{E} = \max\left(\varepsilon + \frac{8\mathcal{A}}{\delta^2} + \mathcal{A}, \frac{16\mathcal{A}}{\delta^2}, \frac{8\mathcal{A}}{\delta^2}\right).$$

Now, for a given $r > 0$, there exists $\varepsilon > 0$ $(\varepsilon < r)$, we get

$$\mathfrak{G}_m(x;r) = \{m : u_n < m \leqq v_n \quad \text{and} \quad |\mathfrak{L}_m(f;x) - f(x)| \geqq r\}.$$

Furthermore, for $j = 0,1,2$, we have

$$\mathfrak{G}_{j,m}(x;r) = \left\{m : u_n < m \leqq v_n \quad \text{and} \quad \left|\mathfrak{L}_m(f;x) - f_j(x)\right| \geqq \frac{r-\varepsilon}{3\mathcal{K}}\right\},$$

so that

$$\mathfrak{G}_m(x;r) \leqq \sum_{j=0}^{2} \mathfrak{G}_{j,m}(x;r).$$

Clearly, we obtain

$$\frac{\|\mathfrak{G}_m(x;r)\|_{C[0,1]}}{v_n - u_n} \leqq \sum_{i=0}^{2} \frac{\|\mathfrak{G}_{j,m}(x;r)\|_{C[0,1]}}{v_n - u_n}. \tag{3.13}$$

Now, using the above assumption about the implications in (3.2) to (3.4) and, by Definition 3, the right-hand side of (3.13) is seen to tend to 0 as $n \to \infty$. Consequently, we get

$$\lim_{n\to\infty} \frac{\|\mathfrak{G}_m(x;r)\|_{C[0,1]}}{v_n - u_n} = 0 \ (\delta, r > 0).$$

Therefore, implication (3.1) holds true. This completes the proof of Theorem 3. □

Next, by using Definition 4, we present the following theorem.

Theorem 4. *Let $\mathfrak{L}_m : C[0,1] \to C[0,1]$ be a martingale sequence of positive linear operators. Also let $f \in C[0,1]$. Then*

$$\text{stat}_{\text{DM}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_\infty = 0 \tag{3.14}$$

if and only if

$$\text{stat}_{\text{DM}} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x) - 1\|_\infty = 0, \tag{3.15}$$

$$\text{stat}_{\text{DM}} \lim_{m\to\infty} \|\mathfrak{L}_m(2x;x) - 2x\|_\infty = 0 \tag{3.16}$$

and

$$\text{stat}_{\text{DM}} \lim_{m\to\infty} \|\mathfrak{L}_m(3x^2;x) - 3x^2\|_\infty = 0. \tag{3.17}$$

Proof. The proof of Theorem 4 is similar to the proof of Theorem 3. We, therefore, choose to skip the details involved. □

We present below an illustrative example for the martingale sequence of positive linear operators that does not satisfy the conditions of the statistical convergence versions of Korovkin-type approximation Theorem 3, and also the result of Srivastava *et al.* [21], but it satisfies the conditions of statistical summability versions of our Korovkin-type approximation Theorem 4. Thus, clearly, our Theorem 4 is stronger than the results asserted by Theorem 3 and also the result of Srivastava *et al.* [21].

We now recall the operator

$$\nu(1+\nu D) \qquad \left(D=\frac{d}{d\nu}\right),$$

which was used by Al-Salam [1] and, more recently, by Viskov and Srivastava [25] (see [16] and the monograph by Srivastava and Manocha [23] for various general families of operators and polynomials of this kind). Here, in our Example 3 below, we use this operator in conjunction with the Bernstein polynomials.

Example 3. Let us consider the *Bernstein polynomials* $\mathcal{B}_m(f;\nu)$ on $C[0,1]$ given by

$$\mathcal{B}_m(f;\nu)=\sum_{m=0}^{n} f\left(\frac{m}{n}\right)\binom{n}{m}\nu^m(1-\nu)^{n-m} \quad (\nu\in[0,1]). \tag{3.18}$$

Next, we present the martingale sequences positive linear operators on $C[0,1]$ defined as follows:

$$\mathfrak{L}_m(f;\nu)=[1+(X_n,\mathcal{F}_n)]\nu(1+\nu D)\mathcal{B}_m(f;\nu) \quad (\forall\, f\in C[0,1]) \tag{3.19}$$

with $(X_n,\mathcal{F}_n)$ mentioned already in Example 2 above.

Now, by using our proposed operators (3.19), calculate the values of the functions 1, 2ν and $3\nu^2$ as follows:

$$\mathfrak{L}_m(1;\nu)=[1+(X_m,\mathcal{F}_m)]\nu(1+\nu D)1=[1+(X_m,\mathcal{F}_m)]\nu,$$

$$\mathfrak{L}_m(2\nu;\nu)=[1+(X_m,\mathcal{F}_m)]\nu(1+\nu D)2\nu=[1+(X_m,\mathcal{F}_m)]\nu(1+2\nu),$$

and

$$\begin{aligned}\mathfrak{L}_m(3\nu^2;\nu)&=[1+(X_m,\mathcal{F}_m)]\nu(1+\nu D)3\left\{\nu^2+\frac{\nu(1-\nu)}{m}\right\}\\&=[1+(X_m,\mathcal{F}_m)]\left\{\nu^2\left(6-\frac{9\nu}{m}\right)\right\},\end{aligned}$$

so that we have

$$\mathrm{stat}_{\mathrm{DM}}\lim_{m\to\infty}\|\mathfrak{L}_m(1;\nu)-1\|_\infty=0,$$
$$\mathrm{stat}_{\mathrm{DM}}\lim_{m\to\infty}\|\mathfrak{L}_m(2\nu;\nu)-2\nu\|_\infty=0$$

and

$$\text{stat}_{\text{DM}} \lim_{m\to\infty} \|\mathfrak{L}_m(3\nu^2;\nu) - 3\nu^2\|_\infty = 0.$$

Consequently, the sequence $\mathfrak{L}_m(f;\nu)$ satisfies the conditions (3.15) to (3.17). Therefore, by Theorem 4, we have

$$\text{stat}_{\text{DM}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;\nu) - f\|_\infty = 0.$$

Here, clearly, the given martingale sequence $(X_m, \mathcal{F}_m)$ of functions in Example 2 is statistically deferred Cesàro summable but not deferred Cesàro statistically convergent. Thus, the martingale operators defined by (3.19) satisfy Theorem 4. However, they do not satisfy Theorem 3.

4. Concluding remarks and observations

In this concluding section of our investigation, we present several further remarks and observations concerning the various results which we have proved in this article.

Remark 1. Let $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ be a martingale sequence given in Example 2. Then, since

$$\text{stat}_{\text{DM}} \lim_{m\to\infty} X_m = \frac{1}{2} \quad \text{on} \quad [0,1],$$

we have

$$\text{stat}_{\text{DM}} \lim_{m\to\infty} \|\mathfrak{L}_m(f_j;x) - f_j(x)\|_\infty = 0 \qquad (j = 0,1,2). \tag{4.1}$$

Thus, by Theorem 3, we can write

$$\text{stat}_{\text{DM}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_\infty = 0, \tag{4.2}$$

where

$$f_0(x) = 1, \quad f_1(x) = 2x \quad \text{and} \quad f_2(x) = 3x^2.$$

Here the martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is neither statistically convergent nor it converges uniformly in the ordinary sense; thus, clearly, the classical and statistical versions of Korovkin-type theorems do not work here for the operators defined by (3.19). Hence, this application indicates that our Theorem 4 is a non-trivial generalization of the classical as well as statistical versions of Korovkin-type theorems (see [6] and [10]).

Remark 2. Let $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ be a martingale sequence given in Example 2. Then, since

$$\text{stat}_{\text{DM}} \lim_{m\to\infty} X_m = \frac{1}{2} \text{ on } [0,1],$$

so (4.1) holds true. Now, by applying (4.1) and Theorem 4, condition (4.2) also holds true. However, since the martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is not deferred Cesàro statistically convergent, but it is statistically deferred Cesàro summable. Thus, Theorem 4 is certainly a non-trivial extension of Theorem 3. Therefore, Theorem 4 is stronger than Theorem 3.

Remark 3. It is sincerely hoped that the various developments, which we have presented in this article as well as the associated bibliographical items, will motivate the interested readers in further related researches involving other approximation operators, other applied aspects of the Laguerre and associated special functions and polynomials which we have used here, the usages of wavelet frames, and so on.

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